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A CENTRAL LIMIT THEOREM ON THE SPACE OF POSITIVE DEFINITE SYMMETRIC MATRICES

by Piotr GRACZYK

0. Introduction.

Central limit theorems for rotation-invariant random variables on the symmetric space $\mathcal{P}_n$ of positive definite symmetric $n \times n$ matrices have been investigated in case $n = 2$ by Karpelevich, Tutubalin and Shur ([6]), Faraut ([1]) and Terras ([9]), in case $n = 3$ by Terras ([10]) and for n arbitrary by Richards ([8]). They find applications in multivariate statistics and in some engineering problems ([9]).

In this paper we prove a central limit theorem of Lindeberg-Feller type on the space $\mathcal{P}_n$. It generalizes a theorem obtained by Faraut ([1]) for $n = 2$. To state and prove this theorem we introduce on $\mathcal{P}_n$ some analogs of the mean and dispersion in the real case.

Sections 1 and 2 provide the basic definitions and facts from the harmonic analysis on $\mathcal{P}_n$ used in the paper. In Section 3 we define and investigate the mean and the dispersions on $\mathcal{P}_n$. In Section 4 we derive in a simple way a Taylor expansion of the spherical functions on $\mathcal{P}_n$. Section 5 contains the main result of the paper.

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1. Preliminaries.

Throughout this paper $G = GL(n, \mathbb{R})$ will denote the general linear group of $n \times n$ nonsingular matrices and $K = O(n)$ the group of $n \times n$ orthogonal matrices. The symmetric space G/K is identified with $\mathcal{P}_n$, the space of real, positive definite symmetric $n \times n$ matrices.

The transitive action of G on $\mathcal{P}_n$ is defined by

$$X \mapsto X[g] = gXg^t$$

where $g \in G$, $X \in \mathcal{P}_n$ and g^t is the transpose of g. The correspondence of G/K and $\mathcal{P}_n$ is given by $gK \mapsto I[g] = gg^t$, where $I \in \mathcal{P}_n$ is the identity matrix. The space $\mathcal{P}_n$ is a Riemannian manifold with the *arc length* $ds^2 = \mathrm{Tr}\big((X^{-1}dX)^2\big)$ for $X = (x_{ij})_{i,j \le n}$, $dX = (dx_{ij})_{i,j \le n}$.

A differential operator L on $\mathcal{P}_n$ is said to be *G-invariant* if it commutes with the action of G, that is for every $g \in G$ and $f \in C^\infty(\mathcal{P}_n)$

$$(Lf)^g = L(f^g)$$

where $f^g(X) = f(X[g])$ when $X \in \mathcal{P}_n$.

The algebra $\mathbb{D}(\mathcal{P}_n)$ of all G-invariant differential operators on $\mathcal{P}_n$ is commutative and isomorphic with the algebra $I(\mathfrak{a})$ of symmetric polynomials on the Cartan space $\mathfrak{a} = \{H | H \text{diagonal}\} \cong \mathbb{R}^n$. By Newton's theorem the algebra $I(\mathfrak{a})$ is generated by the symmetric polynomials :

$$p_j(\mathbf{x}) = \sum_{i=1}^{n} x_i^j, \qquad j = 0, \ldots, n. \tag{1}$$

We will denote $\gamma : \mathbb{D}(\mathcal{P}_n) \to I(\mathfrak{a})$ the isomorphism of $\mathbb{D}(\mathcal{P}_n)$ and $I(\mathfrak{a})$. Remark that the order of $L \in \mathbb{D}(\mathcal{P}_n)$ equals to the degree of $\gamma(L)$ ([4], p. 306).

A function $h : \mathcal{P}_n \to \mathbb{C}$ is *K-invariant* if $h^k = h$ for all $k \in O(n)$. A K-invariant function h is said to be *spherical* if $h(I) = 1$ and h is an eigenfunction of all G-invariant differential operators on $\mathcal{P}_n$. All the spherical functions are given by

$$\Phi_{\mathbf{s}}(X) = \int_K \Delta_1^{s_1 - s_2}(X[k]) \ldots \Delta_{n-1}^{s_{n-1} - s_n}(X[k]) \Delta_n^{s_n}(X[k]) dk \tag{2}$$

where $\mathbf{s} = (s_1, \ldots, s_n) \in \mathbb{C}^n, \Delta_j(Y)$ is the principal minor of order j of Y and dk denotes the normalised Haar measure on K. Formula (2) corresponds to the classical Harish-Chandra integral formula for spherical functions on G with $\mathbf{s} = \dfrac{\lambda+\rho}{2}$, where $\rho = (\rho_1, \ldots, \rho_n)$, $\rho_j = \frac{1}{2}(2j-n-1)$, $\lambda \in \mathbb{C}^n$. We will write $\varphi_\lambda = \Phi_{\mathbf{s}}$. By ([4], p. 418) we have then $L\varphi_\lambda = \gamma(L)(\lambda)\varphi_\lambda$ for $L \in \mathbb{D}(\mathcal{P}_n)$.

Terras ([10],[11]) and Richards ([8]) use the coordinates $\mathbf{r} = (r_1, \ldots, r_n)$ with $\mathbf{r} = \frac{\lambda}{2}$.

Let W denote the Weyl group of permutations. Then $\varphi_\lambda = \varphi_{w\lambda}$ for all $w \in W$. One has also $\Phi_\rho = \varphi_\rho \equiv 1$. For all the details concerning the spherical functions on $\mathcal{P}_n$ see e.g.[11].

The following lemma describes the relationship between the spherical functions on $\mathcal{P}_n$ and the spherical functions on

$$\mathcal{SP}_n = \{X \in \mathcal{P}_n | \det X = 1\}.$$

The space $\mathcal{SP}_n$ may be identified with the symmetric space $SL(n)/SO(n)$.

Lemma 1.

$$\Phi_{\mathbf{s}}(X) = (\det X)^{\frac{1}{n}\sum_{i=1}^n s_i} \Psi_{\mathbf{s}}(X_1) \tag{3}$$

where $X = (\det X)^{\frac{1}{n}} X_1$ *with* $X_1 \in \mathcal{SP}_n$ *and*

$$\Psi_{\mathbf{s}}(X_1) = \int_{SO(n)} \Delta_1^{s_1-s_2}(X_1[k]) \ldots \Delta_{n-1}^{s_{n-1}-s_n}(X_1[k]) dk_{SO(n)}$$

is a spherical function on $\mathcal{SP}_n$.

Proof. — By (2) one gets

$$\Phi_{\mathbf{s}}(X) = (\det X)^{\frac{1}{n}\sum_{i=1}^n s_i} \int_K \Delta_1^{s_1-s_2} \ldots \Delta_{n-1}^{s_{n-1}-s_n}(X_1[k]) dk. \tag{4}$$

Using the fact that $dk|_{SO(n)} = \frac{1}{2} dk_{SO(n)}$ and the invariance of dk it follows that the integral in (4) equals $\frac{1}{2}(\Psi_{\mathbf{s}}(X_1) + \Psi_{\mathbf{s}}(X_1[h]))$ for all $h \in O(n)$ such that $\det(h) = -1$. There exist $k_0 \in SO(n)$ and h_0, $\det(h_0) = -1$, such that $X_1[k_0]$ is diagonal and $X_1[h_0k_0] = X_1[k_0]$. Then $\det(h_0k_0) = -1$ and $\Psi_{\mathbf{s}}(X_1[h_0k_0]) = \Psi_{\mathbf{s}}(X_1)$. □

Theorem of Helgason-Johnson ([4], p. 458) and Lemma 1 imply that the spherical function φ_λ is bounded on $\mathcal{P}_n$ if and only if $\mathrm{Re}\left(\sum_{i=1}^n \lambda_i\right) = 0$

and Re$\lambda \in C(\rho)$, where $C(\rho)$ denotes the convex enveloppe of $\{w\rho | w \in W\}$. Then $|\varphi_\lambda| \leq 1$.

In the sequel we will often use the G-invariant differential operators on $\mathcal{P}_n$ of order 1 and 2. Let us state some of their principal properties now.

All the differential operators of order 1 in $\mathbb{D}(\mathcal{P}_n)$ are given up to a multiplicative constant by the *Euler operator* E which is defined by

$$Ef(X) = \frac{d}{dt} f(tX)|_{t=1}. \tag{5}$$

Note that E is homogeneous of degree 0. Lemma 1 shows that a spherical function $\Phi_{\mathbf{s}}$ is homogeneous of order $\sum_{i=1}^{n} s_i$, so

$$E\Phi_{\mathbf{s}} = \left(\sum_{i=1}^{n} s_i \right) \Phi_{\mathbf{s}}.$$

We denote the eigenvalue of E acting on $\Phi_{\mathbf{s}}$ by $\gamma_1(\mathbf{s})$.

If f is K-invariant on $\mathcal{P}_n$ then by the spectral decomposition of $\mathcal{P}_n$ it suffices to know f on

$$A = \{a \in \mathcal{P}_n | a \text{ diagonal with } a_{ii} > 0\}.$$

The Lie algebra of A is given by $\mathfrak{a}$. One denotes the diagonal entries of $H \in \mathfrak{a}$ by $h_1, \ldots, h_n$. Formula (5) implies

$$Ef(\exp H) = \sum_{i=1}^{n} \frac{\partial \tilde{f}}{\partial h_i} \tag{6}$$

where $\tilde{f}(h_1, \ldots, h_n) = f(\exp H)$.

The G-invariant differential operators of order 2 (without terms of lower order) are given by the linear combinations of E^2 and the *Laplace-Beltrami operator* Δ on $\mathcal{P}_n$. If f is K-invariant then by [2]VI,4.2

$$\Delta f(\exp H) = \sum_j \frac{\partial^2 \tilde{f}}{\partial h_j^2} + \frac{1}{2} \sum_{i<j} \coth\left(\frac{h_i - h_j}{2}\right) \left(\frac{\partial}{\partial h_i} - \frac{\partial}{\partial h_j}\right) \tilde{f}. \tag{7}$$

In order to find the eigenvalues of Δ acting on $\Phi_{\mathbf{s}}$ one may use the horospherical part of Δ (see [2]VI,4.4). If N denotes the nilpotent group of lower triangular matrices with 1 on the diagonal and if f is N-invariant on

$\mathcal{P}_n$, then $f(X) = F(a_1, \ldots, a_n)$ where $(a_i)_{1\leq i\leq n}$ are the eigenvalues of X and

$$\Delta f(X) = \sum_{i=1}^{n} \left(a_i \frac{\partial}{\partial a_i} \right)^2 F - \sum_{i=1}^{n} \rho_i a_i \frac{\partial F}{\partial a_i}.$$

Remark that the function under the integral in (2) is N-invariant. It follows that

$$\Delta \Phi_{\mathbf{s}} = \gamma_2(\mathbf{s}) \Phi_{\mathbf{s}}$$

with $\gamma_2(\mathbf{s}) = (\mathbf{s} - \rho | \mathbf{s}) = \frac{1}{4}(\|\lambda\|^2 - \|\rho\|^2)$.

It is easy to check that Δ is elliptic and E^2 is semi-elliptic on $\mathcal{P}_n$. All the elliptic second order operators in $\mathbb{D}(\mathcal{P}_n)$ are given by

$$L = a\left(\Delta - \frac{1}{n} E^2 \right) + bE^2 + cE + d, \quad a, b > 0. \tag{8}$$

Observe that the operator $\Omega = \Delta - \frac{1}{n} E^2$ restrained to $\mathcal{SP}_n$ is the Laplace-Beltrami operator on $\mathcal{SP}_n$. Ω is semi-elliptic on $\mathcal{P}_n$.

2. K-invariant probability measures on $\mathcal{P}_n$.

A probability measure μ on $\mathcal{P}_n$ is said to be *K-invariant* if for any measurable subset B of $\mathcal{P}_n$ and for all $k \in K$ we have $\mu(kBk^t) = \mu(B)$. We shall then write $\mu \in M^\natural(\mathcal{P}_n)$.

In this paper we consider only K-invariant measures on $\mathcal{P}_n$. We can identify such measures with K-biinvariant measures on G. Then the *convolution* $\mu_1 * \mu_2$ of two measures in $M^\natural(\mathcal{P}_n)$ is defined by the convolution of the corresponding measures on G and then projecting on $\mathcal{P}_n$. This convolution is commutative.

The *spherical Fourier transform* of a measure $\mu \in M^\natural(\mathcal{P}_n)$ is defined by

$$\hat{\mu}(\lambda) = \int_{\mathcal{P}_n} \varphi_\lambda(X) d\mu(X)$$

for λ such that φ_λ is bounded. In Section 1 we have formulated sufficient and necessary conditions for such λ. We also write

$$\hat{\mu}(\mathbf{s}) = \int_{\mathcal{P}_n} \Phi_{\mathbf{s}}(X) d\mu(X).$$

The spherical Fourier transform carries the convolution of K-invariant measures on $\mathcal{P}_n$ into the usual product

$$\widehat{\mu_1 * \mu_2}(\mathrm{s}) = \hat{\mu}_1(\mathrm{s})\hat{\mu}_2(\mathrm{s}).$$

If $\mu \in M^\natural(\mathcal{P}_n)$ is infinitely divisible, the infinitesimal generator of the continuous semigroup of measures $(\mu_t)_{t>0}$ with $\mu_1 = \mu$ is given by the Hunt formula ([5]). It is then natural to call μ *Gaussian* if the generator of $(\mu_t)_{t>0}$ is a second order G-invariant elliptic differential operator on $\mathcal{P}_n$ which annihilates constants (cf.[3]). If the generator is semi-elliptic we say that μ is *Gaussian degenerate*. By (8) the Fourier transform of a Gaussian measure $\mu \in M^\natural(\mathcal{P}_n)$ has the following form

$$\hat{\mu}(\mathrm{s}) = \exp\left[a\gamma_2(\mathrm{s}) + \left(b - \frac{1}{n}a\right)\gamma_1^2(\mathrm{s}) + c\gamma_1(\mathrm{s})\right]$$

with $a, b > 0$, $c \in \mathbb{R}$.

Examples. — (*i*) The Laplace-Beltrami operator Δ is the generator of the *heat semigroup* $(\kappa_t)_{t>0}$ on $\mathcal{P}_n$ (cf.[11]). We have

$$\hat{\kappa}_t(\mathrm{s}) = \exp[t(\mathrm{s} - \rho|\mathrm{s})].$$

Certainly, the measures κ_t are Gaussian on $\mathcal{P}_n$.

(*ii*) The operator $\Omega = \Delta - \frac{1}{n}E^2$ is the generator of the semigroup $(\nu_t)_{t>0}$ on $\mathcal{P}_n$ which extends naturally the heat semigroup $(\tilde{\nu}_t)_{t>0}$ on $\mathcal{SP}_n$, i.e. $\nu_t(B) = \tilde{\nu}_t(B \cap \mathcal{SP}_n)$. We have

$$\hat{\nu}_t(\mathrm{s}) = \exp\left\{t\left[(\mathrm{s} - \rho|\mathrm{s}) - \frac{1}{n}\left(\sum_{i=1}^n s_i\right)^2\right]\right\}$$

and ν_t are Gaussian degenerate.

(*iii*) Let $\mathfrak{n}_t$ be a random variable on $\mathbb{R}$ with normal distribution of mean 0 and variance t. Let η_t be the probability distribution of the random variable $\exp(\mathfrak{n}_t I)$ on $\mathcal{P}_n$. We have then

$$\hat{\eta}_t(\mathrm{s}) = \exp\left[\frac{t}{2}\left(\sum_{i=1}^n s_i\right)^2\right].$$

The generator of the semigroup $(\eta_t)_{t>0}$ equals $\frac{1}{2}E^2$. This operator corresponds to the Laplacian in the $\mathbb{R}^+I$-direction of the space $\mathcal{P}_n$ considered as a product $\mathcal{SP}_n \times \mathbb{R}^+$. The measures η_t are Gaussian degenerate on $\mathcal{P}_n$.

(*iv*) If δ_t is the Dirac delta in $\exp(tI)$ then

$$\hat{\delta}_t(\mathbf{s}) = \exp\left(t\sum_{i=1}^{n} s_i\right)$$

and the generator of $(\delta_t)_{t>0}$ is E.

Note that $\kappa_t = \nu_t * \eta_{2tn^{-1}}$. All the Gaussian measures on $\mathcal{P}_n$ have the form $\nu_t * \eta_u * \delta_w$ for some t, u positive and $w \in \mathbb{R}$.

3. The mean and the dispersions on $\mathcal{P}_n$.

In order to analyse the asymptotic behavior of K-invariant measures on $\mathcal{P}_n$ we need some analogs of the mean and the covariance of a measure on $\mathbb{R}^n$. In this section we introduce in a natural way such analogs and prove some properties of them.

3.1. Choice of dispersions on $\mathcal{P}_n$.

To have an analog of the covariance on $\mathcal{P}_n$ one seeks an application

$$D : M^{\natural}(\mathcal{P}_n) \to [0, \infty]$$

satisfying for all $\mu_1, \mu_2 \in M^{\natural}(\mathcal{P}_n)$ the condition

$$D(\mu_1 * \mu_2) = D(\mu_1) + D(\mu_2). \tag{9}$$

One also assumes that there exists an analytic, K-invariant function Q on $\mathcal{P}_n$ such that

$$D(\mu) = \int Q(X) d\mu(X) \tag{10}$$

for all $\mu \in M^{\natural}(\mathcal{P}_n)$. The function D will be called a *dispersion* on $\mathcal{P}_n$. Observe that the condition (9) is equivalent to

$$\int_K Q(Y[xk]) dk = Q(I[x]) + Q(Y) \tag{11}$$

for all $x \in G$ and $Y \in \mathcal{P}_n$.

In the real case the covariance of a centralised measure may be represented by a second order derivative of its Fourier transform in 0. In the case of $\mathcal{P}_n$ the spherical function $\varphi_\lambda \equiv 1$ if $\lambda = \rho$. It turns out that on $\mathcal{P}_n$ we have the following analogous property.

THEOREM 1. — *If Q is an analytic, K-invariant function on $\mathcal{P}_n$ verifying (11) then there exists a vector $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$ such that*

$$Q(X) = \frac{\partial \varphi_\lambda(X)}{\partial \mathbf{v}}\Big|_{\lambda=\rho}, \quad X \in \mathcal{P}_n. \tag{12}$$

Proof. — First observe that if Q satisfies (11) then for all $L \in \mathbb{D}(\mathcal{P}_n)$ annihilating constants $LQ = \text{const}$. Indeed, let us apply L to (11) for x fixed and then put $Y = I$. One gets $LQ(xx^t) = LQ(I)$ for all $x \in G$, so LQ is constant on $\mathcal{P}_n$.

Next remark that if Q_1 and Q_2 verify (11) and $LQ_1 = LQ_2$ for all $L \in \mathbb{D}(\mathcal{P}_n)$ annihilating constants then $Q_1 = Q_2$. This property comes out from the analyticity of Q_1 and Q_2 and the fact that (11) implies $Q_1(I) = Q_2(I) = 0$.

If $Q = \frac{\partial \varphi_\lambda}{\partial \mathbf{v}}|_{\lambda=\rho}$ for any vector $\mathbf{v}$ then by the convolution property of the spherical Fourier transform on $\mathcal{P}_n$ the condition (9) holds for D corresponding to such Q.

Let now Q be arbitrary satisfying (11). To prove the theorem it suffices to show that there exists a vector $\mathbf{v} = (v_1, \ldots, v_n)$ such that

$$L_j Q = L_j\left(\frac{\partial \varphi_\lambda}{\partial \mathbf{v}}\Big|_{\lambda=\rho}\right) = \mathbf{v} \cdot \operatorname{grad}(L_j \varphi_\lambda)|_{\lambda=\rho}$$

for $L_j = \gamma^{-1}(p_j)$ with p_j as in (1), $j = 1, \ldots, n$. We have $\mathbf{l}_j = \operatorname{grad}(L_j\varphi_\lambda)|_{\lambda=\rho} = \operatorname{grad}\, \gamma(L_j)|_{\lambda=\rho} = \operatorname{grad}\, p_j|_{\lambda=\rho} = j(\rho_1^{j-1}, \ldots, \rho_n^{j-1})$. It follows that $\mathbf{l}_1, \ldots, \mathbf{l}_n$ are independent and the system of equations

$$L_j Q = \mathbf{v} \cdot \mathbf{l}_j, \quad j = 1, \ldots, n$$

has a solution. □

Now we want to choose the directions $\mathbf{v}$ of derivation in (12) so as the function Q was nonnegative. The decomposition (3) shows that this is possible only for $\sum_{i=1}^{n} v_i = 0$.

By reasons of convexity (or more generally by the Helgason-Johnson theorem) the spherical functions verify $0 < \varphi_\lambda \leq 1$ for $\lambda \in C(\rho)$. Thus, if

$\rho + t\mathbf{v} \in C(\rho)$ for t positive sufficiently small then $-\frac{\partial \varphi_\lambda}{\partial \mathbf{v}}|_{\lambda=\rho}$ is nonnegative. One will differentiate in the directions of neighbour vertices of ρ in $C(\rho)$. They are given by the permutations of neighbour entries of ρ :

$$\beta_1 = (\rho_2, \rho_1, \rho_3, \ldots, \rho_n)$$
$$\ldots$$
$$\beta_{n-1} = (\rho_1, \rho_2, \ldots, \rho_n, \rho_{n-1}).$$

Then the vectors $\mathbf{v}_j = \beta_j - \rho = (0, \ldots, 1, -1, \ldots, 0)$ lie on the edges of $C(\rho)$ beginning in ρ. Observe that $\mathbf{v}_j = -\alpha_j$, $j = 1, \ldots, n-1$, where α_j are the simple positive roots corresponding to the Iwasawa decomposition $G = NAK$ with N lower triangular. The vectors $\mathbf{v}_j$, $j = 1, \ldots, n-1$, are independent and span all the hyperplane $\{\lambda | \sum \lambda_i = 0\}$. By reasons of normalisation we will differentiate with respect to the vectors $2\mathbf{v}_j$.

DEFINITION. — *The dispersions D_j on $\mathcal{P}_n$, $j = 1, \ldots, n-1$, are defined by*

$$D_j(\mu) = \int Q_j(X) d\mu(X)$$

where

$$Q_j(X) = -2 \frac{\partial \varphi_\lambda(X)}{\partial \mathbf{v}_j}|_{\lambda=\rho}.$$

Then one has

(13)
$$Q_j(X) = 2\left(\frac{\partial}{\partial \lambda_{j+1}} - \frac{\partial}{\partial \lambda_j}\right) \varphi_\lambda(X)\big|_{\lambda=\rho} = \left(\frac{\partial}{\partial s_{j+1}} - \frac{\partial}{\partial s_j}\right) \Phi_{\mathbf{s}}(X)\big|_{\mathbf{s}=\rho}$$

and for $\mu \in M^\natural(\mathcal{P}_n)$

(14)
$$D_j(\mu) = 2\left(\frac{\partial}{\partial \lambda_{j+1}} - \frac{\partial}{\partial \lambda_j}\right) \hat{\mu}(\lambda)\big|_{\lambda=\rho} = \left(\frac{\partial}{\partial s_{j+1}} - \frac{\partial}{\partial s_j}\right) \hat{\mu}(\mathbf{s})\big|_{\mathbf{s}=\rho}.$$

Example. — A direct calculation using (14) allows to find the dispersions of the measures considered in Section 2 :

$$D_j(\kappa_t) = D_j(\nu_t) = t \;;$$
$$D_j(\eta_t) = 0 \;;$$
$$D_j(\delta_t) = 0.$$

We shall give now some properties of the functions Q_j.

THEOREM 2.

(i) $Q_j(X) = Q_j(tX)$, $j = 1, \dots, n-1$, *for all* $X \in \mathcal{P}_n$ *and* $t > 0$.

(ii) $Q_j(I) = 0$, $j = 1, \dots, n-1$.

(iii) $Q_1(X) + \dots + Q_{n-1}(X) > 0$ *for all* $X \neq tI$.

Proof. — (*i*) follows obviously from (3) and (13). (*ii*) follows from $\Phi_{\mathbf{s}}(I) = 1$ for all $\mathbf{s}$. To prove (*iii*) it suffices to consider $X \in \mathcal{S}P_n$, $X \neq I$. Suppose that $Q_1(X) = \dots = Q_{n-1}(X) = 0$. Then $\frac{\partial \varphi_\lambda}{\partial \mathbf{u}}|_{\lambda=\rho} = 0$ for all the directions $\mathbf{u}$ such that $\sum u_i = 0$. If $\sum \lambda_i = 0$, by (3) $\varphi_\lambda(X) = \psi_\lambda(X)$ where ψ_λ denotes the spherical function on $\mathcal{SP}_n$ in the Harish-Chandra notation. Thus the application $\lambda \mapsto \psi_\lambda(X)$, $\sum \lambda_i = 0$, has a critical point in $\lambda = \rho$, and by W-invariance also in $\lambda = w\rho$ for $w \in W$.

On the other hand ψ_λ is given by the formula of Harish-Chandra :

$$\psi_\lambda(X) = \int_{SO(n)} e^{(\lambda - \rho | \mathcal{H}(ak))} dk_{SO(n)} \tag{15}$$

where $a \in A \cap SL(n)$ is such that $X = a^2[k_0]$ for some $k_0 \in SO(n)$ and $g = k \exp \mathcal{H}(g).n$ is the Iwasawa decomposition of $SL(n)$. Denote by μ_X the image of $dk_{SO(n)}$ by the mapping $k \mapsto \mathcal{H}(ak)$. Then μ_X is a probability measure on $\tilde{\mathfrak{a}} = \{H | H \text{ diagonal and } \mathrm{Tr} H = 0\}$. By (15) we have then for any $\mathbf{u} \in \mathbb{R}^n$

$$\psi_\lambda(X) = \int_{\tilde{\mathfrak{a}}} e^{(\lambda - \rho | H)} d\mu_X(H)$$

$$\frac{\partial \psi_\lambda}{\partial \mathbf{u}}(X) = \int_{\tilde{\mathfrak{a}}} (\mathbf{u}|H) e^{(\lambda - \rho | H)} d\mu_X(H)$$

$$\frac{\partial^2 \psi_\lambda}{\partial \mathbf{u}^2}(X) = \int_{\tilde{\mathfrak{a}}} (\mathbf{u}|H)^2 e^{(\lambda - \rho | H)} d\mu_X(H). \tag{16}$$

By a theorem of Kostant ([7]) supp $\mu_X = C(\log a)$. Since $X \neq I$ we have $a \neq I$ and $\log a \neq 0$. The space $\mathcal{SP}_n$ is irreducible so by [4]IV,10.11 $\dim C(\log a) = \dim \tilde{\mathfrak{a}}$ and (16) implies $\frac{\partial^2 \psi_\lambda}{\partial \mathbf{u}^2}(X) > 0$ for all λ and $\mathbf{u} \neq 0$. It means that $\psi_\lambda(X)$ is strictly convex and in particular it has at most one critical point on $\tilde{\mathfrak{a}}$. For $w \in W$ different from identity $w\rho \neq \rho$. That gives a contradiction. □

COROLLARY 1. — *Let $\mu \in M^\natural(\mathcal{P}_n)$. Then*

$$D_1(\mu) = \cdots = D_{n-1}(\mu) = 0$$

if and only if μ is concentrated on $\{tI|t > 0\}$. □

Example. — In the case $n = 2$ the explicit form of the dispersion density is known ([1]) :

$$Q(a_r) = 2\log\left(\mathrm{ch}\frac{r}{2}\right)$$

where

$$a_r = \begin{pmatrix} e^r & 0 \\ 0 & e^{-r} \end{pmatrix}.$$

For $n \geq 3$ one may give the following integral formula

$$Q_j(X) = \int_K \{\log \Delta_{j-1}(X[k]) - 2\log \Delta_j(X[k]) + \log \Delta_{j+1}(X[k])\} dk$$

but the explicit form of $Q_j(X)$ is not known.

3.2. The mean and the variance in $\mathbb{R}^+ I$-direction.

Theorem 2 and Corollary 1 show that the dispersions D_j do not control the behaviour of measures in $M^\natural(\mathcal{P}_n)$ in the direction $\mathbb{R}^+ I$. In fact, via (3) one may say that D_j are the dispersions in the direction of $\mathcal{SP}_n$. We will introduce now some complementary characteristics of $\mu \in M^\natural(\mathcal{P}_n)$.

Having in mind the decomposition (3) of a spherical function on $\mathcal{P}_n$ and denoting $w = \sum_{i=1}^{n} s_i$ it is natural to have the following definition.

DEFINITION. — *Let $\mu \in M^\natural(\mathcal{P}_n)$. We define :*

the mean of μ by

$$M(\mu) = \frac{\partial}{\partial w}\hat{\mu}\big|_{\mathbf{s}=\rho} = \frac{1}{n}\int \log(\det X) d\mu(X)$$

the second moment of μ by

$$M_2(\mu) = \frac{\partial^2}{\partial w^2}\hat{\mu}\big|_{\mathbf{s}=\rho} = \frac{1}{n^2}\int \log^2(\det X) d\mu(X)$$

the variance of μ *by* $d^2(\mu) = M_2(\mu) - M^2(\mu)$.

Note that any measure $\mu \in M^\natural(\mathcal{P}_n)$ may be centralised by putting

$$\tilde{\mu}(B) = \mu(e^{M(\mu)}B)$$

for B measurable. Then $M(\tilde{\mu}) = 0$ and $d^2(\mu) = M_2(\tilde{\mu})$. Observe that

$$\begin{aligned} M(\mu_1 * \mu_2) &= M(\mu_1) + M(\mu_2) \\ d^2(\mu_1 * \mu_2) &= d^2(\mu_1) + d^2(\mu_2) \end{aligned}$$

but $d^2(\mu) = \int q(X)d\mu(X)$ with $q(X) = \frac{1}{n^2}\log^2(\det X)$ only for centralised measures μ.

The derivative $\frac{\partial}{\partial w}$ in the definition of M and d^2 equals in the (s_i)-coordinates

$$\frac{\partial}{\partial w} = \frac{1}{n}\sum_{i=1}^{n}\frac{\partial}{\partial s_i}.$$

This makes possible to calculate $M(\mu)$ and $d^2(\mu)$ if one knows the spherical Fourier transform of μ.

Example. — For the measures considered in Section 2 we have :

$$\begin{aligned} M(\kappa_t) &= 0, & d^2(\kappa_t) &= 2tn^{-1}; \\ M(\nu_t) &= 0, & d^2(\nu_t) &= 0; \\ M(\eta_t) &= 0, & d^2(\eta_t) &= t; \\ M(\delta_t) &= t, & d^2(\delta_t) &= 0. \end{aligned}$$

COROLLARY 2. — *If* $\mu \in M^\natural(\mathcal{P}_n)$ *and* $M(\mu) = d^2(\mu) = 0$, $D_j(\mu) = 0$ *for* $j = 1, \ldots, n-1$, *then* $\mu = \delta_I$. □

Note that a Gaussian measure μ on $\mathcal{P}_n$ is fully characterized by its mean $M(\mu)$, variance $d^2(\mu)$ and dispersion $D_1(\mu)$.

4. Taylor expansion of spherical functions.

In this section we will derive a useful Taylor expansion of spherical functions on $\mathcal{P}_n$. This expansion will be more detailed than that of Richards ([8]) and we will prove it in a simpler way.

A spherical function $\Phi_{\mathbf{s}}$ is K-invariant so it is enough to consider $\Phi_{\mathbf{s}}(\exp H)$, $H \in \mathfrak{a}$. One may treat $\Phi_{\mathbf{s}}(\exp H)$ as a function of $h_1, \ldots, h_n$. It is then symmetric in $h_1, \ldots, h_n$. Remark that $\Phi_{\mathbf{s}}(\exp H)$ is real analytic since $\Phi_{\mathbf{s}}$ is a solution of an elliptic differential equation with analytic coefficients : $\Delta\Phi_{\mathbf{s}} = \gamma_2(\mathbf{s})\Phi_{\mathbf{s}}$. Making use of the symmetry and analyticity of the function in the h_j we get the following Taylor expansion at $H = 0$:

$$\Phi_{\mathbf{s}}(\exp H) = 1 + a(\mathbf{s})\sum h_i + b(\mathbf{s})\sum h_i^2 + c(\mathbf{s})(\sum h_i)^2 + R_{\mathbf{s}}(H) \tag{17}$$

with

$$R_{\mathbf{s}}(H) = \sum f_\alpha(\mathbf{s})P_\alpha(H) \tag{18}$$

where $P_\alpha(H)$ are symmetric polynomials in $h_1, \ldots, h_n$ homogeneous of order greater or equal to 3.

In order to find the functions a, b, c in (17) let us apply the operators E, E^2 and Δ to (17) at $H = 0$.

For the operators E and E^2 one uses (6). E considered as a differential operator on functions of $h_1, \ldots, h_n$ is homogeneous of order 1 while the polynomials P_α are homogeneous of order at least 3. That implies $ER_{\mathbf{s}}(0) = E^2R_{\mathbf{s}}(0) = 0$ and

$$\begin{aligned}\gamma_1(\mathbf{s}) =& na(\mathbf{s})\\ \gamma_1^2(\mathbf{s}) =& 2nb(\mathbf{s}) + 2n^2c(\mathbf{s}).\end{aligned}$$

For the operator Δ one applies (7). By an argument of homogeneity one obtains $\Delta R_{\mathbf{s}}(0) = 0$. We have then

$$\gamma_2(\mathbf{s}) = (n^2 + n)b(\mathbf{s}) + 2nc(\mathbf{s}).$$

Solving these equations we get

THEOREM 3.

$$\Phi_{\mathbf{s}}(\exp H) = 1 + a(\mathbf{s})\sum h_i + b(\mathbf{s})\sum h_i^2 + c(\mathbf{s})(\sum h_i)^2 + R_{\mathbf{s}}(H)$$

with

$$\begin{aligned}a(\mathbf{s}) =& \frac{1}{n}\gamma_1(\mathbf{s})\\ b(\mathbf{s}) =& \frac{n\gamma_2(\mathbf{s}) - \gamma_1^2(\mathbf{s})}{n(n-1)(n+2)}\\ c(\mathbf{s}) =& \frac{(n+1)\gamma_1^2(\mathbf{s}) - 2\gamma_2(\mathbf{s})}{2n(n-1)(n+2)}\end{aligned} \tag{19}$$

where $\gamma_1(\mathbf{s}) = \sum s_i$, $\gamma_2(\mathbf{s}) = (\mathbf{s} - \rho|\mathbf{s})$ *and* $R_{\mathbf{s}}(H)$ *is as in (18).* □

By (13) and by differentiating of (19) one obtains the following expansion of the functions Q_j at $H = 0$:

COROLLARY 3.

$$Q_j(\exp H) = \frac{1}{(n-1)(n+2)}\sum h_i^2 - \frac{1}{n(n-1)(n+2)}(\sum h_i)^2 + R'_j(H) \tag{20}$$

where $R'_j(H) = \sum c_{\alpha j} P_\alpha(H)$ *with* P_α *as in (18),* $j = 1, \ldots, n-1$. □

Writing as in Section 3 $q(X) = \frac{1}{n^2}\log^2(\det X)$ we have $(\sum h_i)^2 = n^2 q(X)$. Replacing $\sum h_i^2$ in (19) by the expression for $\sum h_i^2$ obtained from (20) we get :

$$\Phi_{\mathbf{s}}(\exp H) = 1 + \frac{1}{n}\gamma_1(\mathbf{s})\sum h_i + \left(\gamma_2 - \frac{1}{n}\gamma_1^2\right) Q_j(\exp H) + \frac{1}{2}\gamma_1^2 q(\exp H) + R_{j,\mathbf{s}}(H) \tag{21}$$

where

$$R_{j,\mathbf{s}}(H) = \sum f_{j\alpha}(\mathbf{s}) P_\alpha(H) \tag{22}$$

with $P_\alpha(H)$ as in (18).

For $H = (h_1, \ldots, h_n)$ we put $\|H\| = \sum |h_i|$. Then we have

$$R_{j,\mathbf{s}}(H) = \mathcal{O}(\|H\|^3) \quad \text{if } H \to 0. \tag{23}$$

In order to estimate $R_{j,\mathbf{s}}(H)$ when $\|H\| \to \infty$ and $\Phi_{\mathbf{s}}$ is bounded one has to estimate Q_j in infinity.

LEMMA 2. — $Q_j(\exp H) \leq \|H\|$ for all $H \in \mathfrak{a}$ and $j = 1, \ldots, n-1$.

Proof. — By Theorem 2(*i*) $Q_j(\exp H) = Q_j(\exp H')$ with $H' = (h_1 - \frac{1}{n}\sum h_i, \ldots, h_n - \frac{1}{n}\sum h_i) \in \tilde{\mathfrak{a}}$. Then $Q_j(\exp H') = -2\frac{\partial \psi_\lambda}{\partial \mathbf{v}_j}|_{\lambda=\rho}$ where ψ_λ is spherical on $\mathcal{SP}_n$. By the Harish-Chandra formula

$$\begin{aligned} Q_j(\exp H') =& 2\int_{SO(n)} \left(-\mathbf{v}_j | \mathcal{H}\left(\exp\frac{1}{2}H'.k\right)\right) dk_{SO(n)} \\ =& 2\int_{SO(n)} \left(\alpha_j | \mathcal{H}\left(\exp\frac{1}{2}H'.k\right)\right) dk_{SO(n)}. \end{aligned}$$

By K-invariance of Q_j one may assume that $H' \in \tilde{\mathfrak{a}}^+$, i.e. $h'_1 \leq \cdots \leq h'_n$. By [4]IV,6.5 $\mathcal{H}(ak) \leq \mathcal{H}(a)$ for $a \in \exp(\tilde{\mathfrak{a}}^+), k \in SO(n)$, so

$$Q_j(\exp H') \leq 2\left(\alpha_j|\mathcal{H}\left(\exp\frac{1}{2}H'\right)\right) = h'_{j+1} - h'_j = h_{j+1} - h_j.$$

Finally

$$Q_j(\exp H) \leq |h_j| + |h_{j+1}| \leq \|H\|.$$

□

COROLLARY 4. — *For every $j = 1, \ldots, n-1$ and* $\mathbf{s}$ *such that $\Phi_{\mathbf{s}}$ is bounded*

$$R_{j,\mathbf{s}}(H) = \mathcal{O}(\|H\|) + \mathcal{O}((\sum h_i)^2)$$

when $\|H\| \to \infty$. □

5. Central limit theorem.

Let $\{\mu_{mj}\}$, $m \in \mathbb{N}$, $1 \leq j \leq k_m$ be a family of K-invariant probability measures on $\mathcal{P}_n$. Put

$$\mu_m = \mu_{m1} * \mu_{m2} * \cdots * \mu_{mk_m}.$$

Denote by $H(X)$ the diagonal matrix of logarithms of eigenvalues of $X \in \mathcal{P}_n$. Then we have the following central limit theorem :

THEOREM 4. — *Suppose that the measures $\{\mu_{mj}\}_{m\in\mathbb{N},1\leq j\leq k_m}$ satisfy the following conditions :*

$$M(\mu_{mj}) = 0$$

(24) $$\lim_{m\to\infty} D_1(\mu_m) = t$$

(25) $$\lim_{m\to\infty} d^2(\mu_m) = u$$

(26) $$\lim_{m\to\infty}\sum_{j=1}^{k_m}\int \frac{\|H\|^3}{1+\|H\|^2}d\mu_{mj} = 0$$

(27) $$\lim_{m\to\infty}\sum_{j=1}^{k_m}\int_{\{\|H\|>1\}} (\mathrm{Tr}H)^2 d\mu_{mj} = 0.$$

Then the measures μ_m converge weakly to the Gaussian measure $\nu_t * \eta_u$.

Proof. — First observe that for any $\varepsilon > 0$

$$\begin{aligned}\int \|H\| d\mu_{mj} &= \int_{\{\|H\|\leq\varepsilon\}} \|H\| d\mu_{mj} + \int_{\{\|H\|>\varepsilon\}} \|H\| d\mu_{mj} \\ &\leq \varepsilon + \frac{1+\varepsilon^2}{\varepsilon^2} \int \frac{\|H\|^3}{1+\|H\|^2} d\mu_{mj}.\end{aligned}$$

Then Lemma 2 and (26) imply that $\lim_m D_1(\mu_{mj}) = 0$ uniformly in j. Similarly,

$$\int (\sum h_i)^2 d\mu_{mj} \leq \int \|H\| d\mu_{mj} + \int_{\{\|H\|>1\}} (\mathrm{Tr} H)^2 d\mu_{mj}$$

and by (27) $\lim_m d^2(\mu_{mj}) = 0$ uniformly in j.

Fix $\mathbf{s}$ such that $\Phi_{\mathbf{s}}$ is bounded. By Corollary 4 and (23) the conditions (26) and (27) imply

$$\lim_m \sum_{j=1}^{k_m} \int |R_{1,\mathbf{s}}(H)| d\mu_{mj} = 0. \tag{28}$$

In particular $\int |R_{1,\mathbf{s}}(H)| d\mu_{mj}$ tends to 0 uniformly with respect to j. Then (21) implies

$$\lim_m \sup_{1\leq j\leq k_m} |\hat{\mu}_{mj}(\mathbf{s}) - 1| = 0. \tag{29}$$

By (21)

$$\sum_j [1-\hat{\mu}_{mj}(\mathbf{s})] = -\left(\gamma_2 - \frac{1}{n}\gamma_1^2\right) D_1(\mu_m) - \frac{1}{2}\gamma_1^2 d^2(\mu_m) - \sum_j \int R_{1,\mathbf{s}}(H) d\mu_{mj}.$$

By (24),(25),(28) and (29) we have then

$$\begin{aligned}&\lim_m \sum_j [1-\hat{\mu}_{mj}(\mathbf{s})] = -\left(\gamma_2 - \frac{1}{n}\gamma_1^2\right) t - \frac{1}{2}\gamma_1^2 u; \\ &\lim_m \sum_j [1-\hat{\mu}_{mj}(\mathbf{s})]^2 = 0.\end{aligned}$$

Using again (29) we get

$$\begin{aligned}\lim_m \hat{\mu}_m(\mathbf{s}) &= \exp\left[\lim_m \sum_j \log \hat{\mu}_{mj}(\mathbf{s})\right] \\ &= \exp\left[\left(\gamma_2 - \frac{1}{n}\gamma_1^2\right) t + \frac{1}{2}\gamma_1^2 u\right] = \widehat{\nu_t * \eta_u}(\mathbf{s}).\end{aligned}$$

By the Lévy continuity theorem on $\mathcal{P}_n$ (look [5]Thm.4.2 in the case of G semisimple; the proof of Gangolli works on $\mathcal{P}_n$) we get $\mu_m \Rightarrow \nu_t * \eta_u$. □

Remark. — Under the hypotheses of Theorem 4 $\lim_m D_l(\mu_m) = t$ for $l = 2, \ldots, n-1$ (see Corollary 3).

COROLLARY 5. — *Let $(\kappa_t)_{t>0}$ be the heat semigroup on $\mathcal{P}_n$. Let the family of measures $\{\mu_{mj}\}$ be centralised and verify (26) and (27). If*

$$\lim_m D_1(\mu_m) = D_1(\kappa_t) = t$$
$$\lim_m d^2(\mu_m) = d^2(\kappa_t) = 2tn^{-1}$$

then

$$\mu_m \Rightarrow \kappa_t. \qquad \square$$

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