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YAKOV ELIASHBERG

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CONTACT 3-MANIFOLDS TWENTY YEARS SINCE J. MARTINET'S WORK

by Yakov ELIASHBERG*

To the memory of Claude Godbillon and Jean Martinet

Twenty years ago Jean Martinet (see [Ma]) showed that any orientable closed 3-manifold admits a contact structure. Three years later after the work of R. Lutz (see [4]) and in the wake of the triumph of Gromov's h-principle, it seemed that the classification of closed contact 3-manifolds was at hand. Ten years later in the seminal work [Be], D. Bennequin showed that the situation is much more complicated and that the classification of contact structures on 3-manifolds, and even on S^3, was not likely to be achieved. My paper [E1] raised the hope that the situation is not so bad. The subject of the present paper is the status of the problem today and some recent progress in this direction including the classification of contact structures on S^3.

I discussed the original plan of this paper with D. Bennequin, I got further inspiration from the work [Gi] of E. Giroux and from numerous discussions with D.B. Fuchs. I am extremely grateful to all of them. The paper was written while I was visiting the Institute of Mathematics of the University of Basel. I want to thank the Institute and, especially, my host Norbert A'Campo for the hospitality.

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1. Classes of contact structures on 3-manifolds.

1.1. *Contact structure.*

A contact structure on a 3-manifold M is a *completely nonintegrable* tangent plane field ξ. The complete nonintegrability of ξ can be expressed by the inequality $\alpha \wedge d\alpha \neq 0$ for a 1-form α which defines (at least locally) the plane field ξ, i.e., $\xi = \{\alpha = 0\}$. Note that the sign of the form $\alpha \wedge d\alpha$ is independent of the sign of α and, therefore, a contact structure ξ defines an orientation of the manifold M. If the manifold M is already oriented then one can distinguish between *positive* and *negative* contact structures. Note that a contact structure itself can be nonorientable.

It is important to mention that a contact structure has no local invariants : the group of contact diffeomorphisms acts transitively on any connected manifold (Darboux). Besides, there are no local invariants of the space of contact structures on closed 3-manifolds : all homotopic contact structures (via a contact homotopy) are isotopic (J. Gray).

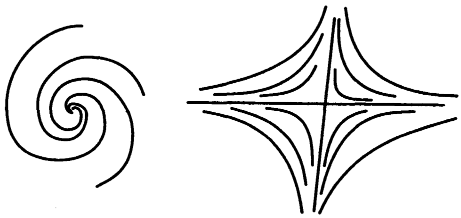
1.2. *Surfaces in a contact manifold.*

Let $F \subset M$ be a 2-surface in a contact 3-manifold (M, ξ). Generically F is tangent to ξ at a finite set $\Sigma = \{p_1, \ldots, p_k\} \subset F$. Outside Σ the contact structure ξ intersects $T(F)$ along a line field K which integrates to a 1-dimensional foliation on F with singularities at points of Σ. This singular foliation on F is called the *characteristic foliation* of F and will be denoted by F_ξ.

If F and ξ are oriented then one can distinguish between *positive* and *negative* points of Σ depending on whether the orientations of F and ξ coincide at these points or not.

The foliation F_ξ is always locally orientable. Therefore, the index of the line field K at its singularities is well defined. Generically, it is equal to ± 1. We will call a singular point $p \in S$ *elliptic* if its index is $+1$, *hyperbolic* if it is -1. The foliation F_ξ has a focus type singularity in an elliptic point and the standard hyperbolic singularity in a hyperbolic one (see Figure 1). By a C^1-small perturbation of F near an elliptic point $p \in \Sigma$, one can always get the picture of the foliation F_ξ as in Figure 2. Note that, topologically, pictures of elliptic points in Figure 1 and Figure 2 are indistinguishable.

The characteristic foliation F_ξ (up to a diffeomorphism fixed at F) uniquely defines the germ of a contact structure along F. Any (properly



Elliptic Hyperbolic

Fig. 1

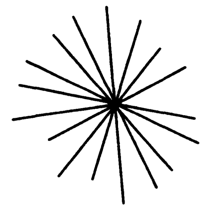


Fig. 2

defined) singular foliation on F is a characteristic foliation for a contact structure defined near F.

Note that the sign of an elliptic singular point p of F_ξ can be seen from the topology of the *oriented* foliation F: the sign depends on whether p is a *sink* or a *source*. The sign of a hyperbolic point is not so easily seen because it is a C^1- rather than C^0-topological invariant.

1.3. *Overtwisted contact structures.*

A contact structure ξ on M is called *overtwisted* (see [E1]) if there exists an embedded 2-disk $\mathcal{D} \subset M$ such that the characteristic foliation $\mathcal{D}_\xi$ contains one closed leaf C and exactly one singular point $p \in \mathcal{D}$ inside C. The point p is automatically elliptic in this case.

Overtwisted contact structures can be easily constructed with the so-called *Lutz twist* (see [Lu], [Be] or [E1]). As it was shown in [E1], *the isotopy classification of overtwisted contact structures on closed* 3-*manifolds coincides with their homotopy classification as tangent plane fields.*

This result shows an extreme flexibility of overtwisted contact structures and make them less interesting for the geometry.

1.4. *Tight contact structures.*

A contact structure ξ will be called *tight* if for any embedded disc $\mathcal{D} \subset M$, the characteristic foliation $\mathcal{D}_\xi$ contains no limit cycles. Certainly, overtwisted contact structures are not tight. It turns out (see Sect. 3 below) that :

1.4.1. THEOREM. — *Nonovertwisted contact structures are tight.*

Thus overtwisted and tight structures form two complementary classes of contact structures on 3-manifolds.

It is not easy to find out if a contact structure is tight or not. Until recently the only known example of a tight structure was provided by Bennequin's theorem (see [Be]) : *The standard contact structure on S^3 is tight.*

1.5. *Fillable structures.*

Any real hypersurface Σ in a 2-dimensional complex manifold carries a canonical 2-dimensional tangent plane field ξ which is formed by its complex tangencies. The (strict) *pseudoconvexity* of Σ ensures the nonintegrability of ξ. In other words, a pseudoconvex hypersurface Σ carries a canonical contact structure ξ. The structure which is defined by this construction on the unit sphere $S^3 \subset \mathbf{C}^2$ is just the *standard contact structure* ξ_0 on S^3.

It is easy to prove that all orientable contact structures can be obtained by this construction (see, for example, [E2]) : *for any contact manifold (M, ξ) there exists a complex structure on $M \times \mathbf{R}$ such that $M \times 0 \subset M \times \mathbf{R}$ is a pseudoconvex hypersurface and the induced contact structure on $M \times 0$ is exactly ξ.* What turns out to be much harder is to extend the complex structure from the neighborhood of M to a compact complex manifold bounded by the contact manifold (M, ξ). In the latter case the contact structure ξ is called *(holomorphically) fillable.* For example, the standard contact structure ξ_0 on S^3 is fillable. Note that a compact complex manifold bounded by M is automatically Kählerian (this is not an obvious fact) and, therefore, symplectic. So one may wish to generalize the definition of a fillable structure as follows (see [E3]).

A contact 3-manifold (M, ξ) is called *(symplectically) fillable* if there exists a symplectic manifold (W, ω) bounded by M such that

– the restriction $\omega \mid \xi$ does not vanish;

– the orientation of M defined by the contact structure ξ (see 1.1) coincides with its orientation as the boundary of the symplectic manifold (W, ω).

As I mentioned above, a holomorphically fillable manifold is symplectically fillable but no example which would show the difference between the two types of fillableness is known. For what follows the reader may assume either of the types.

The following result generalizes Bennequin's theorem (see [Gro] and [E4]) : *A fillable contact structure is tight.*

For S^3 this result just gives Bennequin's theorem : according to [E4], *the only fillable contact structure on S^3 is the standard structure ξ_0*. So far, fillable structures have been the main source of examples of tight structures. We do not know if the notions of tightness and fillableness coincide but we do know examples of tight structures for which no fillings are known. On the other hand there exist many constructions which give fillable manifolds (see, for example, [E5] and [E7]).

1.6. *Convex contact manifold.*

The notion of convexity in the contact geometry was introduced in [EG] and carefully studied by E. Giroux in [Gi].

A contact manifold (M, ξ) is called *convex* if it admits a contact vector field X which is gradient-like for a Morse-function $\varphi \colon M \to \mathbf{R}$. The standard contact sphere (S^3, ξ_0) is obviously convex. Thus convex contact structures exist even on closed manifolds.

E. Giroux proved in [Gi] that

Any orientable 3-manifold admits a convex contact structure.

Giroux showed also that any 3-manifold admits even overtwisted convex contact structure. In fact, we do not know if nonconvex structures do exist. In any case, the contact convexity has proven to be a useful tool in contact geometry. In particular, the technique developed by Giroux in [Gi] plays an important role in the proof of results which are discussed in the next section.

2. Rigidity results for tight structures.

From now on we will consider oriented closed 3-manifolds and *positive* oriented contact structures on them.

2.1. *Classification of tight contact structures on S^3.*

2.1.1. THEOREM. — *A tight contact structure on S^3 is isotopic to the standard contact structure ξ_0.*

Combining 2.1.1 and Theorem 1.6.1 from [E1], we get the complete classification of contact structures on S^3. To formulate the result let us fix the trivialization of $T(S^3)$ defined by vector fields xi, xj, xk for $x \in S^3$ where the sphere S^3 is considered as the unit sphere in R^4 identified with the 1-dimensional quaternion space. Then homotopy classes of 2-plane fields on S^3 can be canonically identified with homotopy classes from $\pi_3(S^2) = \mathbf{Z}$ and, therefore, can be numbered by integers : $\alpha_0, \alpha_{\pm 1}, \dots$. With these notations the standard structure ξ_0 belongs to α_0.

2.1.2. THEOREM. — *The class α_0 contains exactly two nonequivalent* (positive) *contact structures : the standard and the overtwisted. All other classes α_i, $|i| > 0$, contain only* one *contact structure, the overtwisted.*

Theorem 2.1.2 remains true also in the following relative version (see Sect. 5.4 below).

2.1.3. THEOREM. — *Two tight contact structures on the ball B^3 which coincide at ∂B^3 are isotopic relative to ∂B^3.*

The next theorem which can be deduced from 2.1.3 (see Sect. 7 below) shows that there are no invariants of tight contact structures on $\mathbf{R}^3$.

2.1.4. THEOREM. — *All tight contact structures on $\mathbf{R}^3$ are isomorphic.*

It is interesting to compare this theorem with the theorem from [E6] which shows that *there exist moduli of tight contact structures on $S^1 \times \mathbf{R}^2$.*

2.2. *The Euler class of a contact structure.*

Let F be a closed oriented surface in a tight contact manifold (M, ξ). Let $e(\xi) \in H^2(M)$ be the Euler class of ξ considered as a 2-plane bundle over M. By $\chi(F)$ we denote the Euler characteristic of F. Then the following Bennequin-type inequality holds (*cf.* Theorem 4.1.4 in [E4]) :

2.2.1. THEOREM. — *If $F = S^2$ then $e(\xi)[F] = 0$. Otherwise,*

$$|e(\xi)[F]| \leq -\chi(F).$$

Theorem 2.2.1 (which is proved in Sect. 3 below) implies, in particular, that *for any 3-manifold M only a finite number of classes from $H^2(M)$ may be represented as Euler classes of tight contact structures on M* (*cf.* 4.3 in [E4]). Euler class $e(\xi) \in H^2(M)$ is the only homotopy invariant of the restriction of ξ to the 2-skeleton of M. Theorem 2.1.2 then fixes a homotopy class of the extension of a tight structure ξ to M. In particular, we get

2.2.2. THEOREM. — *For any closed manifold M only finite number of homotopy classes of plane distributions can contain tight contact structures.*

It is unclear if the number of *isotopy* classes of tight contact structures on any closed 3-manifold is finite.

2.3. *Tight contact structures on lens spaces $L(p,1)$.*

Surgeries along Legendrian curves (see [E5] and [E7]) allow us to construct, on some manifolds, fillable (and, therefore, tight) contact structures with different values of Euler classes as the following theorem illustrates (see [E7], Sect. 6).

2.3.1. THEOREM. — *All even elements from $H^2(L(p,1)) = \mathbf{Z}_p$ if p is even and all non-zero elements from $H^2(L(p,1))$ if p is odd can be realized as Euler classes of tight contact structures on $L(p,1)$.*

Note that the mod2 reduction of the Euler class of any orientable tangent plane field always vanishes. Therefore, Theorem 2.3.1 gives a complete list of realizable classes from $H^2(L(p,1))$ up to a possible exception of one class for an odd p.

2.4 *Diffeomorphisms and contact diffeomomorphisms of S^3.*

Theorem 2.1.1 implies that any orientation preserving diffeomorphism of S^3 is isotopic to an automorphism of the standard contact structure ξ_0. On the other hand using the technique of filling by holomorphic discs it is easy to show that any automorphism of ξ_0 extends to a diffeomorphism of the 4-ball. Thus we get (see Sect. 6 below) J. Cerf's theorem (see [Ce]).

2.4.1. THEOREM. — *$\Gamma_4 = 0$, i.e. any diffeomorphism of the 3-sphere extends to the 4-ball.*

One can generalize results of this paper to multiparametric families of tight contact structures.

2.4.2. THEOREM. — *The space Tight(S^3) of tight contact structures on S^3 fixed at a point $s \in S^3$ is contractible.*

Let Diff_0 be the group of orientation preserving diffeomorphisms of S^3 which leave the contact plane $\xi_0(s)$ invariant and let Diff_{ξ_0} be its subgroup which consists of automorphisms of the standard contact structure ξ_0. Then Theorem 2.4.2 implies

2.4.3. COROLLARY. — *The inclusion* $\mathrm{Diff}_{\xi_0} \to \mathrm{Diff}_0$ *is a homotopy equivalence.*

3. Tight vs. Overtwisted. Proof of Theorems 1.4.1 and 2.2.1.

3.1. *Invariants* d_+ *and* d_- (*cf.* [HE] and [E4]).

Let F be an oriented generic surface in a contact manifold with an oriented contact structure ξ. We denote, respectively, by $e_\pm$, $h_\pm$ numbers of positive and negative, elliptic and hyperbolic interior singular points of F_ξ. Let $d_\pm = e_\pm - h_\pm$. Suppose that the surface F is either closed or transversal to ξ at the boundary. If F is closed, let us denote by $c(F)$ the value of the Euler class $e(\xi)$ of the bundle ξ evaluated on F. Otherwise, let $c(F)$ be the obstruction to the extention to $\xi\mid_F$ of a vector field tangent to F and ξ along ∂F. Let $\chi(F)$ be the Euler characteristic of F.

Then we have (see [HE] and [E4]) :

3.1.1. PROPOSITION. — $d_\pm = \dfrac{1}{2}\big(\chi(F) \pm c(F)\big)$.

3.2. *Legendrian curves.*

Let F now be a surface bounded by a Legendrian (i.e., tangent to ξ) curve. Let us shift Γ slightly along the normal to ξ vector field. Then the Thurston-Bennequin invariant $tb(\Gamma \mid F)$ is the intersection number of the perturbed curve Γ' and F or, in other words, the linking number between Γ and Γ' with respect to F. Note that $tb(\Gamma \mid F)$ can be considered as the obstruction to a deformation relative to Γ of F to a surface F' transversal to ξ. The invariant $tb(\Gamma \mid F)$ is equivalently well defined in the case when Γ is a piecewise smooth Legendrian curve. In this case the corner points of Γ are necessarily hyperbolic or elliptic points of the characteristic foliation F_ξ. It is easy to see that

3.2.1. PROPOSITION. — *Let F be a surface bounded by a piecewise smooth Legendrian curve Γ. Then F can be deformed to an embedded surface $\tilde F$ bounded by a smooth Legendrian curve $\partial\tilde F = \tilde\Gamma$ such that $tb(\tilde\Gamma \mid \tilde F) = tb(\Gamma \mid F)$ and the characteristic foliations F_ξ and $\tilde F_\xi$ are homeomorphic. In particular, hyperbolic corners of Γ disappear, elliptic corners become smooth elliptic points on $\tilde\Gamma$ and all interior singular points remain the same.*

Finally observe

3.2.2. PROPOSITION. — *If all singular points on* Γ *are of the same sign then* $tb(\Gamma \mid F) = 0$.

3.3. *Elimination Lemma.*

The key point in the proof of 1.4.1 is the following Elimination lemma 3.3.1 (see Sect. 3 in [E7] for the proof). In the present form the lemma is due to D. Fuchs who improved a slightly weaker result of E. Giroux (see 2.3.3 in [Gi]). A similar lemma for fillable structures is contained in my paper [E4] but it is insufficient for purposes of the present paper.

3.3.1. ELIMINATION LEMMA (E. Giroux, D. Fuchs). — *Let* (M, ξ) *and* F *be as above and* Γ *be a trajectory of* F_ξ *whose closure contains an elliptic point* p *and a hyperbolic point* q *of the same sign. Let* U *be a neighborhood of* Γ *in* M *which contains no other singular points of* F_ξ *except* p *and* q. *Then there exists a* C^0*-small isotopy of* F *in* M *which is supported in* U, *fixed at* Γ *and such that the new surface* F' *has no singular points of the characteristic foliation* F'_ξ *inside* U. *If* p *and* q *belong to the Legendrian boundary of* F *then one can kill them leaving* ∂F *fixed.*

It is difficult to kill singular points but it is easy to create them.

3.3.2. LEMMA. — *By a* C^0*-small isotopy of a surface* F *near a nonsingular point of* F_ξ *one can always create a pair of singular points of* F_ξ, *one elliptic and one hyperbolic, having the same pre-specified sign (see Fig. 3).*

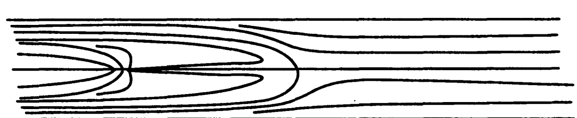


Fig. 3

3.4. *Proof of 1.4.1.*

Suppose that $\mathcal{D}_\xi$ contains a limit cycle C for some embedded disc $\mathcal{D}$. We can assume that there is no other limit cycle inside C; otherwise we can take a smaller C. Let $\mathcal{D}'$ be the disc bounded by C. Note that the sum of indices of singular points of $\mathcal{D}'$ is equal to 1. Therefore, if $\mathcal{D}'$ contains no hyperbolic points, it contains a unique singular elliptic point. But this contradicts the tightness of ξ. Let q be a hyperbolic point of $\mathcal{D}'_\xi$. Let us orient $\mathcal{D}'_\xi$ in such a way that C is an attracting limit cycle of $\mathcal{D}'_\xi$ and remember that elliptic points which are sources are said to be positive. Let q be a hyperbolic point in $\mathcal{D}'_\xi$. Then stable separatrices of $\mathcal{D}'_\xi$ come from positive elliptic points (which may coincide). If the point q is positive itself then it can be killed with one of these elliptic points via Lemma 3.3.1. Suppose that q is negative. If one of the unstable separatrices ends at an elliptic point then this point is necessarily negative, and, therefore, we can apply again 3.3.1. Finally consider the case when both unstable separatrices s_1 and s_2 of q are attracted by the limit cycle C (see Figure 4). Using 3.3.2 one can create an additional pair of negative elliptic and hyperbolic points e and h in such a way that both unstable separatrices s_1 and s_2 of q end at e (see Figure 4). One can achieve also (see 3.2.1) that s_1 and s_2 at e form the angle 180°, i.e. $\gamma = s_1 \cup s_2$ is a smooth Legendrian curve which contains exactly 2 negative singular points e and q. Therefore, $tb(\gamma) = 0$ (see 3.2.2). Let $\mathcal{D}''$ be the disc bounded by γ. Using 3.3.1 one can perturb $\mathcal{D}''$ near γ leaving γ fixed to kill singular points e and q without creating additional points inside $\mathcal{D}''$. The disc $\mathcal{D}''$ contains less hyperbolic points inside than $\mathcal{D}'$ and, therefore, we can continue the process until all hyperbolic points are killed. Q.E.D.

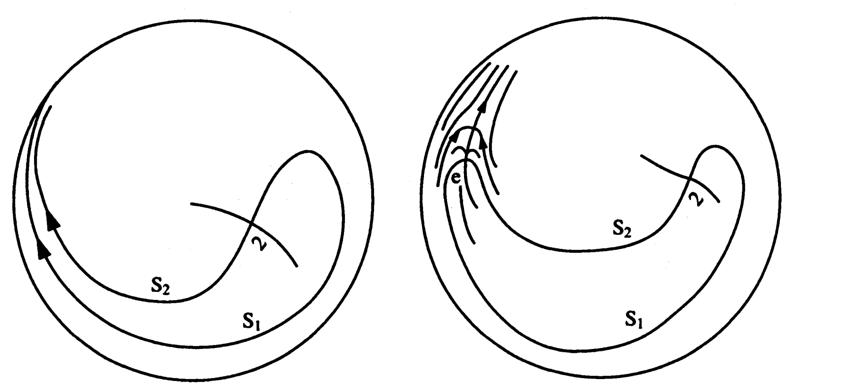


Fig. 4

3.5. *Legendrian polygons.*

When discussing the (C^0-) topology of characteristic foliations there is no difference between pictures of elliptic singularities as on Fig. 1 and Fig. 2. Moreover, as it was mentioned in 1.2. one can always achieve sink-source type pictures as in Fig. 2 by C^0-small perturbation of the surface. From now on we will always assume it is done.

Let Q be an oriented connected surface with piecewise smooth boundary and F be an oriented surface in a contact manifold (M, ξ). A *Legendrian polygon* in F is a pair (Q, α) where α is an orientation preserving immersion $Q \to F$ such that :

- α is injective on Int$Q$;
- corners (vertices) of ∂Q are mapped to singular points of F_ξ, and
- the (smooth) edges of ∂Q are mapped diffeomorphically onto smooth leaves of F_ξ.

As it follows from the definition, the map α can identify either vertices or whole edges of ∂Q.

We call the polygon *simply connected* if Q (and not necessarily $\alpha(Q)$!) is simply connected.

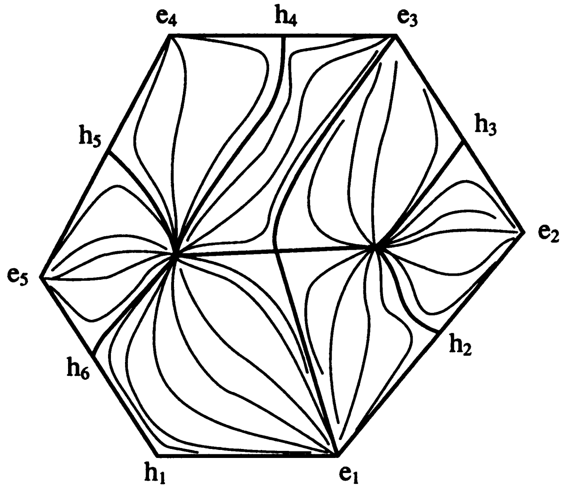
We call the polygon *injective* if either the map α is injective or it identifies only vertices (and not edges) on ∂Q.

We will always count all elliptic points at ∂Q as vertices even if ∂Q is smooth at these points. Sides of ∂Q can contain interior singular points (by the definition, hyperbolic). We call these points *pseudovertices* of Q.

The map α induces on Q a singular foliation from F_ξ. We will denote it by Q_ξ (see Fig. 5). Types and signs of corresponding singular points of F_ξ and Q_ξ coincide.

3.5.1. PROPOSITION. — *If the contact structure ξ on M is tight then for any injective simply connected Legendrian polygon (Q, α) the boundary ∂Q (if not empty) must contain positive as well as negative singular points of Q_ξ.*

Proof. — Suppose first that the map $\alpha : Q \to F$ which defines the polygon is injective. Note that if all singular points of Q_ξ on ∂Q are of the same sign then $tb(\alpha(\partial Q)) = 0$ (see 3.2.2). According to 3.2.1 one can first deform $\alpha(\partial Q)$ into a smooth Legendrian curve with a still vanishing



e_1- e_5 are elliptic vertices,
h_1 is a hyperbolic vertex,
h_2- h_6 are pseudovertices

Fig. 5

Thurston-Bennequin invariant, then, keeping the boundary fixed, make $D = \alpha(Q)$ transversal to ξ along ∂D. In this case the Legendrian curve ∂D becomes a closed leaf of D_ξ, which contradicts the assumption that ξ is tight.

If the map $\alpha : Q \to F$ identifies some of the vertices of ∂Q, then one can first disjoint these vertices applying 3.3.2 near some elliptic points on F in such way as to create a "double-sun" picture (see Fig. 6). This deformation changes F_ξ but leaves Q_ξ the same. Therefore, this case can be reduced to the case of injective α. Q.E.D.

3.6. *Surfaces with boundaries transversal to the contact structure.*

3.6.1. LEMMA. — *Suppose $\mathcal{D}$ is an embedded disc in a tight contact manifold (M, ξ). Suppose that $\partial\mathcal{D}$ is transversal to ξ and the orientations are chosen in such a way that trajectories of $\mathcal{D}_\xi$ exit through $\mathcal{D}_\xi$. Then by a C^0-small isotopy fixed near $\partial\mathcal{D}$ one can kill all positive hyperbolic points of $\mathcal{D}_\xi$.*

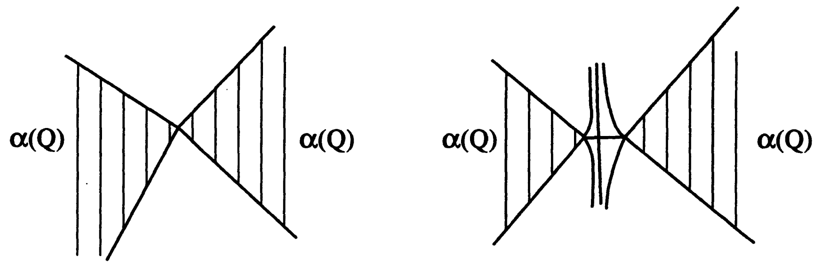


Fig. 6

Proof. — According to the tightness of ξ, the foliation $\mathcal{D}_\xi$ has no closed leaves. Therefore, stable separatrices of positive hyperbolic points must come from positive elliptic points. Hence we can apply 3.3.1 and consequently kill all the hyperbolic points. Q.E.D.

3.7. *Basins.*

Let V be a compact subdomain in a surface F in a contact manifold (M, ξ). Suppose that ∂V is transversal to F_ξ and trajectories of F_ξ *exit* through ∂V. The *basin* $B(V) \subset F$ is the set of points of F which can be reached from V along trajectories of F_ξ. We will consider basins either in the case when F is closed or when F has a boundary but all trajectories originating in V are locked inside V. Generically, these trajectories can be attracted either by negative elliptic points, or by limit cycles or by hyperbolic points of F_ξ. First, observe the following, simple

3.7.1. LEMMA. — *If no trajectory exiting V is attracted by limit cycles then the closure $\overline{B(V)}$ has a natural structure of a Legendrian polygon $(\bar{B}(V), \alpha_V)$. All elliptic vertices on the boundary $\Gamma(V) = \partial\bar{B}(V)$ are negative and no side of $\Gamma(V)$ can contain more than one pseudovertex.*

Note that in the presence of limit cycles one can still define a corresponding Legendrian polygon $(\bar{B}(V), \alpha_V)$ where the map α_V is not defined at some vertices of $\Gamma(V)$. These vertices correspond to attracting limit cycles of F_ξ and the two edges ending at such a vertex P are mapped by α_V to separatrices γ and γ' converging to a limit cycle C. Using 3.3.2 one can always perturb F near C to create a pair E, H of negative elliptic and hyperbolic points in such a way that separatrices γ and γ' in the perturbed surface are attracted by E instead of C. Doing this for each missing vertex of $\Gamma(V)$ one can define $(\bar{B}(V), \alpha_V)$ as a usual Legendrian polygon in the surface F after the perturbation.

We will also consider basins for *positive elliptic points*. By the definition, for such a point E the basin $B(E)$ is the same as the basin of its small round neighborhood.

3.8 *Proof of 2.2.1.*

We will prove that any elliptic point of F can be cancelled via a perturbation of F with a hyperbolic point unless F is a sphere with $d_+ = d_- = 1$. In the latter case this would imply that $c(F) = d_+ - d_- = 0$. In the first case we would have $d_+, d_- \leq 0$ and, in view of 3.1.1,

$$\chi(F) \pm c(F) \leq 0 \qquad \text{or} \qquad |c(F)| \leq -\chi(F) .$$

Let us take an elliptic point $E \in F$. We can think that it is positive and, therefore, a source of trajectories of F_ξ. Let us consider its basin $B(E)$. Using 3.7 we can perturb F, if necessary, to insure that the closure $\overline{B(E)}$ contains no limit cycles. Note that the required perturbation creates additional *negative* points of F_ξ but does not affect numbers of *positive* points. The closure $\overline{B(E)}$ has a structure of a Legendrian polygon $(\bar{B}(E), \alpha_E)$ with the boundary Γ_E. There are two possibilities. Either $\bar{B}(E)$ is a sphere and $\Gamma(E)$ is empty or $\bar{B}(E)$ is a disc. In the first case F_ξ cannot have other singular points than E and the negative elliptic point $\tilde{E} = F \setminus B(E)$. Therefore, we have $d_+ = d_- = 1$ in this case. In the second case, we can assume that F_ξ is generic, i.e. there is no separatrix connection between hyperbolic points. Therefore, all hyperbolic points on $\Gamma(E)$ are pseudovertices. If at least one of these points is positive, then one can cancel E (see 3.3.1) and the hyperbolic point along the separatrix which connects them. If all pseudovertices on the boundary are negative then, again applying 3.3.1, one can consequently cancel all of them with negative elliptic vertices on $\Gamma(E)$ until $\Gamma(E)$ becomes a smooth Legendrian curve without singular points on it, which contradicts to the tightness of ξ because $\bar{B}(E)$ is diffeomorphic to the disc in this case. Q.E.D.

4. Characteristic foliations on spheres in tight contact manifolds.

In this section we study special properties of characteristic foliations on spheres in tight contact manifolds. The main results of the section are lemmas 4.3.1 and 4.4.5 below.

4.1. *Taming functions.*

Let $S \subset M$ be an oriented 2-sphere embedded in the contact manifold (M, ξ). Let X be a vector field generating the characteristic foliation S_ξ. We will assume that all zeroes of X are simple or of "birth-death" type. We say that a function $\varphi : M \to R$ *tames* the foliation S_ξ if the following properties are satisfied :

T1. The field X is gradient-like for φ, i.e. $d\varphi(X) \geq \rho \|d\varphi\|^2$ for a positive constant $\rho > 0$ and a Riemannian metric on S; in particular, singularities of X coincide with critical points of φ;

T2. positive (resp. negative) elliptic points of S_ξ are local minima (resp. maxima) of φ;

T3. passing through a hyperbolic critical value in the positive direction increases the number of components of the level set $\{\varphi = C\}$ if the point is negative and decreases it in the opposite case.

4.2. *Basins in an embedded sphere.*

Basins in a sphere embedded into a tight contact manifold enjoy special properties.

4.2.1. LEMMA. — *Let (M, ξ) be a tight contact manifold, $S \subset M$ a two-sphere, and $V \subset S$ a domain with the boundary ∂V transversal to the foliation S_ξ. Suppose that trajectories of S_ξ exit through ∂V. Then the closure $\overline{B(V)} \subset S$ is covered by a Legendrian polygon $(\bar{B}(V), \alpha_V)$ with the boundary $\Gamma(V)$ such that*

a) *if $\Gamma(V) = \emptyset$ then $\bar{B}(V) = S^2$;*

b) *if the polygon $(\bar{B}(V), \alpha_V)$ is injective (see 3.5) then $\Gamma(V)$ contains at least one positive pseudovertex;*

c) *if $d_+(V) = 1$ then all pseudovertices which are identified by α_V are negative.*

Proof. — Part a) is evident so let us start with b). Suppose that the polygon $(\bar{B}(V), \alpha_V)$ is injective. Deforming S, if necessary, near images of hyperbolic vertices of $\Gamma(V)$ (see 3.2.1), we can make all vertices of $\Gamma(V)$ elliptic (and hence, necessarily) negative points. Because S is a sphere there exists a simply connected injective Legendrian polygon (Q, β) such that $\beta(\partial Q) = \alpha(\Gamma(V))$. If all pseudovertices on $\Gamma(V)$ were negative it would imply that all singular points on $\Gamma(V)$, and therefore, on ∂Q are negative which contradicts 3.5.1.

To prove c) we first apply 3.6.1 and kill all positive hyperbolic points inside V. Then the assumption $d_+(V) = 1$ implies that the new surface, which we still denote by V, contains exactly one positive elliptic interior point E. Each pseudovertex at $\Gamma(V)$ has a separatrix which comes from V. After a small additional perturbation inside V to destroy separatrix connections between hyperbolic points, we can claim that all these separatrices begin at E. Therefore, if two positive pseudovertices $H_1, H_2 \in \Gamma(V)$ are mapped by α_V into one hyperbolic point $H \in S$ then both stable separatrices of H start at E and form in S a piecewise smooth embedded Legendrian circle with exactly two singular points E and H on its boundary. Because both points E and H are positive this contradicts 3.5.1.

Q.E.D.

4.3. *Existence of a taming function.*

4.3.1. LEMMA. — *Let S be a sphere embedded into a tight contact manifold (M, ξ). Suppose that all singularities of S_ξ are simple. Then S_ξ admits a taming function $S \to \mathbf{R}$.*

Note : We do not assume genericity of S_ξ and admit separatrix connections between hyperbolic points.

Proof. — Let the required function φ be equal to 0 at all positive elliptic points. For $0 < c \leq 1$ the level set $\{\varphi = c\}$ consists of small circles transversal to S_ξ. Note that each component of the set $V_1 = \{\varphi \leq 1\}$ has $d_+ = 1$. The closure $\overline{B}(V_1)$ equals S because any point belongs to the closure of a trajectory starting at some positive elliptic point. We are going to extend our function consequently to sets V_k, $k = 2, \cdots$, in such a way that all critical values of φ are even integers, the set V_k coincides with $\{\varphi \leq 2k-1\}$ and contains at least k singular points of S_ξ, $\overline{B(V_k)} = S$ and each component of V_k has $d_+ = 1$. Because the number of singular points of S_ξ is finite then $V_N = S$ for some $N > 0$.

Suppose that the set V_k and the function $\varphi|_{V_k}$ with the required properties are already constructed. For each component W of V_k consider its basin $B(W)$, the Legendrian polygon $(\bar{B}(W), \alpha_W)$ and its boundary $\Gamma(W)$ (see 3.7). Note that W satisfies the conditions of Lemma 4.2.1. Therefore, either

a) $\bar{B}(W) = S^2$; or

b) $\Gamma(W)$ contains a positive pseudovertex H which is not identified with other pseudovertices on $\Gamma(W)$; or

c) $\Gamma(W)$ contains a pair of negative pseudovertices H_1 and H_2 which are identified by α_W.

In the case a) φ admits an obvious extension to S with the maximum at the negative elliptic point in $\overline{B(W)} \setminus B(W)$. In the case b) note that the condition $\overline{B(V_k)} = S$ implies that the pseudovertex $H \in \Gamma(W)$ is identified with a pseudovertex $H' \in \Gamma(W')$ for another component W' of V_k. Let $h = \alpha_W(H) = \alpha_{W'}(H') \in S$. We can extend the function φ to compact subset $V_{k+1} \subset S$, $V_k \in \mathrm{Int}V_{k+1}$, with the following properties :

- $\varphi\ |_{\partial V_{k+1}} = 2k+2$, $\varphi\ |_{\mathrm{Int}V_{k+1}} < 2k+2$;
- φ tames the foliation S_ξ restricted to V_{k+1};
- $\varphi\ |_{V_{k+1} \setminus V_k}$ has exactly one critical point $h \in S$ and it is nondegenerate and hyperbolic;
- if $\tilde{W}$ is a component of V_{k+1} which contains h, then $\overline{B(\tilde{W})} = \overline{B(W)} \cup \overline{B(W')}$; for any other component $\tilde{\tilde{W}}$ of V_{k+1} we have $\overline{B(\tilde{\tilde{W}})} = B(V_k \cap \tilde{\tilde{W}})$.

Note that $\overline{B(V_{k+1})} = S$ and we have $d_+ = 1$ for any component of V_{k+1}.

The case c) can be treated analogically and, therefore, the construction of φ can be continued by induction.

Q.E.D.

4.4. *Families of taming functions.*

We start with two obvious lemmas, 4.4.1 and 4.4.2.

4.4.1. LEMMA. — *Suppose that ξ is a contact structure near $S \subset M$ and $\varphi : S \to \mathbf{R}$ is a taming function for S_ξ. Suppose that the structure $\tilde{\xi}$ is sufficiently C^0-close to ξ and coincides with ξ near singularities of S_ξ. Then φ tames the characteristic foliation $S_{\tilde{\xi}}$ as well.*

We say that the characteristic foliation S_ξ is *generic* if all its singularities are nondegenerate and there are no separatrix connections between hyperbolic points.

4.4.2. LEMMA. — *Let $\xi_t, t \in [-1,1]$, be a family of contact structures near S. Suppose that S_{ξ_t} is generic for $t \neq 0$, S_{ξ_0} has a birth-death type singular point p and outside of an arbitrarily small neighborhood of $p \in S$ the foliations S_{ξ_t} for $t \in [-1,0]$ are isomorphic. (In other words, p is* *a birth type singularity when t grows.) Suppose that $S_{\xi_{-1}}$ admits a taming function φ_{-1}. Then φ_{-1} can be included into a smooth family $\varphi_t : S \to \mathbf{R}$, $t \in [-1,1]$, such that φ_t tames S_{ξ_t} for all $t \in [-1,1]$.*

4.4.3. LEMMA. — *Suppose that for a contact structure ξ near S the singularities of the characteristic foliation S_ξ are nondegenerate. Let φ_0, φ_1 be two taming functions for S_ξ. Then they can be included into a smooth family $\varphi_t : S \to \mathbf{R}$, $t \in [0,1]$, of functions which tame S_ξ.*

Proof. — Note that functions φ_0 and φ_1 have the same critical points. Therefore, φ_0 and φ_1 would be obviously homotopic as taming functions if they had had the same ordering of corresponding critical values. Now observe that if $c_1 < c_2$ are 2 consecutive critical values of a taming function φ such that c_2 corresponds to a positive hyperbolic point than whichever type has the point corresponding c_1 one can always change the order of these two critical values via a homotopy of taming functions. Therefore both functions φ_0 and φ_1 can be deformed as taming functions to the same taming function φ.

Q.E.D.

4.4.4. *Remark.* — The analysis of the above argument shows that, in fact, *the space of functions taming a given characteristic foliation S_ξ is contractible.*

Now we are ready to prove the main result of the section.

4.4.5. LEMMA. — *Let $\xi_t, t \in [0,1]$, be a family of tight contact structures near the sphere $S \subset M$. Suppose that for all $t \in [0,1]$ except finite number of values $0 < t_1 < \dots t_n < 1$ the characteristic foliation S_{ξ_t} is generic and for $t = t_i, i = 1,\dots,n$, the foliation S_{ξ_t} contains either a "death-birth" type singularity or a separatrix connection between hyperbolic points. Then any two functions φ_0 and $\varphi_1 : S \to \mathbf{R}$, which tame S_{ξ_0} and S_{ξ_1} respectively, can be included into a smooth family of functions $\varphi_t, t \in [0,1]$, such that φ_t tames S_{ξ_t} for each $t \in [0,1]$.*

Proof. — First of all note that the topology of the foliation S_{ξ_t} does not change for $t \in]t_i, t_{i+1}[$, $i = 0,\dots,n$ (we let $t_0 = 0, t_{n+1} = 1$). Therefore, a function $\varphi_{t'}$, which tames $S_{\xi_{t'}}$, for a particular value $t' \in]t_i, t_{i+1}[$ is automatically included into a family φ_t of functions taming S_{ξ_t} for all $t \in]t_i, t_{i+1}[$. Moreover, using 4.4.3 one can construct the family φ_t,

$t \in]t_{i-1}, t_{i+1}[$ to include two given taming functions $\varphi_{t'}$ and $\varphi_{t''}$ for two different values $t', t'' \in]t_i, t_{i+1}[$.

We start the construction of the family φ_t for values $t = t_{i_1}, \dots, t_{i_k}$ for which foliations $S_{\xi_{t_{i_l}}}, l = 1, \dots k$, have simple zeroes. This is possible according to 4.3.2. Then using 4.4.1 we include $\varphi_{t_{i_l}}$ into families φ_t defined for t close to $t_{i_l}, l = 1, \dots k$. Now we apply 4.4.2 to construct φ_t for t close to critical values $t = t_{j_1}, \dots t_{j_s}$ for which $S_{\xi_{j_l}}, l = 1, \dots, s$, has a birth-death type singularity. Finally, using the remark at the beginning of the proof we extend the family for all $t \in [0, 1]$.

Q.E.D.

5. Extension of contact structures from the sphere to the ball.

5.1. *Germs of contact structures and their characteristic foliations.*

The following simple lemma is a relative version of Darboux's theorem. It shows that there is no difference in problems of extending germs of contact structures or their characteristic foliations.

5.1.1. LEMMA. — *The canonical projection of the space G of germs at $S = \partial B$ of contact structures to the space* Fol *of their characteristic foliations on S is a Serre fibration with a contractible fiber.*

5.2. *From functions to embeddings.*

Let $\mathbf{C}^2$ be the space with coordinates $(z_1 = x_1 + iy_1, z_2 = x_2 + iy_2)$, $\mathbf{R}^3 \subset \mathbf{C}^2$ the subspace $\{y_2 = 0\} \subset \mathbf{C}^2$ and $p : \mathbf{R}^3 \to \mathbf{R}$ the projection $(x_1, y_1, x_2) \longrightarrow x_2$. Let Emb be the space of embeddings $\alpha : B \to \mathbf{R}^3$ such that : a) $p \circ \alpha|_{\partial B}$ has only nodegenerate or birth-death type critical point; b) mean curvature of $\alpha(\partial B)$ at critical points of the function $p|_{\alpha(\partial B)}$ is positive (for $\alpha(\partial B)$ oriented as the boundary of $\alpha(B)$); c) all nonsingular components of level-sets of $p \circ \alpha$ are 2-discs.

We denote by Funct the space of functions $S = \partial B \to \mathbf{R}$ with nondegenerate or death-birth type critical points. Let r : Emb $\to$ Funct be the restriction map.

5.2.1. LEMMA. — *The map r is surjective and for any path $f_t \in$* Funct, *$t \in [0, 1]$, and for any embeddings $\alpha_0, \alpha_1 \in$* Emb *with $r(\alpha_0) = f_0$, $r(\alpha_1) = f_1$ there exists a covering isotopy $\alpha_t \in$* Emb *with $r \circ \alpha_t = f_t$, $t \in [0, 1]$, which connects α_0 and α_1.*

Proof. — With a function $f \in$ Funct we associate a tree Γ_f identifying each component of level-sets $\{f = \text{const}\}$ to a point. The function f defines a function $\bar{f} : \Gamma_f \to \mathbf{R}$. First observe that for any family $f_t \in$ Funct, $t \in [0,1]$, there exists a family of embeddings $Q_t : \Gamma_{f_t} \to \mathbf{R}^3$, $t \in [0,1]$, such that $p \circ Q_t = \bar{f}_t$. Taking regular neighborhoods of embedded graphs $Q_t(\Gamma_{f_t}) \subset \mathbf{R}^3$, $t \in [0,1]$, we get, after maybe a small perturbation, a family of embeddings $\beta_t : B \to \mathbf{R}^3$, $t \in [0,1]$, from Emb such that $r \circ \beta_t = f_t$. Finally note that the fibers $r^{-1}(f) \subset$ Emb, $f \in$ Funct, are obviously connected and, therefore, one can deform the path $\beta_t : B \to \mathbf{R}^3$ to the path $\alpha_t : B \to \mathbf{R}^3$, $t \in [0,1]$, which connects α_0 and α_1 and still covers f_t, $t \in [0,1]$.

Q.E.D.

5.2.2 *Remark.* — One can easily modify the proof of 5.2.1 to show that the map r : Emb $\to$ Funct is a Serre fibration with a connected (but not simply connected!) fiber.

5.3. *Pseudoconvex embeddings.*

Consider now spaces Conv, Conv^∂ and Conv_1^∂ which consist of pairs (α, γ) where $\alpha : B \to \mathbf{R}^3$ is an embedding from Emb and γ is a function $\alpha(B) \to \mathbf{R}$ in the case of Conv, the germ of the function along $\alpha(S) = \alpha(\partial B)$ in the case of Conv^∂ or the 1-jet along S of the function in the case of Conv_1^∂. We suppose that γ equals 0 on $\alpha(S)$, is positive on $\text{Int}(\alpha(B))$ and has no critical points at the boundary $\alpha(S)$. Thus the graph $\Gamma_\gamma = \{y_2 = \gamma(u),\ u \in \alpha(B)\} \subset \mathbf{C}^2$ (or the germ or the 1-jet of this graph) is transversal to $\mathbf{R}^3 = \{y_2 = 0\} \subset \mathbf{C}^2$. In the case of spaces Conv and Conv^∂ we assume in addition that Γ_γ is strictly pseudoconvex being cooriented as the boundary of the domain $\{y_2 \le \gamma(u)\} \subset \mathbf{C}^2$. Let EF (resp. FF) be the subspace of Emb $\times$ Fol (resp. Funct $\times$ Fol) which consists of pairs $(\alpha, \mathcal{F})$ where $\mathcal{F}$ is a characteristic foliation on $S = \partial B$ tamed by the function $p \circ \alpha$ (resp. α).

Let $\tilde{r} : EF \to FF$ be the restriction of the map $r \times \text{id}$: Emb $\times$ Fol $\to$ Funct $\times$ Fol and let Cont be the space of contact structures on B. If $(\alpha, \gamma) \in$ Conv then the distribution of complex tangencies on Γ_γ defines a contact structure $\text{cont}(\alpha, \gamma)$ on B. A pair $(\alpha, \gamma) \in \text{Conv}_1^\partial$ defines a characteristic foliation $\text{fol}(\alpha, \gamma)$. We denote by $\overline{\text{fol}}$ the map $(\alpha, \gamma) \to (\alpha, \text{fol}(\alpha, \gamma))$ of Conv_1^∂ to EF.

First note the following simple fact.

5.3.1. LEMMA. — *The map* $\overline{\text{fol}} : \text{Conv}_1^\partial \to EF$ *is a homeomorphism.*

Now we have

5.3.2. LEMMA. — *The natural maps* $\pi_1 : \text{Conv} \to \text{Conv}^\partial$ *and* $\pi_2 : \text{Conv}^\partial \to \text{Conv}_1^\partial$ *are Serre fibrations with contractible fibers.*

Proof. — The statement about π_2 is standard. To prove that π_1 is a Serre fibration with a contractible fiber observe first that for any compact set K a map $F : K \to \text{Conv}^\partial$ admits a canonical lift $\hat{F} : K \to \text{Conv}$ as follows. Let $F(x) = (\alpha_x, \gamma_x) \in \text{Conv}^\partial$. Suppose that all germs γ_x, $x \in K$ are represented by functions defined in neighborhoods $U_x \supset \alpha_x(S)$. There exists $R > 0$ and $\epsilon > 0$ such that

– the ball $B_R \subset \mathbf{R}^3$ of radius R centered at the origin contains all images $\alpha_x(B)$, $x \in K$;

– the convex surface $\Sigma = \{y_2 = \delta_\epsilon(x_1, y_1, x_2) = \epsilon\sqrt{R^2 - x_1^2 - x_2^2 - y_2^2}$, $(x_1 y_2, x_2) \in B_R\}$ intersects transversely all hypersurfaces $\Gamma_{\gamma_x} = \{y_2 = \gamma_x(u)$, $u = (x_1, y_1, x_2) \in U_x\}$ along spheres $\tilde{S}_x$ close to $\alpha_x(S)$.

Now take $\tilde{\gamma}_x = \min(\gamma_x, \delta_\epsilon)$ and note that the piecewise smooth hypersurface $\Gamma_{\tilde{\gamma}_x} = \{y_2 = \tilde{\gamma}_x(u), u \in B\}$ is pseudoconvex. To get the required extension $\hat{\gamma}_x$ of γ_x from U_x to the ball B we apply to $\tilde{\gamma}_x$, $x \in K$, any canonical smoothing procedure P which preserves pseudoconvexity.

Let us now consider a map $G : K \to \text{Conv}$ and note that if $F = \pi_1 \circ G : K \to \text{Conv}^\partial$ then G is homotopic among covering maps to the map $\hat{F} : K \to \text{Conv}$, $\hat{F}(x) = (\alpha_x, \hat{\gamma}_x)$, constructed above. Indeed, let $F(x) = (\alpha_x, \gamma_x)$, $R > 0$, $\epsilon > 0$ and the smoothing operator P be as above. Suppose, in addition, that $\gamma_x(u) < \delta_1(u)$ for all $u \in \alpha_x(B)$, $x \in K$. Then the required homotopy F_t, $t \in [\epsilon, 1]$, which connects $F_\epsilon = \hat{F}$ and $F_1 = F$ is given by the formula $F_t(x) = P(\min(\gamma_x, \delta_t))$.

Q.E.D.

5.3.3. COROLLARY. — *The composition* $q = \text{fol} \circ \pi_2 \circ \pi_1 : \text{Conv} \to EF$ *is a fibration with contractible fiber.*

Combining 5.3.3 and 5.2.1 we get

5.3.4. COROLLARY. — *The map* $\tilde{q} = \tilde{r} \circ q : \text{Conv} \to FF$ *is surjective and for any path* $f_t = (\varphi_t, \mathcal{F}_t) \in FF$, $t \in [0,1]$, *and any points*

$a_0 = (\alpha_0, \gamma_0)$, $a_1 = (\alpha_1, \gamma_1) \in$ Conv with $\tilde{q}(a_i) = (\varphi_i, \mathcal{F}_i)$ for $i = 0, 1$, there exists a path $a_t \in$ Conv with $\tilde{q}(a_t) = f_t, t \in [0, 1]$.

Finally, we get

5.3.5. COROLLARY. — *Let $\mathcal{L} \in$ Fol be a generic foliation admitting a taming function $\varphi : S \to \mathbf{R}$ and let $A \in$ Conv be such that $\tilde{q}(A) = (\varphi, \mathcal{L})$. Then the contact structure* cont A *which extends $\mathcal{L}$ is independent of the choice of φ and A up to an isotopy fixed at S.*

Proof. — Let φ', A' be another choice for φ and A. According to 4.4.5 the functions φ and φ' are homotopic as taming functions for $\mathcal{L}$. Let $\varphi_t, t \in [0, 1]$, be the homotopy. According to 5.3.4 the path $(\varphi_t, \mathcal{L}) \in FF$, $t \in [0, 1]$, can be lifted to Conv as a path A_t connecting A and A'. Therefore, $\operatorname{cont} A_t$, $t \in [0, 1]$, is a homotopy between contact structures $\operatorname{cont} A$ and $\operatorname{cont} A'$. The homotopy is fixed at the boundary and, therefore, $\operatorname{cont} A$ and $\operatorname{cont} A'$ are isotopic.

Q.E.D.

5.4. *Proof of Theorem 2.1.3.*

Let ξ be a tight contact structure on the ball B of radius 1. Let S_t, $t \in [0, 1]$, be the family of concentric spheres of radii t. Near the center of the ball the structure ξ is standard. Therefore, we can assume that for $t = \epsilon$ the sphere S_ϵ bounds the standard contact ball and, in particular, the characteristic foliation $(S_\epsilon)_\xi$ has exactly two singular points which are elliptic. Therefore S_ϵ admits the standard taming function f_ϵ with exactly two critical points. According to 4.4.5 there exists a family $f_t : S_t \to \mathbf{R}, t \in [\epsilon, 1]$, of taming functions for foliations $(S_t)_\xi$, which starts at f_ϵ. Now according to 5.3.4 there exists a lift of the path $((S_t)_\xi, f_t) \in FF$ to the path $F : [\epsilon, 1] \to$ Conv. Then the structure $\operatorname{cont} F(t)$ extends $(S_t)_{\xi_t}$ to the ball B_t. Now consider the following family ζ_t, $t \in [\epsilon, 1]$, of contact structures on B

$$\zeta_t = \begin{cases} \xi & \text{on} \quad B \setminus B_t \\ \operatorname{cont} F(t) & \text{on} \quad B_t \end{cases}$$

Note that $\zeta_\epsilon = \xi$, $\zeta_1 = \operatorname{cont} F(1)$. Therefore, structures ξ and $\operatorname{cont} F(1) = \zeta_1$ are homotopic relative to the boundary and therefore, isotopic. But according to 5.3.5 the structure ζ_1 depends only on S_ξ. Therefore all tight contact structures on B extending S_ξ are isotopic to ζ_1 and, therefore, isotopic among themselves.

Q.E.D.

6. $\Gamma_4 = 0$.

In this section we prove Cerf's theorem 2.4.1 (see [Ce]).

6.1. LEMMA. — *Let $f : S^3 \to S^3$ be an orientation preserving diffeomorphism and ξ_0 be the standard contact structure on S^3. Then f is isotopic to a contact automorphisim g of ξ_0, i.e. $dg(\xi_0) = \xi_0$.*

Proof. — The contact structure $f^*(\xi_0)$ is positive and tight. Therefore, according to 2.1.3 there exists an isotopy $\varphi_t : S^3 \to S^3$ such that $\varphi_0 = \mathrm{id}$ and $\varphi_1^*(f^*\xi_0) = \xi_0$. Then $h_t = \varphi_t \circ f$, $t \in [0,1]$, is an isotopy between $h_0 = f$ and a contact diffeomophism $g = \varphi_1 \circ f$.

Q.E.D.

6.2. LEMMA. — *Any contact automorphism g of (S^3, ξ_0) extends to the 4-ball B^4, $\partial B^4 = S^3$.*

Proof. — We consider S^3 as the unit sphere in $\mathbf{C}^2$. Let $l : S^3 \to \mathbf{R}$ be the restriction to S^3 of the coordinate function $y_2 = \mathrm{Im} z_2$. For $t \in]-1,1[$ let us denote by Σ^t the two-sphere $\{l = t\}$. Let $\tilde{\Sigma}_t = g(\Sigma^t)$. We can think that the diffeomorphism g is fixed near the poles $p_\pm = \{l = \pm 1\} \in S^3$. Then $\Sigma^t = \tilde{\Sigma}^t$ for t close to ± 1. For any $t \in]-1,1[$ the characteristic foliation $\mathcal{F}_t = \Sigma^t_{\xi_0}$ has exactly two singular points q_0^t, $q_1^t \subset \Sigma^t$ which are elliptic. Slightly C^0-perturbing the function l we can arrange that the leaves of characteristic foliations $\mathcal{F}_t, t \in]-1,1[$ are meridians joining q_0^t and q_1^t (rather than spirals; *cf.* 3.5). The diffeomorphism g preserves the contact structure ξ_0 and, therefore, for $t \in]-1,1[$ it maps diffeomorphically the characteristic foliation $\mathcal{F}_t$ onto the charateristic foliation $\tilde{\mathcal{F}}_t = \tilde{\Sigma}^t_{\xi_0}$. Let $\tilde{q}_0^t = g(q_0^t)$, $\tilde{q}_1^t = g(q_1^t)$. The punctured spheres $\Sigma^t \backslash \{q_0^t \cup q_1^t\}$ and $\tilde{\Sigma}^t \backslash \{\tilde{q}_0^t \cup \tilde{q}_1^t\}$ admit foliations $\mathcal{L}_t$ and $\tilde{\mathcal{L}}_t$ by circles transversal to $\mathcal{F}_t$ and $\tilde{\mathcal{F}}_t$, and spanning holomorphic discs in the unit 4-ball $B \subset \mathbf{C}^2$; the foliations $\mathcal{L}_t$ and $\tilde{\mathcal{L}}_t$ depend smoothly on the parameter t and the corresponding holomorphic discs for all $t \in]-1,1[$ form diffeomorphic foliations $\mathcal{D}$ and $\tilde{\mathcal{D}}$ of the ball B (see [Gro] and [E4]). Reparametrizing g along leaves of $\mathcal{F}_t$ we can arrange that g is a diffeomorphism of $\mathcal{L}_t$ onto $\tilde{\mathcal{L}}_t$ for all $t \in]-1,1[$. The group of diffeomorphisms of a 2-disc which are fixed at its boundary is contractible. Therefore g extends to a diffeomorphism between the foliations $\mathcal{D}$ and $\tilde{\mathcal{D}}$ and, therefore $g : S^3 \to S^3$ extends to a diffeomorphism $G : B \to B$.

Q.E.D.

Now Theorem 2.4.1 follows immediately from 6.1 and 6.2.

7. Tight contact structures on $\mathbf{R}^3$. Proof of Theorem 2.1.4.

We start with

7.1. LEMMA. — *Let $S \subset (M,\xi)$ be an embedded 2-sphere in a tight contact manifold (M,ξ) and $U \supset S$ any small open neighborhood in M. Then there exists an isotopy $h_t : M \to M$, $t \in [0,1]$, $h_0 = \mathrm{id}$, which is supported in U and such that the characteristic foliation $\tilde{S}_\xi$ on the sphere $\tilde{S} = h_1(S)$ is isomorphic to the characteristic foliation on the boundary of the round unit ball in the standard contact $(\mathbf{R}^3, \xi_0)$.*

Proof. — According to 2.2.1 and 3.1.1 we have $d_+(S) = d_-(S) = 1$ and using 3.3.1 we can kill all hyperbolic points of S_ξ by an isotopy supported in U. Then the resulting sphere $\tilde{S}$ carries the characteristic foliation $\tilde{S}_\xi$ which is *homeomorphic* to the characteristic foliation $\mathcal{F}$ on the round unit sphere in the standard contact $(\mathbf{R}^3, \xi_0)$. But then $\tilde{S}_\xi$ can be made *diffeomorphic* to $\mathcal{F}$ by an additional C^0-perturbation of $\tilde{S}$ in U (see 2.1.5.1 in [E1]).

Q.E.D.

Now we have

7.2. LEMMA. — *Let ξ be a tight contact structure on $\mathbf{R}^3$. Then there exist compact domains V_k, $k = 1, \cdots$, such that $\bigcup\limits_1^\infty V_k = \mathbf{R}^3$ and for each $k = 1, \cdots$, we have $V_k \subset \mathrm{Int} V_{k+1}$ and (V_k, ξ) is isomorphic to the round unit ball (B, ξ_0) in the standard contact $(\mathbf{R}^3, \xi_0)$.*

Proof. — Take any exhaustion $\mathbf{R}^3 = \bigcup\limits_1^\infty V_k$, $V_k \subset \mathrm{Int}\ V_{k+1}$, $k = 1, \cdots$, of $\mathbf{R}^3$ by compact domains diffeomorphic to B. Applying 7.1 we will get an exhaustion $\mathbf{R}^3 = \bigcup \tilde{V}_k$, $\tilde{V}_k \subset \mathrm{Int}\ \tilde{V}_{k+1}$, $k = 1, \cdots$, by embedded balls such that characteristic foliations $(\tilde{S}_k)_\xi$ on their boundaries $\partial \tilde{V}_k = \tilde{S}_k$ are isomorphic to $(\partial B)_{\xi_0}$. But then according to 2.1.3 $(\tilde{V}_k, \xi)$ is isomorphic to (B, ξ_0) for all $k = 1, \ldots$.

Q.E.D.

We will need also the following obvious fact

7.3. Lemma. — *The space of contact embeddings $(B, \xi_0) \to (B, \xi_0)$ is connected.*

Proof of Theorem 2.1.4. — Let $\mathbf{R}^3 = \bigcup_1^\infty V_k$ be an exhaustion of the tight contact $(\mathbf{R}^3, \xi)$ constructed in 7.2 and $\mathbf{R}^3 = \bigcup_1^\infty B_k$ be a similar exhaustion for the standard contact $(\mathbf{R}^3, \xi_0)$. By the construction there exists a contact isomorphism $h_1 : (B_1, \xi_0) \to (V_1, \xi)$. We are going to extend it inductively to all $V_k, k = 2, \ldots$. Suppose that the contact diffeomorphism $h_k : (B_k, \xi_0) \to (V_k, \xi)$ is already constructed. Both contact manifolds (B_{k+1}, ξ_0) and (V_{k+1}, ξ_0) are isomorphic to the unit round ball (B, ξ_0) and, therefore, there exists a contact diffeomorphism $g : (B_{k+1}, \xi_0) \to (V_{k+1}, \xi_0)$. According to 7.3 the contact embeddings $g|_{B_k}, h_k : (B_k, \xi_0) \to (V_{k+1}, \xi)$ are isotopic as contact embeddings. Hence there exists an ambient contact diffeotopy $H_t : (V_{k+1}, \xi) \to (V_{k+1}, \xi)$, $t \in [0,1]$, such that $H_0 = \mathrm{id}$, $H_t|_{\partial V_{k+1}} = \mathrm{id}$ and $H_1 \circ g|_{B_k} = h_k$. Then the contact diffeomorphism $h_{k+1} = H_1 \circ g$ is the required extension of h_k.

Q.E.D.

8. Open questions.

8.1. *Fillable and tight.*

8.1.1 Find examples of s-fillable structures which are not h-fillable and tight structures which are not fillable.

8.1.1'. For example, does $M \times S^1$, where M is a closed surface of genus > 1, admit an h-fillable structure? (An s-fillable structure on $M \times S^1$ does exist.)

8.2. *Which manifolds admit fillable or tight structure?*

8.2.1. For example, does $V\#(-V)$, where V is a 3-manifold without orientation reversing diffeomorphism, admit a fillable (tight) contact structure?

8.2.1'. Let V be as in 8.2.1. Suppose V admits a positive tight structure. Does it admit a negative tight contact structure?

8.2.2. Let V be an irreducible 3-manifold. Does it admit a tight or fillable contact structure?

8.2.3. Which *open* 3-manifolds admit a tight structure?

To answer this question one should understand when a tight contact structure on a 3-manifold with boundary can be extended to an attached handle.

8.3. *Cobordism.*

Let W be a complex manifold with two boundary components, V_1 and V_2. Suppose that V_1 is pseudoconvex and V_2 is pseudoconcave. Let ξ_1 on ξ_2 be contact structures on V_1 and V_2 formed by complex tangencies. If (V_2, ξ_2) is fillable then (V_1, ξ_1) is fillable.

8.3.1. Let (V_2, ξ_2) be tight. Is (V_1, ξ_1) tight?

8.3.2. Let (V_1, ξ_1) be fillable or tight. Is (V_2, ξ_2) fillable or tight?

8.3.2'. Suppose (V_1, ξ_1) is tight. Can (V_2, ξ_2) be overtwisted?

8.4. *Partial order vs. Equivalence.*

The construction in 8.3 defines a partial order on the set of contact 3-manifolds. Consider an equivalence relation generated by this order.

8.4.1. Is it trivial? For example, can an overtwisted structure be equivalent in this sense to a tight one?

8.5. *Number of fillings.*

8.5.1. Is it true that a fillable structure can be filled only in a finite number of topologically distinct (up to blowing up) ways?

This is true for S^3 (see [E4]) and for standard contact structures on lens spaces $L(p, 1)$ (see [McD]).

8.6. *Counting a number of fillable or tight contact structures on a given 3-manifold.*

8.6.1 *Conjecture* : Any 3-manifold admits, up to isotopy, only finitely many tight contact structures.

8.6.1'. More precisely, let ξ_1 and ξ_2 be tight contact structures on a 3-manifold M such that their Euler classes $e(\xi_1)$, $e(\xi_2) \in H^2(M)$ coincide. Are ξ_1 and ξ_2 isotopic?

According to 2.2 the number of cohomology classes from $H^2(M)$, which can be represented as Euler classes of tight structures, is finite. Therefore 8.6.1$'$ would imply 8.6.1.

8.6.2. Which classes from $H^2(M)$ can be realized as Euler classes of tight contact structures?

8.6.2'. For example, can the trivial class from $H^2(L(p,1))$, for an odd p, be realized (see 2.3.1 above)?

8.6.3. Is conjecture 8.6.1 true in a stronger form : *The space of tight contact structures with fixed Euler class is contractible*? This would imply that a group of diffeomorphisms of M which leave invariant the Euler class of a tight contact structure ξ, is contractible to the subgroup of contact automorphisms of ξ.

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Yakov ELIASHBERG,
Stanford University
Department of Mathematics
Stanford, CA 94305(USA).