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On the characteristic power series of the $U$ operator


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ON THE CHARACTERISTIC POWER SERIES
OF THE U OPERATOR

by F.Q. GOUVÊA and B. MAZUR

Let $p$ be a prime number, and let $k$ be an integer. Atkin’s $U$ operator acts in a completely continuous manner on the $p$-adic space of overconvergent modular forms of weight $k$. The goal of this note is to show that the “Fredholm” characteristic power series of $U$ varies “$p$-adically continuously” in the weight $k$, in the following sense. If $a_m(k)$ is the $m$-th coefficient of the characteristic power series of $U$ acting on overconvergent forms of weight $k$, we show that if $k_1 \equiv k_2 \pmod{p^n(p-1)}$ then $a_m(k_1) \equiv a_m(k_2) \pmod{p^{n+1}}$ for every $m \geq 0$. We then extend this to “higher order differences” of the function $k \mapsto a_m(k)$, in the spirit of [Ser2], Thm. 14.

Our $p$-adic continuity result leads us to hope that there is a notion of “overconvergent $p$-adic modular form of weight $k$” not only for rational integers $k$, but for $k$ in the $p$-adic space

$$\mathcal{X} = \lim_{n \to \infty} \mathbb{Z}/(p-1)p^n\mathbb{Z},$$

and that the $U$ operator preserves overconvergence and is completely continuous (and therefore has a spectral theory) for all $k \in \mathcal{X}$. If so, our result would suggest that this spectral theory is uniformly continuous in $k$. At present, however, it is not evident to us how to define overconvergence for $p$-adic modular forms of general $p$-adic weights.

The methods we use are a direct extension of those in [Gou2], and our main result answers one of the “Further Questions” posed there. This
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1. Introduction.

To describe our main result precisely, let $p$ be a prime number, and assume $p \geq 5$. Fix a "tame level" $N$ not divisible by $p$; we will be working with $p$-adic modular forms of integral weight on $\Gamma_1(N)$. Let $B$ be a $p$-adically complete and separated ring, and let $r \in B$. We will let $M_k(N, B; r)$ denote the space of $r$-overconvergent $p$-adic modular forms of weight $k$ on $\Gamma_1(N)$ defined over $B$ (for definitions and properties of these spaces, whose importance was first realized by Dwork, we refer to the accounts in [Kat] and [Gou2]). If $B$ is a discrete valuation ring and $K$ is its field of fractions, we write

$$
M_k(N, K; r) = M_k(N, B; r) \otimes K;
$$

this is a $p$-adic Banach space over $K$ with respect to the norm determined by making $M_k(N, B; r)$ the unit ball. This space contains the classical spaces considered in [GM].

We fix a discrete valuation ring $B$, let $K$ be its field of fractions, and write, for simplicity, $M_k(r) = M_k(N, K; r)$. When $0 < \text{ord}(r) < p/(p+1)$, the Atkin U operator is a completely continuous linear operator on the $p$-adic Banach space $M_k(r)$, and hence has a spectral theory. In particular we can consider the characteristic power series $F_k(t) = \det(1 - tU|M_k(r))$ and, for each rational number $\alpha$, the "slope $\alpha$ subspace" $M_{k,\alpha}$ which is spanned by all the forms $f \in M_k(r)$ such that $(U - \lambda)^m(f) = 0$ for some integer $m > 0$ and some $\lambda \in \overline{K}$ such that $\text{ord}(\lambda) = \alpha$. It is a basic result in the spectral theory of the U operator that the space $M_{k,\alpha}$ is finite-dimensional and independent of the choice of $r$ (provided $0 < \text{ord}(r) < p/(p+1)$).

We can now state our main result.
THEOREM 1. — Let \( p \geq 5 \) be a prime number, \( N \) an integer not divisible by \( p \), \( B \) a \( p \)-adically complete and separated discrete valuation ring, and \( K \) its field of fractions. Choose any \( r \in B \) satisfying \( 0 < \text{ord}(r) < p/(p + 1) \). Let \( P_k(t) \) be the characteristic power series of the \( U \) operator acting on the space \( M_k(N, K; r) \) of \( r \)-overconvergent \( p \)-adic modular forms of weight \( k \) and level \( N \). Write \( P_k(t) = \sum a_m(k)t^m \). If \( k_1 \) and \( k_2 \) are integers such that

\[
k_1 \equiv k_2 \pmod{p^n(p - 1)},
\]

then we have, for each \( m \),

\[
a_m(k_1) \equiv a_m(k_2) \pmod{p^{n+1}}.
\]

Much of the technical complication in the proof of such a result is due to the fact that there are two natural topologies on the Banach spaces \( M_k(r) \). For the first topology, recall that elements of \( M_k(r) \) can be interpreted as functions of “not too supersingular” elliptic curves \( E \) defined over some \( p \)-adically complete and separated \( B \)-algebra \( A \). The restriction on the curve \( E \) is that \( E_{p-1}(E, \omega) \) should be a divisor of \( r \in B \). (See [Kat] and [Gou2] for details.) A “test-object of level \( N \) and growth condition \( r \)” is simply such a curve together with a level structure. The first topology is just the natural topology on such “functions”: its norm \( \| \cdot \|_{\text{mod}} \) is characterized by

\[
\|f\|_{\text{mod}} \leq 1 \quad \text{if and only if} \quad f(E/A, \omega, \iota, Y) \in A
\]

for any test-object \( (E/A, \omega, \iota, Y) \) of level \( N \) and growth condition \( r \). We call this topology the modular topology; its unit ball is precisely the space \( M_k(N, B; r) \). The second topology, which we call the \( q \)-expansion topology, is induced by the \( q \)-expansion map; its norm \( \| \cdot \|_{\text{q-exp}} \) can be described by saying that \( \|f\|_{\text{q-exp}} \leq 1 \) if and only if all the coefficients of the \( q \)-expansion of \( f \) are in \( B \) (i.e., are integral). It is a basic fact that the modular topology on \( M_k(1) \) (i.e., for \( r = 1 \)) coincides with the \( q \)-expansion topology, so that the unit ball in \( M_k(r) \) with respect to the \( q \)-expansion topology can also be described as the intersection \( M_k(r) \cap M_k(N, B; 1) \). (A proof can be found in [Kat].) This shows, in particular, that \( M_k(1) \) is isomorphic to Serre’s space of \( p \)-adic modular forms of weight \( k \), which is defined in [Ser2] in terms of limits of \( q \)-expansions.\(^1\) We have an inclusion of the “closed” unit balls

\[
M_k(N, B; r) \subset M_k(r) \cap M_k(N, B; 1),
\]

\(^1\) In other words, given a sequence of classical forms \( f_i \) whose \( q \)-expansions \( f_i(q) \) converge, coefficient-by-coefficient, to \( f(q) \in B[[q]] \), there always exists a form \( f \) in
but the set on the right is unbounded with respect to the modular topology.

It is sometimes convenient to use the $q$-expansion map to identify $M_k(r)$ with its image in $K \otimes B[[q]]$. (Except in the case when $r$ is a unit in $B$, the image will not be closed with respect to the "natural" topology on $K \otimes B[[q]]$.) From this point of view, the “unit ball with respect to the $q$-expansion topology” is just the intersection $M_k(r) \cap B[[q]]$.

2. Proof of Theorem 1.

As usual, there are Hecke operators $T_\ell$ for each prime number $\ell \neq p$ which act on $M_k(N, B; r)$; these have the expected action on $q$-expansions. (See [Kat] or [Gou2] for the definitions.) For $\ell = p$, however, the relevant operator is not $T_p$ (even though $p \nmid N$), but Atkin’s $U$ operator, which acts on $q$-expansions by the formula

$$U(\sum a_n q^n) = \sum a_{np} q^n.$$  

This is defined on $M_k(N, K; r)$ as $1/p$ times the trace of the Frobenius operator $\text{Frob}$, which acts on $q$-expansions as

$$\text{Frob}(\sum a_n q^n) = \sum a_{n^p} q^n.$$  

The theory of these two operators is described in detail in Chapter II of [Gou2]. We will recall here only the most important points for our purposes. To begin with,

**Proposition 1.** — If $\text{ord}(r) < 1/(p + 1)$, then we have

$$U(M_k(N, K; r)) \subset M_k(N, K; r^p).$$

See [Gou2] for a proof; we refer to this result by the code phrase “$U$ improves overconvergence.” As Dwork was the first to point out, the fact that $U$ improves overconvergence implies that $U$ is a completely continuous endomorphism of $M_k(N, K; r)$ for any $r$ satisfying $0 < \text{ord}(r) < p/(p + 1)$. What this means is that for any integer $n$ one can find a finite-dimensional $M_k(1)$ (here $k$ may be a $p$-adic weight) whose $q$-expansion is $f(q)$. Conversely, any such form is obtained in this way. A form defined by such a limit may or may not be overconvergent, since it is an element of $M_k(1)$, which properly contains $M_k(r)$, and there seems to be no direct way of deciding if it is from the existence of such a construction.
subspace $V_n \subset M_k(N, k; r)$ such that the image of the unit ball $M_k(N, B; r)$ is contained in $V_n + p^n M_k(N, K; r)$. In our case, one can find $V_n$ quite explicitly: it is generated by the $p$-adic modular forms obtained as quotients $f/E_{p-1}^i$, where $f$ is a classical modular form of level $N$ and weight $k+i(p-1)$, for $0 \leq i < (n+1)/((p-1)\text{ord}(r))$. It is straightforward to estimate that we have $\dim V_n = O(n^2)$ as $n$ tends to infinity.\(^{(2)}\)

The fact that $U$ is overconvergent implies that it has a spectral theory, as explained in [Ser] and [Mon] (see also the discussion in [Gou2]). In particular, we emphasize the following three facts:

1. The $U$ operator has a characteristic power series

$$P_k(t) = \det(1 - tU|M_k(r)) \in B[[t]]$$

which is independent of $r$ and defines a $p$-adic entire function whose reciprocal roots are the eigenvalues of $U$ on $M_k(r)$ and form a sequence tending to zero in $B$. In particular, we can write

$$P_k(t) = \prod_i (1 - \lambda_it)$$

with $\lambda_i$ ranging through the nonzero eigenvalues of $U$ (taken in the algebraic closure of $K$). We know that $\text{ord}(\lambda_i) \geq 0$ and $\lambda_i \to 0$.

2. It is possible to define the exterior powers $\bigwedge^n U$ of any completely continuous operator; they are again completely continuous, hence have traces. Then, if we write

$$P_k(t) = \sum a_n(k)t^n,$$

we have

$$a_n(k) = \text{trace}(\bigwedge^n U).$$

See [Ser2], [Lan], Chapt. 15, §5 and [Gou2] for more information on this.

3. Fix $\alpha \geq 0$, and define $M_{k,\alpha}$ to be the subspace of $M_k(r)$ spanned by the forms $f$ such that we have

$$(U - \lambda)^m(f) = 0$$

for some integer $m > 0$ and some $\lambda \in \overline{K}$ with $\text{ord}(\lambda) = \alpha$. $M_{k,\alpha}$ is then a finite-dimensional vector space, and there exists a closed Banach subspace

\(^{(2)}\) After a conversation with G. Stolzenberg, we have come to think of an estimate for $\dim V_n$ as giving a “modulus of complete continuity” for our operator.
such that we have a $U$-equivariant decomposition of $M_k(r)$ as a direct sum:

$$M_k(N,K;r) = M_{k,\alpha} \oplus F_{k,\alpha}.$$  

We call $M_{k,\alpha}$ the slope $\alpha$ eigenspace for $U$ acting on forms of weight $k$.

Recall that a $\mathbb{Z}_p$-lattice $D \subset V$ in a $p$-adic vector space $V$ is a free $\mathbb{Z}_p$-submodule of $V$ such that $D \otimes \mathbb{Q}_p = V$.

**Lemma 2.** Let $\Phi_1$ and $\Phi_2$ be completely continuous operators on a $p$-adic Banach space $V$, and let $D \subset V$ be any $\mathbb{Z}_p$-lattice in $V$. If $\Phi_1(D) \subset D$, $\Phi_2(D) \subset D$ and

$$(\Phi_1 - \Phi_2)(D) \subset p^n D,$$

then

$$P(t, \Phi_1) \equiv P(t, \Phi_2) \pmod{p^n},$$

where we understand the congruence coefficient-by-coefficient.

**Proof.** Put

$$P(t, \Phi_1) = \sum a_i t^i \quad \text{and} \quad P(t, \Phi_2) = \sum b_i t^i.$$  

We have $a_0 = b_0 = 1$, and we want to show that $a_i \equiv b_i \pmod{p^n}$ for each $i \geq 1$.

Let $\Psi = \Phi_1 - \Phi_2$. Clearly, $\Psi$ is completely continuous, and $\Psi(D) \subset p^n D$ implies that every eigenvalue of $\Psi$ is divisible by $p^n$. Hence we have

$$\text{trace}(\Phi_1) - \text{trace}(\Phi_2) = \text{trace}(\Psi) = \sum \lambda \equiv 0 \pmod{p^n},$$

where the sum is over the eigenvalues of $\Psi$. Since $a_1 = \text{trace}(\Phi_1)$ and $b_1 = \text{trace}(\Phi_2)$, this proves the first congruence.

For the remaining congruences, recall that we have

$$a_m = \text{trace} \left( \bigwedge^m \Phi_1 \right) \quad \text{and} \quad b_m = \text{trace} \left( \bigwedge^m \Phi_2 \right),$$

so we need to look at $\Psi = \bigwedge^m \Phi_1 - \bigwedge^m \Phi_2$. These are operators on $\bigwedge^m V$, which contains the $\mathbb{Z}_p$-lattice $D' = \bigwedge^m D$. Then, noting that

$$\bigwedge^m \Phi_1 - \bigwedge^m \Phi_2 = \left( \bigwedge^{m-1} \Phi_1 \right) \wedge (\Phi_1 - \Phi_2) + \left( \bigwedge^{m-1} \Phi_1 - \bigwedge^{m-1} \Phi_2 \right) \wedge \Phi_2,$$
we prove by induction that $\Psi(D')$ is contained in $p^nD'$. Thus, $a_m \equiv b_m \pmod{p^n}$, as claimed.

Now assume $k_1 \equiv k_2 \pmod{p^n(p-1)}$, and let $\mathcal{E} : M_{k_1}(N, K; r) \to M_{k_2}(N, K; r)$ denote multiplication by $E_{p-1}^{(k_2-k_1)/(p-1)}$. This is easily seen to be an isomorphism of Banach spaces. (One needs only check that the inverse map preserves overconvergence; for this, note that if $f \in M_{k_2}(B, N; r)$ then one sees directly from the definition that $r^{(k_2-k_1)/(p-1)}\mathcal{E}f \in M_{k_1}(B, N; r)$.)

Write $U_k$ for the $U$ operator acting on forms of weight $k$. We consider the operators

$$\Phi = U_{k_1} \quad \text{and} \quad \Psi = \mathcal{E}^{-1}U_{k_2}\mathcal{E},$$

both acting on $M_{k_1}(N, K; r)$. Note, first, that both are completely continuous, because both $U$ operators are. Furthermore, our two series may be computed using them:

$$P_1(t) = \det(1 - tU|M_{k_1}(N, K; r)) = \det(1 - t\Phi)$$

and, since conjugate operators have the same characteristic series,

$$P_2(t) = \det(1 - tU|M_{k_2}(N, K; r))$$

$$= \det(1 - t(\mathcal{E}^{-1}U\mathcal{E})|M_{k_1}(N, K; r)) = \det(1 - t\Psi).$$

Now we are in position to invoke Lemma 2. We take

$$D = M_{k_1}(N, K; r) \cap M_{k_1}(N, B; 1) = \{f \in M_{k_1}(N, K; r) | f(q) \in B[[q]]\}.$$

This is a lattice in $M_k(r)$, since the $q$-expansions of modular forms have bounded denominators. To apply the lemma, we need to see that $(\Phi - \Psi)D \subset p^nD$.

**Lemma 3.** — Let $W$ be a vector space over $K$, and let $L$ be a lattice in $W$. Suppose $E : W \to W$ satisfies $E = I + p^tT$, where $I$ is the identity map and $T : W \to W$ is a linear map stabilizing $L$. Set $F = E^{-1}$.

If $\Upsilon : W \to W$ is a linear operator mapping $L$ into $\nu L$ for some $\nu \in K$, then the linear operator $F\Upsilon E - \Upsilon$ maps $L$ into $p^t\nu L$.

**Proof.** — Simply note that

$$F\Upsilon E - \Upsilon = F\Upsilon(E - I) + (F - I)\Upsilon,$$

that both $E - I$ and $F - I$ map $L$ to $p^tL$, and that $F$ preserves $L$.  \[\square\]
In our situation, we take $W = K \otimes B[[q]]$, $L = B[[q]]$, $E = \mathcal{E}$ to be multiplication by $E_{p-1}^{(k_2-k_1)/(p-1)}$, and $Y = U$, so that $v = 1$. Applying the lemma, we get

$$(\Phi - \Psi)B[[q]] \subset p^nB[[q]].$$

Since we already know that the operator $\Phi - \Psi$ preserves $M_{k_1}(r)$, it follows that $(\Phi - \Psi)(D) \subset p^nD$, as claimed.

Thus, the hypotheses of Lemma 2 are satisfied, and this completes the proof of the theorem.

3. Higher order differences.

Given what has just been proved, it is natural to ask whether the coefficients $a_m(k)$ are Iwasawa functions, i.e., if there exist power series $A_m \in \mathbb{Z}_p[[T]]$ such that we have $a_m(k) = A_m((1 + p)^k - 1)$. We cannot yet answer this question. We can, however, move a few more steps in the direction of an answer by obtaining further congruence relations among the coefficients $a_m(k)$. In fact, Iwasawa functions can be completely characterized (as in [Ser2], Theorem 14) in terms of congruence properties; what we will show is that at least some of the congruences in Serre’s characterization are indeed satisfied.

To state these congruences, let $k \mapsto a(k)$ be any function from $\mathbb{Z}$ to $\mathbb{Z}_p$. Fix an $n$, set $s = p^n(p-1)$, and construct difference functions as follows:

$$
\delta_1(a,k) = a(k+s) - a(k)
$$

$$
\delta_2(a,k) = \delta_1(a,k+s) - \delta_1(a,k)
= a(k+2s) - 2a(k+s) + a(k)
$$

and, in general, for $i > 1$,

$$
\delta_i(a,k) = \delta_{i-1}(a,k+s) - \delta_{i-1}(a,k).
$$

What Serre shows is that if there exists a power series $A \in \mathbb{Z}_p[[T]]$ such that $a(k) = A((1 + p)^k - 1)$ for all $k \equiv k_0 \pmod{p-1}$, then we must have

$$
\delta_i(a,k_0) \equiv 0 \pmod{p^{i(n+1)}}.
$$

Theorem 1 is the special case of $a(k) = a_m(k)$ and $i = 1$. The basic idea of the proof, however, easily extends to handle the general case, as follows.
THEOREM 2. — Let $p \geq 5$ be a prime number, $N$ an integer not divisible by $p$, $B$ a $p$-adically complete and separated discrete valuation ring, and $K$ its field of fractions. Let $P_k(t)$ be the characteristic power series of the $U$ operator acting on the space $M_k(N, K, r)$ of $r$-overconvergent $p$-adic modular forms of weight $k$ and level $N$. Write $P_k(t) = \sum a_m(k)t^m$. Let $\delta_i$ be as above; then we have, for each $m$ and $k$,

$$\delta_i(a_m, k) \equiv 0 \pmod{p^{i(n+1)}}.$$ 

Proof. — Fix an integer $m$, and recall that $a_m(k)$ is the trace of the exterior power $\Lambda^m U$ acting on (the $m$-th exterior power of) forms of weight $k$. We use this fact to express $\delta_i(a_m, k)$ as the trace of an operator.

Consider first the case when $i = 2$. Let $E$ be the map $\Lambda^m M_k(r) \to \Lambda^m M_{k+s}(r)$ which is the $m$-th exterior power of the map given by multiplication by $E^{p^{n-1}}$. Then, as we saw above,

$$\delta_1(a_m, k) = \text{trace}\left( E^{-1} \Lambda^m U \circ E - \Lambda^m U \right).$$

Similarly, we have

$$\delta_2(a_m, k) = \text{trace}\left( E^{-2} \Lambda^m U \circ E^2 - 2E^{-1} \Lambda^m U \circ E + \Lambda^m U \right).$$

But since

$$E^{-2} \Lambda^m U \circ E^2 - 2E^{-1} \Lambda^m U \circ E + \Lambda^m U = E^{-1} \left( E^{-1} \Lambda^m U \circ E - \Lambda^m U \right) \circ E - \left( E^{-1} \Lambda^m U \circ E - \Lambda^m U \right),$$
we can apply Lemma 3 twice: once with

$$\Upsilon = \Lambda^m U$$

and $v = 1$,

and once with

$$\Upsilon = E^{-1} \Lambda^m U \circ E - \Lambda^m U$$

and $v = p^{n+1}$.

We conclude that $E^{-2} \Lambda^m U \circ E^2 - 2E^{-1} \Lambda^m U \circ E + \Lambda^m U$ maps $\Lambda^m D$ to $p^{2(n+1)} \Lambda^m D$, and therefore that its trace is congruent to zero modulo $p^{2(n+1)}$, as desired.
The general case follows in an analogous way, by repeated application of Lemma 3.

4. Open questions.

What about the other congruences given by Serre in [Ser2]? Specifically, we would like to know the answer to the following:

**QUESTION.** — Let \( c_{ij} \) be defined by the equation

\[
Y(Y - 1) \cdots (Y - j + 1) = \sum c_{ij} Y^i,
\]

and, with notations as above, let

\[
\gamma_j(a, k_0) = \sum_{i=1}^{j} c_{ij} p^{-i(n+1)} \delta_i(a, k_0).
\]

Is it true that we have

\[
\text{ord}_p(\gamma_j(a_m, k)) \geq \text{ord}_p(j!)
\]

for every \( m \) and \( k \)?

The point is that, according to [Ser2], this extra series of congruences, along with the congruences already proven, would be sufficient to guarantee that the \( a_m(k) \) are Iwasawa functions of \( k \).

There is a connection between Theorem 1 and the conjectures about “\( p \)-adic families” of modular eigenforms which we proposed in [GM]. In that paper, we considered the classical spaces \( M_k(K, Np) \) of modular forms of weight \( k \) on \( \Gamma_1(N) \cap \Gamma_0(p) \). On these spaces, there is an action of the \( U \) operator; thus, for each rational number \( \alpha \geq 0 \) we can look at the subspace \( M_{k, \alpha} \) spanned by the eigenforms for the \( U \) operator whose eigenvalues had valuation \( \alpha \). We write \( d(k, \alpha) \) for the dimension of this space. In [GM], we made the following conjecture:

**CONJECTURE 1.** — Let \( k_1 \) and \( k_2 \) be integers. Suppose both \( k_1 \) and \( k_2 \) are bigger than \( 2\alpha + 2 \), and that \( k_1 \equiv k_2 \pmod{p^n(p - 1)} \) for some integer \( n \geq \alpha \). Then \( d(k_1, \alpha) = d(k_2, \alpha) \).

In attempting to prove this conjecture, it seems natural to embed the classical spaces into the corresponding spaces of overconvergent \( p \)-adic
modular forms, which should be the “correct” context for studying $p$-adic properties of modular forms. Recall that we have an inclusion

$$M_k(K, Np) \hookrightarrow M_k(B, N; r) \otimes K,$$

which therefore gives an inclusion $M_{k, \alpha} \hookrightarrow M_{k, \alpha}$ of the slope $\alpha$ subspaces. Writing $d_p(k, \alpha) = \dim M_{k, \alpha}$ for the dimension of the $p$-adic slope $\alpha$ subspace, one might then consider a $p$-adic variant of our conjecture:

**CONJECTURE 2.** — Let $k_1$ and $k_2$ be integers such that $k_1 \equiv k_2 \pmod {p^n(p-1)}$ for some integer $n \geq \alpha$. Then $d_p(k_1, \alpha) = d_p(k_2, \alpha)$.

Both of these conjectures are known to be true when $\alpha = 0$, by the work of Hida on ordinary modular forms.\(^{(3)}\) In Hida’s work, proving Conjecture 2 is the first step in the proof of Conjecture 1, so that it is not unreasonable to expect the two conjectures to be similarly connected in general.

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\(^{(3)}\) The case $\alpha = 0$ of Conjecture 2 also follows from the main theorem in this note, as one can see by considering the Newton polygons of characteristic power series in weights $k_1$ and $k_2$. The dimension in question is the length of the first (horizontal) segment of the polygon, which is clearly the same when the coefficients of two power series are congruent modulo $p$. 

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