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NOTES ON INTERPOLATION OF HARDY SPACES

by Quanhua XU

1. Introduction and the main result.

Let D denote the unit disc of the complex plane and $\mathbf{T}$ the unit circle, the boundary of D. $\mathbf{T}$ is equipped with its normalized Lebesgue measure m. Let H_p $(0 < p \leq \infty)$ denote the usual classical Hardy space of analytic functions in D. Functions in H_p are identified with their radial limits on $\mathbf{T}$ so that H_p is a closed subspace of $L_p = L_p(\mathbf{T})$. Thus for $1 \leq p \leq \infty$, $H_p = \{f \in L_p \ : \ \hat{f}(n) = 0, \ n < 0\}$, where $\{\hat{f}(n)\}_{n\in\mathbf{Z}}$ is the sequence of Fourier coefficients of f (see [6] for more information).

Let f be a measurable function on $\mathbf{T}$. We denote by f^* the non-increasing rearrangement of $|f|$. The main result of this note is the following

THEOREM. — *Let $1 \leq p < \infty$. Let f and g be functions in $H_p \, (= H_p + H_\infty)$ such that*

$$\int_0^t \left(g^*(s)\right)^p ds \leq \int_0^t \left(f^*(s)\right)^p ds \ , \ \forall\, 0 < t \leq 1 \, .$$

Then there exists a linear operator T defined on $L_p \, (= L_p + L_\infty)$ such that $T(f) = g$, which maps L_p into H_p and L_∞ into H_∞ and whose norms on these spaces are dominated by a constant depending only on p.

In particular, for any $f \in H_p$ there exists such an operator T such that $T(f) = f$.

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This result is closely related to the interpolation theory of Hardy spaces. Before giving its consequences, let us first recall the following similar theorem due to P.W. Jones [9].

THEOREM J. — *Let f and g be two functions in H_1 $(= H_1 + H_\infty)$ such that*

$$\int_0^t g^*(s)ds \leq \int_0^t f^*(s)ds \ , \ \forall\, 0 < t \leq 1 \ .$$

Then there exists a linear operator T defined on H_1 such that $T(f) = g$, which maps boundedly H_1 into H_1 and H_∞ into H_∞.

The essential new point in our theorem, as compared with Jones' theorem, is that the required operator T such that $T(f) = g$ is defined on L_p with values in H_p, instead of being defined on H_p as in Jones' theorem. This advantage will allow us to deduce very easily at the same time all interpolation results for H_p-spaces from the corresponding known L_p-space results. For instance, this theorem gives immediately the real and complex interpolation results on Hardy spaces of Jones [8]. It includes, of course, Theorem J above, which says that (H_1, H_∞) is a Calderón-Mitjagin couple (see the discussions below).

The particular case of the preceding theorem where two functions coincide is of special interest. In this case the operator T given by the theorem for f leaves f invariant. For simplicity, we shall say that a linear operator T is a "one-point projection" at f if it maps measurable functions to analytic functions such that $T(f) = f$. Thus our theorem says that for any $f \in H_p$ there exists a one-point projection at f which is bounded simultaneously from L_p to H_p and from L_∞ to H_∞. In fact, together with known results about L_p-spaces, the existence of simultaneously bounded one-point projections already ensures the theorem in its full generality (see Section 4 below).

For stating consequences of our theorem, we shall need some basic notions from the interpolation theory. We now recall them very briefly. The reader is refered to [2] for more information.

We shall always denote by F an interpolation functor for Banach spaces, that is, F is a functor on the category of interpolation couples of Banach spaces into the category of Banach spaces such that for any two interpolation couples (A_0, A_1) and (B_0, B_1), $F(A_0, A_1)$ and $F(B_0, B_1)$ are interpolation spaces with respect to $(A_0, A_1), (B_0, B_1)$ and such that for any bounded operator T from A_j into B_j $(j = 0, 1)$ we have $F(T) = T$.

The most useful interpolation functors are those constructed from the real and complex interpolation methods. Let $0 < \theta < 1, 1 \le q \le \infty$. We shall denote by $(\cdot\,,\,\cdot)_{\theta\,q}$, $(\cdot\,,\,\cdot)_\theta$ real and complex interpolation spaces constructed respectively by the real and complex interpolation methods. By interpolation theorem, they are interpolation functors.

Let (A_0, A_1) be an interpolation couple of Banach spaces. Let $t > 0$ and $x \in A_0 + A_1$, we define

$$K(t, x; A_0, A_1) = \inf\{\|x_0\|_{A_0} + t\|x_1\|_{A_1} \ :\ x = x_0 + x_1\ x_j \in A_j\ ,\ j = 0, 1\}\ .$$

This is the so-called Peetre's K-functional. (A_0, A_1) is called a Calderón-Mitjagin couple if for any $x, y \in A_0 + A_1$ such that

$$K(t, y; A_0, A_1) \le K(t, x; A_0, A_1)\ , \forall t > 0\ ,$$

there exists a linear operator T from A_0+A_1 to itself which maps boundedly A_j into A_j $(j = 0, 1)$ such that $Tx = y$.

Let us also recall the following well-known results (*cf.* [2], [5], [14], [17]). Let $0 < \theta < 1$, $1 \le p_0, p_1 \le \infty$. Then

$$(L_{p_0}, L_{p_1})_{\theta\,p} = L_p \quad \text{and} \quad (L_{p_0}, L_{p_1})_\theta = L_p\ ,$$

where $\dfrac{1-\theta}{p} = \dfrac{\theta}{p_0} + \dfrac{1}{p_1}$. Furthermore, (L_{p_0}, L_{p_1}) is a Calderón-Mitjagin couple. Finally, we have for $1 \le p < \infty$

$$K(t, f; L_p, L_\infty) \approx \Big(\int_0^{t^p} (f^*(s))^p\, ds\Big)^{1/p}\ , \tag{1}$$

where the equivalence constants depend on p only.

We can now state the first consequence of the main theorem. For a Banach space X of integrable functions on $\mathbf{T}$ we denote by $H(X)$ the closed subspace of X defined by

$$H(X) = \{f \in X\ :\ \hat{f}(n) = 0,\ \forall\, n < 0\}.$$

COROLLARY 1. — *Let F be an interpolation functor and $1 \le p < \infty$. Then $F(H_p, H_\infty) = H(F(L_p, L_\infty))$.*

Proof. — It is evident that $F(H_p, H_\infty) \subset H(F(L_p, L_\infty))$. For the reverse inclusion, we take a function f in $H(F(L_p, L_\infty))$. Then $f \in H_1$ and thus by our theorem there is a one-point projection T at f which is bounded from L_p to H_p and from L_∞ to H_∞. Therefore T is also bounded from $F(L_p, L_\infty)$ to $F(H_p, H_\infty)$; so that $f = T(f) \in F(H_p, H_\infty)$ and

$$\|f\|_{F(H_p,H_\infty)} = \|T(f)\|_{F(H_p,H_\infty)} \le C\, \|f\|_{F(L_p,L_\infty)}\ ,$$

where C is a constant depending only on F and p. This concludes the proof.

Corollary 1 simply says that F commutes with H. Applying it to the real and complex functors $(\cdot\,,\,\cdot)_{\theta\, q}$ and $(\cdot\,,\,\cdot)_\theta$, we recover the two interpolation theorems of Jones on Hardy spaces, as we have already noted :

$$(H_p, H_\infty)_{\theta\, q} = H_q \quad \text{and} \quad (H_p, H_\infty)_\theta = H_q\ ,$$

where $\frac{1}{q} = \frac{1-\theta}{p}$ (*cf.* [8]).

The existence of bounded one-point projections also yields immediately the following more precise result on K-functional, which is also due to Jones [8], and reformulated by Sharpley (*cf.* [1]) as follows :

(2) $$K(t, f; H_p, H_\infty) \leq CK(t, f; L_p, L_\infty)\ , \quad \forall f \in H_p,\ \forall t > 0\ ,$$

where C is a constant depending on p only.

Combining (1) and (2), we get for any $f \in H_p\,(1 \leq p < \infty)$ and any $t > 0$

$$K(t, f; H_p, H_\infty) \approx \Big(\int_0^{t^p} (f^*(s))^p\, ds\Big)^{1/p}\ .$$

Therefore, Theorem J says that (H_1, H_∞) is a Calderón-Mitjagin couple. Our main theorem shows that the same is true for $(H_p, H_\infty)\,(1 < p < \infty)$. We shall see more general results of this nature.

The paper is organized as follows. In Section 2, using techniques recently developed by S.V. Kisliakov, we prove a technical lemma, which is of interest in itself and will be crucial in the proof of our theorem. The proof of the theorem is presented in Section 3. It is very elementary and only involves the boundedness of the Hilbert transform. Section 4 contains some further consequences of the theorem. Here, we prove that if (E_0, E_1) is an interpolation couple of rearrangement invariant Banach spaces on $\mathbf{T}$, then $F(H(E_0), H(E_1)) = H(F(E_0, E_1))$ for any interpolation functor F . Moreover, if (E_0, E_1) is a Calderón-Mitjagin couple, so is $(H(E_0), H(E_1))$.

We shall denote by C a constant which is independent of functions in consideration and may vary from lines to lines.

2. Lemma.

The following lemma is a strengthened form of a lemma of Kisliakov [10]. A variant of it may be obtained by adapting a construction of

Bourgain[3], as was pointed out to us by Kisliakov. The only improvement of the following lemma on those of Bourgain and Kisliakov is the property (ii) below, which will be crucial later. The constructions of this type also have applications in the other contexts (*cf.* [3], [10]-[12]).

LEMMA. — *Let a be a function on $\mathbf{T}$ such that*

$$2^{k_0} < a < 2^{k_1} \text{ a.e. on } \mathbf{T} \tag{3}$$

for two integers k_0, k_1 in $\mathbf{Z}$. Then there exist a function b on $\mathbf{T}$ and a sequence $\{\varphi_k\}_{k=k_0+1}^{k_1}$ in H_∞ satisfying the following properties :

(i) $b \geq a$ a.e. on $\mathbf{T}$;

(ii) *for any* $0 < t \leq 1$

$$\int_0^t b^*(s)ds \leq C \int_0^t a^*(s)ds \;;$$

(iii) $\|\varphi_k\|_\infty \leq C \quad (k_0 + 1 \leq k \leq k_1)$;

(iv) $|\varphi_k|^{1/8} b \leq C 2^k$ a.e. $(k_0 + 1 \leq k \leq k_1)$;

(v) $\sum_{k=k_0+1}^{k_1} |\varphi_k|^{1/8} 2^k \leq Cb$ a.e.;

(vi) $\sum_{k=k_0+1}^{k_1} \varphi_k = 1$ a.e.,

where C is an absolute constant.

Proof. — We define by induction three sequences $\{E_k\}_{k=k_0+1}^{k_1}$ of measurable subsets of $\mathbf{T}$, $\{G_k\}_{k_0+1}^{k_1} \subset H_\infty$ and $\{b_k\}_{k_0+1}^{k_1} \subset L_\infty$ as follows. Let first

$$E_{k_1} = \emptyset, \quad G_{k_1} \equiv 1, \quad b_{k_1} = a \,.$$

Then for $k_0 + 1 \leq k \leq k_1 - 1$, we define inductively

$$\begin{aligned} E_k &= \{b_{k+1} > 2^k\} \\ \alpha_k &= \max\Big\{1, \Big(\frac{b_{k+1}}{2^k}\Big)^{1/4}\Big\} \\ F_k &= \frac{1}{\alpha_k + i\mathcal{H}(\alpha_k)}\,, \qquad G_k = 1 - (1 - F_k^{32})^{16} \\ b_k &= b_{k+1} + \delta |G_{k+1} - G_k|^{1/8} 2^{k+1}\,, \end{aligned}$$

where $\mathcal{H}$ stands for the Hilbert transform and where δ $(0 < \delta \leq 1)$ is a constant to be determined later.

It is easy to check inductively that for $k_0+1 \le k \le k_1$, $b_k \in L_\infty$, F_k, $G_k \in H_\infty$ and

$$|F_k| \le \frac{1}{\alpha_k} \le 1 \text{ a.e.}; \tag{4}$$

$$|G_k| \le C|F_k|^{32} \le C \text{ a.e.} . \tag{5}$$

Now let

$$\begin{aligned} \varphi_{k_0+1} &= G_{k_0+1}, \ \varphi_k = G_k - G_{k-1} \quad \text{for } k_0+2 \le k \le k_1 ; \\ b &= b_{k_0+1} + \delta|\varphi_{k_0+1}|^{1/8} 2^{k_0+1} . \end{aligned}$$

Then for these functions b, φ_k, we clearly have (i), (iii) and (vi). It is also clear that $b \ge b_k \ge b_{k+1} \ge a$ for $k_0+1 \le k \le k_1-1$ so that $E_k \supset E_{k+1}$ $(k_0+1 \le k \le k_1-1)$. Let additionally

$$E_{k_0} = \{b_{k_0+1} > 2^{k_0}\} .$$

From (3), $E_{k_0} = \mathbf{T}$. Let

$$e_k = E_k \setminus E_{k+1}, \ k_0 \le k \le k_1 - 1 .$$

Then $\{e_k\}_{k=k_0}^{k_1-1}$ is a sequence of disjoint measurable subsets of $\mathbf{T}$ such that $\mathbf{T} = \bigcup_{k=k_0}^{k_1-1} e_k$.

By the definitions of b_k and b we get for $k_0+1 \le k \le k_1$

$$b = b_k + \delta \sum_{j=k_0+1}^{k} |\varphi_j|^{1/8} 2^j = a + \delta \sum_{j=k_0+1}^{k_1} |\varphi_j|^{1/8} 2^j, \tag{6}$$

which, together with (5), gives for $k_0+1 \le k \le k_1$

$$\begin{aligned} |\varphi_k|^{1/8} b &\le |G_k|^{1/8} b_{k+1} + |G_{k-1}|^{1/8} b_k + C2^k \\ &\le C2^k . \end{aligned}$$

This is (iv). (6) will also yield (v) (with $C = \delta^{-1}$) after appropriately choosing δ in order that (ii) is verified. Therefore it remains to check (ii) for an appropriate δ. For this we first claim that

$$2^k \le b \le C2^k \text{ on } e_k \quad (k_0 \le k \le k_1 - 1) . \tag{7}$$

Indeed, for a fixed $k_0 \le k \le k_1 - 1$, the first inequality in (7) is evident since by (6)

$$b \ge b_{k+1} > 2^k \text{ on } E_k .$$

For the second inequality, by (6) and the definition of E_k

$$b \le b_{k+1} + C\delta 2^k \le C2^k \text{ on } \mathbf{T} \setminus E_k ;$$

whence

$$b \leq C2^k \quad \text{on} \quad e_{k-1} \text{ for } k_0+1 \leq k \leq k_1 - 1\ ,$$

which proves the second inequality of (7).

Now define

$$d = \sum_{k=k_0}^{k_1-1} 2^k \chi_{e_k},$$

where χ_e denotes the characteristic function of a subset e. Then by (7)

$$d \leq b \leq Cd,\ a.e. \text{ on } \mathbf{T}\ . \tag{8}$$

Therefore for proving (ii), we may consider d instead of b. Put $t_{k_1} = 0$, $t_k = \sum_{j=k}^{k_1-1} m(e_j)$ for $k_0 \leq k \leq k_1 - 1$. Then $t_{k_1} = 0 \leq t_{k_1-1} \leq \cdots \leq t_{k_0} = 1$ and

$$d^* = \sum_{k=k_0}^{k_1-1} 2^k \chi_{((t_{k+1}, t_k])}\ . \tag{9}$$

Now fix $k_0 \leq k \leq k_1 - 1$. We are going to estimate $\int_0^{t_k} d^*(s)ds$. We have

$$\begin{aligned}\int_0^{t_k} d^*(s)ds &= \sum_{j=k}^{k_1-1} 2^j m(e_j)\\ &= \int_{E_k} d \leq \int_{E_k} b\ .\end{aligned}$$

It follows by (6) that

$$\begin{aligned}\int_0^{t_k} d^*(s)ds &\leq \int_{E_k} a + C\delta 2^k m(E_k) + \delta \sum_{j=k+1}^{k_1} 2^j \int_{E_k} |\varphi_j|^{1/8}\\ &\leq \int_0^{t_k} a^*(s) + C\delta \int_0^{t_k} d^*(s)ds + \delta \sum_{j=k+1}^{k_1} 2^j \int_{\mathbf{T}} |\varphi_j|^{1/8}\ .\end{aligned}$$

Next we majorize

$$\int_{\mathbf{T}} |\varphi_j|^{1/8} \quad \text{for } k+1 \leq j \leq k_1\ .$$

By the definitions of φ_j, G_j and (4) and the L_2-boundedness of $\mathcal{H}$, we have

$$\begin{aligned}\int_{\mathbf{T}} |\varphi_j|^{1/8} &\le \int_{\mathbf{T}} |1-G_j|^{1/8} + \int_{\mathbf{T}} |1-G_{j-1}|^{1/8}\\ &\le C\Big(\int_{\mathbf{T}} |1-F_j|^2 + \int_{\mathbf{T}} |1-F_{j-1}|^2\Big)\\ &\le C\Big(\int_{\mathbf{T}} |\alpha_j-1|^2 + \int_{\mathbf{T}} |\alpha_{j-1}-1|^2\Big)\\ &\le C\Big(2^{-j/2}\int_{E_j} b_{j+1}^{1/2} + 2^{-(j-1)/2}\int_{E_{j-1}} b_j^{1/2}\Big)\\ &\le C(2^{-j/2}\int_{E_j} b^{1/2} + 2^{-(j-1)/2}\int_{E_{j-1}} b^{1/2}\Big)\ .\end{aligned}$$

Combining the preceding inequalities with (7), we obtain :

$$\begin{aligned}\sum_{j=k+1}^{k_1} 2^j \int_{\mathbf{T}} |\varphi_j|^{1/8} &\le C\sum_{j=k}^{k_1} 2^{j/2}\sum_{l=j}^{k_1-1} 2^{l/2} m(e_l)\\ &\le C\sum_{l=k}^{k_1-1} 2^l m(e_l) = C\int_0^{t_k} d^*(s)ds\ .\end{aligned}$$

Therefore

$$\int_0^{t_k} d^*(s)ds \le \int_0^{t_k} a^*(s)ds + C\delta\int_0^{t_k} d^*(s)ds\ .$$

Choosing $\delta = \dfrac{1}{2C}$, we then deduce that

$$\int_0^{t_k} d^*(s)ds \le 2\int_0^{t_k} a^*(s)ds,\ k_0 \le k \le k_1 - 1\ .$$

Now the function $\int_0^t d^*(s)ds$ is linear for $t \in [t_{k+1}, t_k]$ and the function $\int_0^t a^*(s)ds$ is concave for $t \in [0,1]$. Hence, by the last inequalities we get

$$\int_0^t d^*(s)ds \le 2\int_0^t a^*(s)ds,\ \forall\, 0 < t \le 1\ . \tag{10}$$

This proves (ii) in virtue of (8) and completes the proof of the lemma.

3. Proof of the theorem.

We shall prove the theorem in this section. Let $1 \le p < \infty$. Let f, g be two functions as in the theorem. We shall show the existence of the required operator T only in the case where g is in H_∞ and $|g|$ is bounded away from zero so that the lemma in the previous section can be applied to $a = |g|$. The general case will easily follow from this special one by a standard limit argument.

Let $a = |g|$ and k_0, $k_1 \in \mathbf{Z}$ such that (3) is satisfied. Applying the lemma in Section 2 to a we get b and $\{\varphi_k\}_{k=k_0+1}^{k_1} \subset H_\infty$ which satisfy the properties (i) - (vi). We retain all the notations in that lemma and its proof. Recall the function d is defined by

$$d = \sum_{k=k_0}^{k_1-1} 2^k \chi_{e_k} \, .$$

By (10) and a well-known theorem of Hardy-Littlewood-Pólya (*cf.* [7]), we have

$$\int_0^{t^p} (d^*(s))^p ds \le 2^p \int_0^{t^p} (a^*(s))^p ds, \quad \forall\, 0 < t \le 1 \, .$$

Then by the assumption on f and g in the theorem

$$\int_0^{t^p} (d^*(s))^p ds \le 2^p \int_0^{t^p} (f^*(s))^p ds, \quad \forall\, 0 < t \le 1 \, .$$

Replacing f by $2f$, we may assume

$$\int_0^{t^p} (d^*(s))^p ds \le \int_0^{t^p} (f^*(s))^p ds, \quad \forall 0 < t \le 1 \, .$$

Then it follows from Lemma 1 of [4] that there exist a finite sequence $\{\lambda_n\}_{n=0}^N$ of positive numbers and a sequence $\{\sigma_n\}_{n=0}^N$ of measure preserving transformations of $[0,1]$ onto itself such that

$$\sum_{n=0}^N \lambda_n = 1 \tag{11}$$

$$\sum_{n=0}^N \lambda_n \big(f^*(\sigma_n(s))\big)^p \ge (d^*(s))^p, \; 0 \le s \le 1 \, .$$

Therefore

$$(d^*(s))^p = \sum_{n=0}^N \lambda_n \big(f^*(\sigma_n(s))\big)^p w(s), \; 0 \le s \le 1, \tag{12}$$

where $0 \le w(s) \le 1$ $(0 \le s \le 1)$. Set

$$\tilde{e}_{k,n} = \sigma_n((t_{k+1}, t_k]), k_0 \le k \le k_1 - 1, \; 0 \le n \le N,$$

$$w_k(s) = \sum_{n=0}^N \lambda_n w(\sigma_n^{-1}(s)) \chi_{\tilde{e}_{k,n}}(s), \quad 0 \le s \le 1 \, .$$

Then by (11) and (12), we easily check for $k_0 \le k \le k_1 - 1$ that

$$\int_0^1 (f^*(s))^p w_k(s)ds = 2^{pk} m(e_k) \tag{13}$$

$$\int_0^1 w_k(s)ds \le m(e_k) \tag{14}$$

$$\sum_{k=k_0}^{k_1-1} w_k(s) \le 1, \quad s \in [0,1] . \tag{15}$$

Now for each $k_0 + 1 \le k \le k_1$, we let

$$v_k = \Big(\sum_{j=k-1}^{k_1-1} 2^{pj} m(e_j) \Big)^{1/p} ,$$

$$u_k(s) = \sum_{j=k-1}^{k_1-1} 2^{p(j-k+1)} w_j(s) , \quad s \in [0,1] .$$

By (13) we find a positive function g_k on $[0,1]$ such that

$$\int_0^1 f^*(s) g_k(s) u_k(s) ds = \Big(\sum_{j=k-1}^{k_1-1} 2^{p(j-k+1)} \cdot 2^{pj} m(e_j) \Big)^{1/p} , \tag{16}$$

$$\int_0^1 (g_k(s))^q u_k(s) ds \le 1 , \quad (1/q + 1/p = 1). \tag{17}$$

From [4] Lemma 2 we deduce that there exists a linear operator S from $L_p(\mathbf{T})$ to $L_p(0,1)$ and from $L_\infty(\mathbf{T})$ to $L_\infty(0,1)$ whose norms on these spaces are bounded by 1, such that $Sf = f^*$.

Then we define for every $h \in L_p$

$$T_k(h) = \frac{1}{v_k} \int_0^1 Sh(s) g_k(s) u_k(s) ds .$$

By (16)

$$T_k(f) = \frac{1}{v_k} \Big(\sum_{j=k-1}^{k_1-1} 2^{p(j-k+1)} \cdot 2^{pj} m(e_j) \Big)^{1/p} \ge 1 .$$

Let $a_k = \dfrac{1}{T_k(f)}$. Then $0 < a_k \le 1$ for $k_0 + 1 \le k \le k_1$. The required operator T is then defined by

$$T(h) = \sum_{k=k_0+1}^{k_1} a_k T_k(h) \varphi_k g, \quad h \in L_p .$$

Clearly, $T(f) = g$ and T takes values in H_p. Given $h \in L_\infty$, we have by the Hölder inequality and (17) and (14)

$$\begin{aligned}|T_k(h)| &\leq \frac{\|Sh\|_\infty}{v_k}\Big(\sum_{j=k-1}^{k_1-1} 2^{p(j-k+1)}m(e_j)\Big)^{1/p}\\ &\leq 2^{-k+1}\|h\|_\infty\,.\end{aligned}$$

Then by (i), (iii)-(v)

$$\begin{aligned}\|T(h)\|_\infty &\leq \Big\|\sum_{k=k_0+1}^{k_1} a_k 2^{-k+1}|\varphi_k|^{1/2}b\cdot|\varphi_k|^{1/2}\Big\|_\infty\|h\|_\infty\\ &\leq C\|h\|_\infty\Big\|\sum_{k=k_0+1}^{k_1}|\varphi_k|^{1/4}b^{-1}\cdot|\varphi_k|^{1/4}b\Big\|_\infty\\ &\leq C\|h\|_\infty\,.\end{aligned}$$

Therefore T maps L_∞ into H_∞ boundedly. To prove that T maps L_p into H_p boundedly, we first claim that for $k_0 + 1 \leq k \leq k_1$

$$\int_{\mathbf{T}} |\varphi_k|^{p/2} \leq C2^{-pk}v_k^p\,. \tag{18}$$

Indeed, by the definition of φ_k, we have

$$\begin{aligned}\int_{\mathbf{T}}|\varphi_k|^{p/2} &\leq C\Big(\int_{\mathbf{T}}|1-G_k|^{p/2}+\int_{\mathbf{T}}|1-G_{k-1}|^{p/2}\Big)\\ &\leq C\Big(\int_{\mathbf{T}}|1-F_k|^{4p}+\int_{\mathbf{T}}|1-F_{k-1}|^{4p}\Big)\,.\end{aligned}$$

Then by the L_{4p}-boundedness of $\mathcal{H}$ and by (7)

$$\begin{aligned}\int_{\mathbf{T}}|\varphi_k|^{p/2} &\leq C\Big(\int_{\mathbf{T}}|\alpha_k-1|^{4p}+\int_{\mathbf{T}}|\alpha_{k-1}-1|^{4p}\Big)\\ &\leq C2^{-pk}\sum_{j=k-1}^{k_1-1}\int_{e_j}b^p\\ &\leq C2^{-pk}\sum_{j=k-1}^{k_1-1}2^{pj}m(e_j) = C2^{-pk}v_k^p\,,\end{aligned}$$

thus proving (18). From the Hölder inequality, (i) and (iii)-(v), we deduce that

$$|T(h)|^p \leq \sum_{k=k_0+1}^{k_1}|T_k(h)|^p 2^{pk}|\varphi_k|^{p/2}\text{ , a.e. on }\mathbf{T}.$$

Combining the preceding inequalities with the Hölder inequality and (17), we obtain for $h \in L_p$

$$\begin{aligned}
\|T(h)\|_p^p \leq & C \sum_{k=k_0+1}^{k_1} \Big(\int_0^1 |Sh(s)| g_k(s) u_k(s) ds\Big)^p \\
\leq & C \sum_{k=k_0+1}^{k_1} \int_0^1 |Sh(s)|^p u_k(s) ds \\
= & C \sum_{k=k_0+1}^{k_1} \sum_{j=k-1}^{k_1-1} 2^{p(j-k+1)} \int_0^1 |Sh(s)|^p w_j(s) ds \\
= & C \sum_{k=k_0+1}^{k_1} \sum_{j=k-1}^{k_1-1} 2^{p(j-k+1)} \int_0^1 |Sh(s)|^p \sum_{n=0}^{N} \lambda_n w(\sigma_n^{-1}(s)) \chi_{\tilde{e}_{j,n}}(s) ds \\
= & C \sum_{n=0}^{N} \lambda_n \sum_{j=k_0}^{k_1-1} 2^{p(j+1)} \int_0^1 |Sh(s)|^p w(\sigma_n^{-1}(s)) \chi_{\tilde{e}_{j,n}}(s) ds \sum_{k=k_0+1}^{j+1} 2^{-pk} \\
\leq & C \sum_{n=0}^{N} \lambda_n \sum_{j=k_0}^{k_1-1} \int_0^1 |Sh(s)|^p w(\sigma_n^{-1}(s)) \chi_{\tilde{e}_{j,n}}(s) ds \\
\leq & C \sum_{n=0}^{N} \lambda_n \|Sh\|_p^p \leq C \|h\|_p^p \quad \text{by (11)} .
\end{aligned}$$

This proves the boundedness of T from L_p to H_p and thus completes the proof of the theorem.

4. Further results.

Let E be a rearrangement invariant (r.i. for short) space on $\mathbf{T}$ (*cf.* [13] for the definition of r.i. spaces and their basic properties). Then E is an interpolation space between L_1 and L_∞. We recall that $H(E)$ is the closed subspace of E consisting of all those functions whose Fourier coefficients vanish on negative integers.

COROLLARY 2. — *Let (E_0, E_1) be a couple of r.i. spaces on $\mathbf{T}$. Then*

(i) *for any functor F*

$$F(H(E_0), H(E_1)) = H(F(E_0, E_1)).$$

In particular, for $0 < \theta < 1$, $1 \leq q \leq \infty$

$(H(E_0), H(E_1))_{\theta\, q} = H((E_0, E_1)_{\theta\, q})$ *and* $(H(E_0), H(E_1))_\theta = H((E_0, E_1)_\theta)$.

(ii) *there exists an absolute constant C such that for any $f \in H(E_0) + H(E_1)$ and any $t > 0$*

$$K(t, f; H(E_0), H(E_1)) \leq CK(t, f; E_0, E_1).$$

Proof. — It is similar to that of Corollary 1. Let $f \in H(E_0)+H(E_1)$. By the theorem in Section 1, there exists a one-point projection T at f which is bounded simultaneously from L_1 to H_1 and from L_∞ to H_∞. Then T is bounded on E_j (*cf.* [4], [13]). Hence T is also bounded from E_j to $H(E_j)$ $(j = 0, 1)$, so the corollary follows.

Remark. — Let $f \in H_1$. Then the one-point projection T at f given by the theorem is bounded on any r.i. space E (into $H(E)$). Note that T depends only on f and its norm from E to $H(E)$ is uniformly bounded.

COROLLARY 3. — *Let (E_0, E_1) be a Calderón-Mitjagin couple. Then for any $f, g \in H(E_0) + H(E_1)$ such that*

$$K(t, g; H(E_0), H(E_1)) \leq K(t, f; H(E_0), H(E_1)) \ , \quad \forall t > 0 \ ,$$

there exists a linear operator T defined on $E_0 + E_1$ which maps boundedly E_j into $H(E_j)$ $(j = 0, 1)$, such that $T(f) = g$.

Consequently, $(H(E_0), H(E_1))$ is a Calderón-Mitjagin couple.

Proof. — Let $f, g \in H(E_0) + H(E_1)$ be as above. Then by Corollary 2(ii)

$$K(t, g; E_0, E_1) \leq CK(t, f; E_0, E_1) \ , \quad \forall t > 0 \ .$$

(E_0, E_1) being a Calderón-Mitjagin couple, there exists an operator T_1 bounded from E_j to E_i $(j = 0, 1)$, such that $T_1(f) = g$. Now applying our theorem to g we find a one-point projection T_2 at g bounded from E_j to $H(E_j)$ $(j = 0, 1)$. Hence the operator $T = T_2T_1$ satisfies the requirement of Corollary 3.

Remark. — Corollary 3 in particular shows that (H_{p_0}, H_{p_1}) $(1 \leq p_0, p_1 \leq \infty)$ is a Calderón-Mitjagin couple. This is the H_p-space version of the well-known L_p-space result cited in Section 1.

Remark. — We have considered Hardy spaces of analytic functions only in the unit disc. Our results, however, also hold for Hardy spaces of analytic functions in the upper half-plane. With some minor modifications, all the above proofs go through to the upper half-plane case.

Remark. — The origin of this note is a preprint from June, 1990 (see Pub. IRMA, LILLE - 1990, Vol. 22, n° III). There we gave elementary proofs for the three interpolation theorems of P.W. Jones on Hardy spaces, which we have already cited previously. These proofs are essentially the same as that of the main theorem of the present note. Let us also mention that there exist two other new proofs (independent of ours) for the two first theorems of Jones concerning the real and complex interpolations, found respectively by P.F.X. Müller [15] and G. Pisier [16].

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