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ON ACTIONS OF C^* ON ALGEBRAIC SPACES

by Andrzej BIALYNICKI-BIRULA^(*)

A theorem of Luna [L] says that any torus embedding which is a smooth complete algebraic space (i.e. a smooth Moisèzon space) is an algebraic variety. This result is a consequence of the following theorem proved in the present paper.

THEOREM. — *Let C^* act on a smooth complete algebraic space X . Let X_1 be the source of the action. If X_1 is an algebraic variety, then X_1 is contained in the set of all schematic points of X .*

As a corollary of the theorem we obtain not only the theorem of Luna, but also a result saying that any smooth and complete algebraic space with an action of a reductive group G , such that there exists only one closed G -orbit in X , is a projective variety.

For basic properties of algebraic spaces see [Kn].

We begin with the following

LEMMA 1. — *Let an algebraic group G act on a complete algebraic space X . Then the action is meromorphic.*

Proof. — Let X_0 be a projective model of the field $C(X)$ of meromorphic functions on X . Then the action of G on X leads to an action of G on $C(X)$ and to the induced action of G on X_0 . Moreover by Hironaka Resolution Theorem we may assume that X_0 is smooth and that we have

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a holomorphic G -equivariant map $X_0 \rightarrow X$. Let G_1 be a projective variety containing G as an open subset. Since X_0 is projective with an action of G , the action of G on X_0 is meromorphic and the graph of the action $\Gamma_0 \subseteq G \times X_0 \times X_0$ has an analytic subvariety Γ_1 in $G_1 \times X_0 \times X_0$ as its closure. Let Γ be the closure in $G_1 \times X \times X$ of the graph of the action of G on X . Now, the map $X_0 \rightarrow X$ induces a map $\Gamma_1 \rightarrow \Gamma$. Since the image of a compact analytic subvariety is an analytic subvariety, Γ is an analytic subvariety of $G_1 \times X \times X$ and thus the proof is complete.

Assume now that we have a meromorphic action of C^* on a compact manifold X . Let $X_1 \cup \dots \cup X_r$ be the decomposition of the fixed point set of the action of C^* on X into connected components. For $i = 1, \dots, r$, let $X_i^+ = \{x \in X; \lim_{t \rightarrow 0} tx \in X_i\}$. It follows from [B-BS] Appendix to §0, that there exists exactly one $i = 1, \dots, r$, such that X_i^+ is open and Zariski dense in X . X_i with this property is called the source of the action. Assume that X_1 is the source. Again by [B-BS] Appendix to §0 the map $\tau : X_1^+ \rightarrow X_1$ defined by $x \rightarrow x_0 = \lim_{t \rightarrow 0} tx$ is holomorphic.

We are going to show the following

LEMMA 2. — *Let X be a smooth algebraic space X . Then the map $\tau : X_1^+ \rightarrow X_1$ defined above is a holomorphic bundle with fiber being an affine space C^p with a linear action of C^* such that all weights of the action are positive.*

Proof. — Take $x \in X_1$. Then there exists an open neighborhood U of 0 in the tangent space $T_{x,x}$ invariant under the induced action of S^1 (where $S^1 = \{z \in C^*; |z| = 1\}$) and a S^1 -invariant biholomorphic map ϕ of U onto an open neighborhood of $x \in X$ (see e.g. [Ka], Satz 4.4). We may assume that $T_{x,X} = C^n$, and that the induced action of C^* is diagonal

$$t(z_1, \dots, z_n) = (\kappa_1(t)z_1, \dots, \kappa_n(t)z_n),$$

where $\kappa_1, \dots, \kappa_n$ are characters of C^* and hence can be identified with integers. Moreover we may assume that

$$U = \{z = (z_1, \dots, z_n) \in C^n; |z_i| \leq \varepsilon, \text{ for } i = 1, \dots, n\},$$

where ε is a sufficiently small positive real number. Consider an open connected subset V of $C^* \times U$ composed of all such points (t, u) that $tu \in U$. On V we define two holomorphic mappings :

$$(t, u) \mapsto \phi(tu)$$

$$(t, u) \mapsto t\phi(u).$$

For $t = s \in S^1$, we have $\phi(su) = s\phi(u)$. Hence the above mappings are equal on $S^1 \times U$. Since V is connected and the mappings are holomorphic, they coincide. Thus $\phi(tu) = t\phi(u)$, whenever $u, tu \in U$. Now define $\psi : C^* \times U \rightarrow X$ by $\psi(t, u) = t\phi(u)$. We claim that $t_1u_1 = tu$ implies $\psi(t_1, u_1) = \psi(t, u)$, i.e. that ψ induces a holomorphic map on C^*U . In order to prove this claim notice that if $t_1u_1 = tu$, then $t^{-1}t_1u_1 = u$ and by the above, $\phi(u) = \phi(t^{-1}t_1u_1) = (t^{-1}t_1)\phi(u_1)$. Hence $\psi(t, u) = t\phi(u) = t(t^{-1}t_1)\phi(u_1) = t_1\phi(u_1) = \psi(t_1, u_1)$.

So we have obtained a holomorphic C^* -invariant map $\psi : C^*U \rightarrow X$. Since x belongs to the source of X , the weights κ_i , $i = 1, \dots, n$, are nonnegative and we may assume that $\kappa_1 \geq \dots \geq \kappa_p > \kappa_{p+1} = \dots = \kappa_n = 0$. Since on U the map is an open immersion into X , it is an open immersion of C^*U into X . In fact, assume that $\psi(tu) = \psi(t_1u_1)$. Then, since the weights κ_i are nonnegative, there exists $t_0 \in T$ such that $t_0tu, t_0t_1u_1 \in U$ and $\psi(t_0tu) = t_0\psi(tu) = t_0\psi(t_1u_1) = \psi(t_0t_1u_1)$. Hence $t_0tu = t_0t_1u_1$ and $tu = t_1u_1$.

Let $\pi : C^n \rightarrow C^{n-p} \subset C^n$ be the projection map $\pi(z_1, \dots, z_n) = (0, \dots, 0, z_{p+1}, \dots, z_n)$. Then for $z = (z_1, \dots, z_n) \in C^*U$, $\psi\pi(z) = \tau\psi(z)$. Thus $\tau|_{\psi(C^*U)}$ is a trivial bundle with fiber C^p . This finishes the proof of the lemma.

The gluing functions of the bundle $X_1^+ \rightarrow X_1$ have values in the automorphism group $\text{Aut}_{C^*}(C^p)$ of holomorphic automorphisms of C^p commuting with the action of C^* .

LEMMA 3. — *Let $\tau : X_1^+ \rightarrow X_1$ be as in Lemma 2. Then the bundle is algebraic.*

Proof. — By theorem 3 in [Se2] (compare also [Se1]), it is enough to show that $\text{Aut}_{C^*}(C^p)$ is a linear algebraic group. Any $\alpha \in \text{Aut}_{C^*}(C^p)$ is of the form $\alpha(z) = (\alpha_1(z), \alpha_2(z), \dots, \alpha_p(z))$, where $\alpha_1, \dots, \alpha_p$ are holomorphic functions in p variables. Moreover since α commutes with action of C^* , α_i , for $i = 1, \dots, p$, is homogeneous of weight κ_i when we attach weight κ_j to variable x_j , for $j = 1, \dots, p$.

Since the weights κ_j are strictly positive, α_i for $i = 1, \dots, p$, is a polynomial and there exists an integer N such that degrees of all polynomials α_i , for all $\alpha \in \text{Aut}_{C^*}(C^p)$, are bounded by N . On the other hand $\alpha \in \text{Aut}_{C^*}(C^p)$ if and only if coefficients of the corresponding polynomials α_i satisfy some fixed polynomial identities. This shows that

$\text{Aut}_{C^*}(C^p)$ is an affine and hence a linear group. The proof is complete.

It follows from Lemma 3 that $X_1^+ - X_1/C^* \rightarrow X_1$ is an algebraic bundle with fiber $C^p - \{0\}/C^* - a$ weighted projective space.

LEMMA 4. — *Any C^* -invariant meromorphic function on X_1^+ is meromorphic on X .*

Proof. — The field of C^* -invariant meromorphic functions on X_1^+ can be identified with the field $C(X_1^+ - X_1/C^*)$ of meromorphic (hence rational) functions on a complete algebraic variety $X_1^+ - X_1/C^*$. On the other hand the field of C^* -invariant meromorphic functions on X can be identified with a subfield L of $C(X_1^+ - X_1/C^*)$. Since both have transcendence degree over C^* equal to $n - 1$, the extension $L \subseteq C(X_1^+ - X_1/C^*)$ is algebraic.

Let $U \subseteq X$ be an open C^* -invariant subset of X composed of all schematic points. Let $U_1 \subseteq U$ be an open C^* -invariant algebraic subvariety such that there exists space of orbits U_1/C^* . Then $U_1 \cap (X_1^+ - X_1)$ is open dense in X and $U_1 \cap (X_1^+ - X_1)/C^*$ is open dense in $(X_1^+ - X_1)/C^*$. Rational C^* -invariant functions on U_1 are meromorphic on X and separate points of U_1/C^* . Hence functions from L separate points belonging to an open dense subset $U_1 \cap (X_1^+ - X_1)/C^* \subseteq (X_1^+ - X_1)/C^*$. This shows that the degree of $C(X_1^+ - X_1)/C^*$ over L is equal to 1 and hence $L = C((X_1^+ - X_1)/C^*)$. The proof of the lemma is finished.

We say that a complex valued function g defined on a space Y with an action of C^* is C^* -semi-invariant if, for any $y \in Y$ and $t \in C^*$, $g(ty) = \kappa(t)g(y)$, where $\kappa : C^* \rightarrow C^*$ is a character of C^* . Then κ is called the weight of the semi-invariant function g .

LEMMA 5. — *Let f be a C^* -semi-invariant meromorphic function on X_1^+ . Then f is a meromorphic function on X .*

Proof. — Let U be as in the proof of Lemma 4. Then the field $C(U)$ of rational functions on U coincides with the field of meromorphic functions on X . One can find a function $g \in C(U)$ of the same weight as f . Then f/g is C^* -invariant and meromorphic on X_1^+ . Hence by Lemma 4 f/g is meromorphic on X . Since g is meromorphic on X , f is meromorphic on X .

Proof of the theorem. — Let $x \in X_1$. In order to prove that x is a schematic point in X it is sufficient to show that in the local ring of

holomorphic functions at x there exists a system of parameters composed of functions meromorphic on X (compare [L]). It follows from Lemma 5 that it suffices to find such a system of parameters composed of C^* -semi-invariant function meromorphic on X_1^+ . Since X_1^+ is an algebraic variety, there exists a system of parameters at x composed of C^* -semi-invariant functions which are regular at x and hence rational on X_1^+ . The functions are then meromorphic on X and thus the proof is finished.

COROLLARY 6. — *Let a compact and smooth algebraic space X be a torus embedding of a torus T . Then X is an algebraic variety.*

Proof follows from the theorem and the fact that (since X is a torus embedding) any fixed point of the action of T on X is a source of the induced action of a one parameter subgroup $C^* \rightarrow T$ (this can be seen similarly as in the proof of Lemma 1 by considering a T -invariant birational morphism of a smooth projective variety $X_0 \rightarrow X$). Notice also that any T -orbit contains a fixed point in its closure.

COROLLARY 7. — *Let X be a smooth and compact algebraic space with an action of a reductive group G . Assume that there exists only one closed G -orbit in X . Then X is a projective variety.*

Proof. — By Sumihiro Theorem [Su] any point of a normal algebraic variety X with an action of a connected algebraic group is contained in an invariant open quasi-projective subset. Hence if this variety is complete and contains only one closed orbit it has to be projective (the only open invariant subset containing a point from the closed orbit is the whole space). Thus it suffices to show that the space X is an algebraic variety. Since any G -orbit contains a closed orbit in its closure and the set of schematic points is open G -invariant, it suffices to show that the only closed G -orbit in X contains a schematic point. Therefore it follows from the theorem that it suffices to prove that the source of a one parameter subgroup in G is contained in the closed G -orbit.

Let T be a maximal torus in G . Let $C^* = T_0 \subseteq T$ be a subtorus of T such that the sets of fixed points of T and of T_0 coincide. Let P be the parabolic subgroup corresponding to $C^* = T_0$. Let x belongs to the source of the action of T_0 on X . Then x belongs to the source of the action of T_0 on the closure of Gx in X . Hence the opposite P^- of the parabolic P has to be contained in the stabilizer subgroup of x . Thus the stabilizer is parabolic and the orbit Gx is projective. Hence Gx is the only closed orbit

in X . It means that source of T_0 in X is contained in the only closed orbit and the proof is complete.

COROLLARY 8. — *Let X be a smooth algebraic space with an action of C^* . Let X_1 be the source of the action. If $x \in X_1$ is schematic in X_1 , then it is schematic in X .*

Proof. — Assume that $x \in X_1$ is schematic in X_1 . If $X = X_1$, then the corollary is trivial. Assume that $X \neq X_1$. Then (by [M]) there exists $\rho_1 : Y_1 \rightarrow X_1$, where Y_1 is a smooth algebraic variety and ρ_1 is a composition of blow ups of ideals on X_1 and its transforms supported by smooth centers not containing x . Let $\rho : Y \rightarrow X$ be the composition of the blow-ups of the corresponding ideals on X and its transforms. Then Y is smooth with the induced action of C^* and Y_1 is the source of the action. Since Y_1 is an algebraic variety, any point of Y_1 (by the theorem) is schematic in Y . In particular x is schematic in Y . Since ρ restricted to a Zariski open neighborhood of x is an isomorphism, x is schematic in X .

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