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# TRANSVERSELY AFFINE FOLIATIONS OF SOME SURFACE BUNDLES OVER $S^1$ OF PSEUDO-ANOSOV TYPE

by Hiromichi NAKAYAMA

## Introduction.

E. Ghys and V. Sergiescu classified codimension one foliations without compact leaves of torus bundles over $S^1$ whose monodromy matrices are hyperbolic automorphism ([2]). They cut the manifold along some fiber transverse to the foliation $\mathcal{F}$ and modified the resulting foliation $\mathcal{F}|(T^2 \times I)$ ($I = [0,1]$) so that $\mathcal{F}|(T^2 \times I)$ is tangent to each $\{*\} \times I (* \in T^2)$. Then $\mathcal{F}|(T^2 \times \{0\})$ is equal to $\mathcal{F}|(T^2 \times \{1\})$. However it is difficult to classify foliations without compact leaves of higher genus surface bundles over $S^1$ because it is difficult to find a fiber $S$ so that the singular foliation $\mathcal{F}|(S \times \{0\})$ coincides with $\mathcal{F}|(S \times \{1\})$ and to classify the foliation of $\Sigma \times I$. In this paper, we restrict our attention to transversely affine foliations without compact leaves of some higher genus surface bundles over $S^1$ of pseudo-Anosov type and obtain the following results :

MAIN THEOREM. — *Let $\Sigma$ be a closed orientable surface with genus greater than 1 and let $\pi : M \to S^1$ be an oriented $\Sigma$-bundle over $S^1$ of pseudo-Anosov type such that the real eigenvalues of its monodromy matrix are $\lambda$ and $\frac{1}{\lambda}$, and the eigenspace with respect to $\lambda$ (resp. $\frac{1}{\lambda}$) is one dimensional, where $\lambda$ ($> 1$) is the dilatation number of $M$. Suppose that $\mathcal{F}$ is a transversely oriented and transversely affine codimension one*

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*foliation of $M$ without compact leaves satisfying the Euler class equality $\chi(T\mathcal{F}) = \pm\chi(T\pi)$ ($\in H^2(M;\mathbb{Z})$), where $T\mathcal{F}$ and $T\pi$ denote the tangent bundles of the foliation $\mathcal{F}$ and the bundle foliation of $\pi$ respectively. Then there is a finite covering of $\mathcal{F}$ which is $C^0$ isotopic to a suspension foliation of a pseudo-Anosov diffeomorphism.*

PROPOSITION. — *There is an orientable $\Sigma$-bundle over $S^1$ of pseudo-Anosov type satisfying the conditions of the main theorem. (I.e. the real eigenvalues of its monodromy matrix are $\lambda$ and $\frac{1}{\lambda}$, and the eigenspace with respect to $\lambda$ (resp. $\frac{1}{\lambda}$) is one dimensional, where $\lambda$ is the dilatation number.)*

In Section 1, we give a precise definition of suspension foliations of pseudo-Anosov diffeomorphisms introduced by Meigniez [8], and prove the above proposition. For each bundle structure of pseudo-Anosov type, there exist suspension foliations of the pseudo-Anosov diffeomorphism. The hypothesis of the main theorem on the real eigenvalues of the monodromy and their eigenspaces restricts the bundle structures of $M$. S. Matsumoto showed the author examples of transversely affine foliations of $M$ which are not isotopic to the suspension foliations of pseudo-Anosov diffeomorphisms and have the same holonomy representation as the suspension foliations have ($\chi(T\mathcal{F}) \neq \pm\chi(T\pi)$), which we also describe. In Section 2, we show the existence of a finite covering $\widehat{p} : \widehat{M} \to M$ and an embedding $\widehat{g} : \Sigma \to \widehat{M}$ isotopic to a fiber of the $\Sigma$-bundle $\widehat{M}$ over $S^1$ such that $\widehat{g}^*\widehat{p}^*\mathcal{F}$ is $C^0$ isotopic to a stable or unstable foliation of a pseudo-Anosov diffeomorphism which is $C^0$ isotopic to the monodromy map of $\widehat{M}$ (Theorem 2). We prove the main theorem in Section 3.

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## 1. Pseudo-Anosov diffeomorphisms and their suspension foliations.

Let $\Sigma$ be a closed orientable surface with genus greater than 1. A *pseudo-Anosov diffeomorphism* $f : \Sigma \to \Sigma$ ([1]) is a homeomorphism with two measured foliations $(\mathcal{G}^s, \mu^s)$ and $(\mathcal{G}^u, \mu^u)$ such that $\mathcal{G}^s$ and $\mathcal{G}^u$ are mutually transverse with the same saddle singularities, $f(\mathcal{G}^s, \mu^s) =$

$(\mathcal{G}^s, \frac{1}{\lambda}\mu^s)$ $(\lambda > 1)$ and $f(\mathcal{G}^u, \mu^u) = (\mathcal{G}^u, \lambda\mu^u)$, where we adopt the definition of measured foliations written in [1] and $f$ is supposed to be a $C^\infty$ diffeomorphism except at the saddle singularities of $\mathcal{G}^s$. The measured foliation $(\mathcal{G}^s, \mu^s)$ (resp. $(\mathcal{G}^u, \mu^u)$) is called the *stable* (resp. *unstable*) *foliation* of $f$, and $\lambda$ is called the *dilatation number* of $f$.

W. Thurston showed that every diffeomorphism of $\Sigma$ is $C^0$ isotopic to a "reducible" diffeomorphism or a periodic map or a pseudo-Anosov diffeomorphism ([1], [16]), and a pseudo-Anosov diffeomorphism is $C^0$ isotopic to neither a "reducible" diffeomorphism nor a periodic map.

Throughout this paper, we assume that $\mathcal{G}^\sigma$ $(\sigma = s, u)$ is transversely oriented and $f$ preserves the transverse orientation of $\mathcal{G}^\sigma$. In particular, the number of separatrices passing through each saddle singularity is an even number.

A surface bundle $M$ over $S^1$ is of *pseudo-Anosov type* if its monodromy map is $C^0$ isotopic to a pseudo-Anosov diffeomorphism. The *dilatation number* $\lambda$ of $M$ is defined by that of the pseudo-Anosov diffeomorphism. By the arguments of Exposé 12 of [1], $\lambda$ does not depend on the choice of pseudo-Anosov diffeomorphisms $C^0$ isotopic to the monodromy map of $M$. The *monodromy matrix* of $M$ is the linear automorphism of $H_1(\Sigma)$ induced by $f$. Since we assume that $f$ preserves the transverse orientation of $\mathcal{G}^\sigma$, $\lambda$ and $\frac{1}{\lambda}$ are eigenvalues of the monodromy matrix.

Next we define suspension foliations of pseudo-Anosov diffeomorphisms. Let $M$ be an oriented $\Sigma$-bundle over $S^1$ of pseudo-Anosov type and let $f$ be a pseudo-Anosov diffeomorphism $C^0$ isotopic to the monodromy map of $M$. Denote by $(\mathcal{G}^s, \mu^s)$ and $(\mathcal{G}^u, \mu^u)$ the stable and unstable foliations of $f$ respectively, and denote by $K$ the set of saddle singularities of $\mathcal{G}^s$. Since $\mathcal{G}^\sigma$ $(\sigma = s, u)$ is transversely oriented, there exists a non-singular closed 1-form $\omega^\sigma$ of $\Sigma - K$ *defining* the measured foliation $(\mathcal{G}^\sigma, \mu^\sigma)$. (I.e. the kernel of $\omega^\sigma$ coincides with the tangent bundle of $\mathcal{G}^\sigma$ and $\int_\gamma \omega^\sigma = \mu^\sigma(\gamma)$, where $\gamma$ is a transverse arc of $\mathcal{G}^\sigma$ oriented by the transverse orientation of $\mathcal{G}^\sigma$). Let $\mathcal{H}(\sigma, \alpha, \mathcal{G}^\sigma, \mu^\sigma)$ $(\sigma = s, u,$ $\alpha \neq 0)$ denote the foliation of $(\Sigma - K) \times \mathbf{R}$ *defined* by the non-singular 1-form $\lambda^{\varepsilon(\sigma)t}\omega^\sigma + \alpha dt$ $(t \in \mathbf{R})$, where $\varepsilon(s) = 1$ and $\varepsilon(u) = -1$. (I.e. $T\mathcal{H}(\sigma, \alpha, \mathcal{G}^\sigma, \mu^\sigma) = \mathrm{Ker}(\lambda^{\varepsilon(\sigma)t}\omega^\sigma + \alpha dt)$.) The completion of $\mathcal{H}(\sigma, \alpha, \mathcal{G}^\sigma, \mu^\sigma)$ in $\Sigma \times \mathbf{R}$ is denoted by $\widehat{\mathcal{H}}(\sigma, \alpha, \mathcal{G}^\sigma, \mu^\sigma)$. For the $\mathbf{Z}$-action $\theta$ of $\Sigma \times \mathbf{R}$ given by $\theta_n(x,t) = (f^{-n}(x), t + n)$ $(n \in \mathbf{Z})$, the quotient space of $\Sigma \times \mathbf{R}$ by

$\theta$ is $C^0$ isotopic to $M$. Since $\theta_n^*(\lambda^{\varepsilon(\sigma)t}\omega^\sigma + \alpha dt) = \lambda^{\varepsilon(\sigma)t}\omega^\sigma + \alpha dt$ (here $f^*\omega^\sigma = \lambda^{\varepsilon(\sigma)}\omega^\sigma$), $\widehat{\mathcal{H}}(\sigma,\alpha,\mathcal{G}^\sigma,\mu^\sigma)/\theta$ is a transversely orientable minimal $C^0$ foliation of $M$ with holonomy (having a locally dense resilient leaf [4]), denoted by $\mathcal{F}(\sigma,\alpha,\mathcal{G}^\sigma,\mu^\sigma,f)$.

PROPOSITION. — *Let $f$ and $\overline{f}$ be pseudo-Anosov diffeomorphisms $C^0$ isotopic to the monodromy map of $M$, and let $(\mathcal{G}^\sigma,\mu^\sigma)$ and $(\overline{\mathcal{G}}^\sigma,\overline{\mu}^\sigma)$ be the (un-)stable foliations of $f$ and $\overline{f}$ respectively ($\sigma = s,u$). Then $\mathcal{F}(\sigma,\alpha,\overline{\mathcal{G}}^\sigma,\overline{\mu}^\sigma,\overline{f})$ is $C^0$ isotopic to $\mathcal{F}(\sigma,\pm 1,\mathcal{G}^\sigma,\mu^\sigma,f)$ for any non-zero number $\alpha$.*

*Proof.* — Since $f$ and $\overline{f}$ are $C^0$ isotopic pseudo-Anosov diffeomorphisms, there is a diffeomorphism $g$ of $\Sigma$ isotopic to the identity map satisfying $gf = \overline{f}g$ and $g(\mathcal{G}^\sigma,\mu^\sigma) = (\overline{\mathcal{G}}^\sigma, k\overline{\mu}^\sigma)$ ($\sigma = s,u$) for some $k > 0$ ([1], Exposé 12). Denote by $\omega^\sigma$ (resp. $\overline{\omega}^\sigma$) the closed 1-form defining $(\mathcal{G}^\sigma,\mu^\sigma)$ (resp. $(\overline{\mathcal{G}}^\sigma,\overline{\mu}^\sigma)$), which is defined except at the saddle singularities of $\mathcal{G}^\sigma$ (resp. $\overline{\mathcal{G}}^\sigma$). Then $g^*\overline{\omega}^\sigma = \pm\frac{1}{k}\,\omega^\sigma$. We define the diffeomorphism $h : \Sigma\times\mathbf{R} \to \Sigma\times\mathbf{R}$ by $h(x,t) = \left(g(x), t + \dfrac{\varepsilon(\sigma)\log(k|\alpha|)}{\log\lambda}\right)$ ($(x,t) \in \Sigma\times\mathbf{R}$). Then $h$ satisfies that

$$h^*(\lambda^{\varepsilon(\sigma)t}\overline{\omega}^\sigma + \alpha dt) = \pm|\alpha|\left(\lambda^{\varepsilon(\sigma)t}\omega^\sigma \pm (\alpha/|\alpha|)dt\right) \quad \text{and}$$
$$h\theta_n = \overline{\theta}_n h,$$

where $\theta_n(x,t) = (f^{-n}(x), t+n)$ and $\overline{\theta}_n(x,t) = \left(\overline{f}^{-n}(x), t+n\right)$ ($n \in \mathbf{Z}$). This implies that $\mathcal{F}(\sigma,\alpha,\overline{\mathcal{G}}^\sigma,\overline{\mu}^\sigma,\overline{f})$ is $C^0$ isotopic to $\mathcal{F}(\sigma,\pm 1,\mathcal{G}^\sigma,\mu^\sigma,f)$. □

We call $\mathcal{F}(\sigma,\pm 1,\mathcal{G}^\sigma,\mu^\sigma,f)$ ($\sigma = s,u$) the *suspension foliations of the pseudo-Anosov diffeomorphism* of $M$, denoted by $\mathcal{F}_\pm^\sigma$. By the above proposition, the definition of the suspension foliations of the pseudo-Anosov diffeomorphism of $M$ does not depend on the choice of pseudo-Anosov diffeomorphisms $C^0$ isotopic to the monodromy map of $M$.

Next we construct a smooth model of $\mathcal{F}_\pm^\sigma$, where $\mathcal{F}_\pm^\sigma$ is a $C^\infty$ foliation except at $(K\times\mathbf{R})/\theta$, denoted by $K'$. First we choose a small closed tubular neighborhood $V$ of $K'$ in $M$ such that $\mathcal{F}_\pm^\sigma|\partial V$ is the union of $C^\infty$ product foliations of tori whose leaves are isotopic to $\partial V \cap \left((\Sigma\times\{t\})/\theta\right)$ ($t \in \mathbf{R}$). By attaching the copies of the product foliation $\{D^2\times\{*\}; * \in S^1\}$ of $D^2\times S^1$

to $\mathcal{F}^\sigma_\pm|(M - \operatorname{int} V)$ along the leaves of $\partial D^2 \times S^1$ and $\partial V$, we obtain a $C^\infty$ foliation of $M$, denoted by $\widetilde{\mathcal{F}}^\sigma_\pm$. The foliation $\widetilde{\mathcal{F}}^\sigma_\pm$ is $C^0$ isotopic to $\mathcal{F}^\sigma_\pm$.

The transverse orientation of $\widetilde{\mathcal{F}}^\sigma_+$ (resp. $\widetilde{\mathcal{F}}^\sigma_-$) is given by the positive orientation of $\lambda^{\varepsilon(\sigma)t}\omega^\sigma + dt$ (resp. $\lambda^{\varepsilon(\sigma)t}\omega^\sigma - dt$). Then the Euler class $\chi(T\widetilde{\mathcal{F}}^\sigma_+)$ (resp. $\chi(T\widetilde{\mathcal{F}}^\sigma_-)$) is equal to $\chi(T\pi)$ (resp. $-\chi(T\pi)$). By using this fact and Seke's theorem ([12]), Meigniez ([8]) showed that $\widetilde{\mathcal{F}}^\sigma_+$ is not isotopic to $\widetilde{\mathcal{F}}^\sigma_-$.

We say that a transversely orientable codimension one foliation $\mathcal{F}$ is *transversely affine* if there exists a system of transition functions consisting of elements of $\mathrm{Aff}^+\mathbf{R} = \{x \mapsto ax + b; a > 0\}$. By Seke's theorem ([12]), transversely affine structures are characterized by the pairs $(\omega, \omega_1)$ of 1-forms of $M$ such that

1) $\omega$ defines the foliation $\mathcal{F}$,

(i.e. the tangent bundle of $\mathcal{F}$ coincides with $\ker \omega$.)

2) $d\omega = \omega \wedge \omega_1$,

3) $d\omega_1 = 0$,

modulo the identifications $(\omega, \omega_1) \sim (g\omega, \omega_1 - \dfrac{dg}{g})$ where $g$ is a non-zero function of $M$.

For example, $\widetilde{\mathcal{F}}^\sigma_\pm$ is a transversely affine foliation. In fact, $\widetilde{\mathcal{F}}^\sigma_\pm|(M - \operatorname{int} V)$ has the transversely affine structure $(\lambda^{\varepsilon(\sigma)t}\omega^\sigma \pm dt, -\varepsilon(\sigma) \log \lambda \cdot dt)$, and this transversely affine structure extends to $M$.

Next we define the holonomy representation of a transversely affine foliation $\mathcal{F}$. Let $x_0$ denote the base point of $M$ and let $p : (\widetilde{M}, \widetilde{x}_0) \to (M, x_0)$ be a universal covering of $M$ with the base point $\widetilde{x}_0$ ($p(\widetilde{x}_0) = x_0$). Then there exist two functions $k : (\widetilde{M}, \widetilde{x}_0) \to (\mathbf{R}, 0)$ and $h : (\widetilde{M}, \widetilde{x}_0) \to (\mathbf{R}^*_+, 1)$ ($\mathbf{R}^*_+ = \{t > 0\}$) satisfying $p^*(\omega, \omega_1) = \left(\dfrac{dk}{h}, \dfrac{dh}{h}\right)$ ([12]). For each element $\gamma \in \pi_1(M, x_0)$, there is an element $(a, b) \in \mathbf{R}^*_+ \times \mathbf{R}$ such that $k \cdot \gamma = ak + b$ and $h \cdot \gamma = ah$. We define the *holonomy representation* $\mathrm{hol}_{\mathcal{F}} : \pi_1(M, x_0) \to \mathrm{Aff}^+\mathbf{R}$ of $\mathcal{F}$ by $\mathrm{hol}_{\mathcal{F}}(\gamma) = (x \mapsto ax + b)$. The holonomy representation is uniquely determined up to an inner automorphism of $\mathrm{Aff}\,\mathbf{R}(= \{x \mapsto ax + b; a \neq 0\})$.

For example, the holonomy representation of $\widetilde{\mathcal{F}}^\sigma_\pm$ is as follows (up to an inner automorphism of $\mathrm{Aff}\,\mathbf{R}$). Let $\beta$ be a section of $\pi : M \to S^1$ passing through the base point $x_0$ and oriented by the positive orientation of $S^1$.

Then $\mathrm{hol}_{\widetilde{\mathcal{F}}^\sigma_\pm}([\beta])$ is equal to $(x \mapsto \lambda^{-\varepsilon(\sigma)}x)$. Let $\iota : \Sigma \to M$ denote the inclusion map of the fiber passing through $x_0$ and let $y_0 = \iota^{-1}(x_0)$. Then $\mathrm{hol}_{\widetilde{\mathcal{F}}^\sigma_\pm}(\iota_*\pi_1(\Sigma, y_0))$ is contained in the group of translations $\{x \mapsto x + b\}$, identified with $\mathbf{R}$, and $[\mathrm{hol}_{\widetilde{\mathcal{F}}^\sigma_\pm} \cdot \iota_*]$ $(\in H^1(\Sigma; \mathbf{R}))$ is cohomologous to $[\mathrm{Per}_\mu \sigma]$, where $\mathrm{Per}_\mu\sigma : \pi_1(\Sigma, y_0) \to \mathbf{R}$ is defined by $\mathrm{Per}_\mu\sigma(\gamma) = \displaystyle\int_\gamma \omega^\sigma$.

S. Matsumoto constructed examples of transversely affine foliations of $M$ which are not isotopic to the suspension foliations of the pseudo-Anosov diffeomorphisms.

THEOREM (S. Matsumoto). — *Let $\Sigma$ be a closed orientable surface with genus greater than 1 and let $\pi : M \to S^1$ be an orientable $\Sigma$-bundle over $S^1$ of pseudo-Anosov type such that the saddle singularities of the (un-)stable foliation $\mathcal{G}^\sigma$ $(\sigma = s, u)$ of the pseudo-Anosov diffeomorphism $f$ isotopic to the monodromy map of $M$ are the fixed points of $f$ and have 4 separatrices (4-saddle singularities). Then, for each $k \in \mathbb{Z}$ satisfying $|k| \leq -\chi(\Sigma)/2$, there exists a transversely affine foliation $\mathcal{F}^\sigma_k$ of $M$ satisfying the following conditions :*

*1) $\langle \chi(T\mathcal{F}^\sigma_k), [\Sigma] \rangle = 2k$ where $[\Sigma] \in H_2(M; \mathbb{Z})$ denotes the homology class represented by the fiber of $\pi$.*

*2) $\mathrm{hol}_{\mathcal{F}^\sigma_k}$ is equal to $\mathrm{hol}_{\widetilde{\mathcal{F}}^\sigma_\pm}$ up to an inner automorphism of* $\mathrm{Aff}\,\mathbf{R}$.

*3) $\mathcal{F}^\sigma_k$ has no compact leaves.*

*Proof.* — Let $K = \{s_1, s_2, s_3, \ldots, s_n\}$ denote the set of the saddle singularities of the (un-)stable foliation $\mathcal{G}^\sigma(\sigma = s, u)$ of $f$. The foliation of $(\Sigma - K) \times \mathbf{R}$ defined by the non-singular 1-form $\lambda^{\varepsilon(\sigma)t}\omega^\sigma$ is denoted by $\mathcal{H}^\sigma_v$. Since $\mathcal{H}^\sigma_v$ is invariant under the $\mathbb{Z}$-action $\theta$ $(\theta_n(x,t) = (f^{-n}(x), t+n)$, $n \in \mathbb{Z})$, $\mathcal{H}^\sigma_v/\theta$ is the foliation of $M - K'$ $(K' = (K \times \mathbf{R})/\theta)$, denoted by $\mathcal{F}^\sigma_v$. The transverse orientation of $\mathcal{F}^\sigma_v$ is given by the positive orientation of $\lambda^{\varepsilon(\sigma)t}\omega^\sigma$.

Denote by $\sigma^i_j$ $(j = 1, 2, 3, 4)$ the separatrices of $\mathcal{G}^\sigma$ passing through the saddle singularity $s_i (1 \leq i \leq n)$. To simplify the explanation, we assume that $f(\sigma^i_j) = \sigma^i_j$ $(1 \leq j \leq n,\ 1 \leq i \leq 4)$.

The leaf $(\sigma^i_j \times \mathbf{R})/\theta$ of $\mathcal{F}^\sigma_v$ is diffeomorphic to $S^1 \times \mathbf{R}$ and has holonomy. Hence there exists a small closed tubular neighborhood $V_i$ of $(\{s_i\} \times \mathbf{R})/\theta$ in $M$ such that $\partial V_i$ is transverse to $\mathcal{F}^\sigma_v$ and $\mathcal{F}^\sigma_v|\partial V_i$ consists of four 2-dimensional Reeb components (Fig. 1).

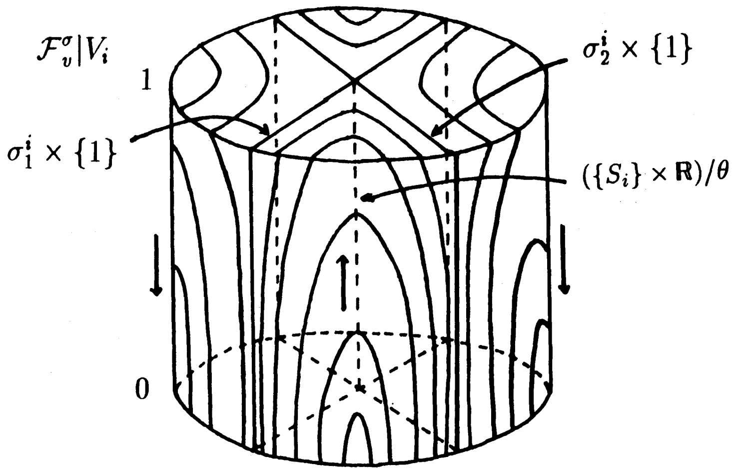


Figure 1

Next we construct two transversely oriented foliations $\mathcal{K}_+$ and $\mathcal{K}_-$ of $S^1 \times D^2$ satisfying the following conditions (Fig. 2) :

1) $\mathcal{K}_\pm | (S^1 \times \partial D^2)$ is isotopic to $\mathcal{F}_v^\sigma | \partial V_i$ with the same transverse orientation.

2) $\mathcal{K}_\pm$ has two annular leaves tangent to $S^1 \times \{*\}$ $(* \in D^2)$, and the other leaves of $\mathcal{K}_\pm$ are transverse to $S^1 \times \{*\}$ (any $* \in D^2$).

3) The transverse orientation of $S^1 \times \{0\}$ $(0 \in D^2)$ induced by the transverse orientation of $\mathcal{K}_+$ (resp. $\mathcal{K}_-$) coincides with the positive (resp. negative) orientation of $S^1$.

($\mathcal{K}_\pm$ consists of two plus half Reeb components [14] and one dead-end component of $D^1 \times S^1 \times S^1$.)

By attaching $\mathcal{F}_v^\sigma | (M - \bigcup_{i=1}^n \operatorname{int} V_i)$ with $k - \dfrac{\chi(\Sigma)}{2}$ copies of $\mathcal{K}_+$ and $-k - \dfrac{\chi(\Sigma)}{2}$ copies of $\mathcal{K}_-$ along the leaves of $\mathcal{F}_v^\sigma | (\bigcup_{i=1}^n \partial V_i)$, $\partial \mathcal{K}_+$ and $\partial \mathcal{K}_-$, we obtain a transversely orientable $C^\infty$ foliation of $M$, denoted by $\mathcal{F}_k^\sigma$. By Thurston's proposition of [15], $\langle \chi(T\mathcal{F}_k^\sigma), [\Sigma] \rangle = 2k$. Furthermore, $\mathcal{F}_k^\sigma$ has no compact leaves, because all the leaves of $\mathcal{F}_v^\sigma | (M - \bigcup_{i=1}^n \operatorname{int} V_i)$ are non-compact.

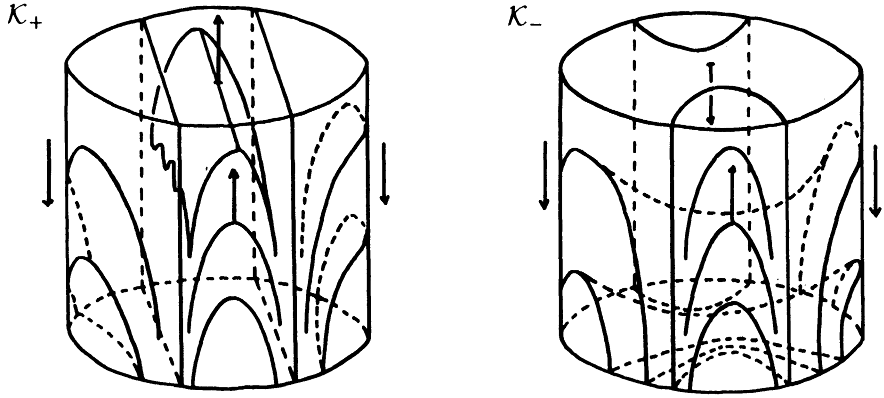


Figure 2

The transversely affine structure of $\mathcal{F}_k^\sigma$ is given as follows. First we define the transversely affine structure of $\mathcal{F}_v^\sigma|(M - \bigcup_{i=1}^n \operatorname{int} V_i)$ by $(\lambda^{\varepsilon(\sigma)t}\omega^\sigma, -\varepsilon(\sigma)\log\lambda \cdot dt)$. The foliation $\mathcal{K}_\pm$ also has a transversely affine structure. By Seke's theorem ([12]), which shows the uniqueness of the transversely affine structure of a foliation with holonomy, the transversely affine structures of $\mathcal{F}_v^\sigma|(\bigcup_{i=1}^n \partial V_i)$ and $\partial\mathcal{K}_\pm$ are unique. Therefore the transversely affine structure of $\mathcal{K}_\pm$ can be attached to that of $\mathcal{F}_v^\sigma|(M - \bigcup_{i=1}^n \operatorname{int} V_i)$. For this transversely affine structure of $\mathcal{F}_k^\sigma$, the holonomy representation is equal to $\operatorname{hol}_{\widetilde{\mathcal{F}}_\pm^\sigma}$ up to an inner automorphism of $\operatorname{Aff}\mathbb{R}$. □

*Remark.* — If $2k \neq \pm\chi(\Sigma)$, then $\mathcal{F}_k^\sigma$ is not homotopic to $\widetilde{\mathcal{F}}_\pm^\sigma$. Therefore $\mathcal{F}_k^\sigma$ is not isotopic to $\widetilde{\mathcal{F}}_\pm^\sigma$.

In the end of this section, we prove the proposition in the introduction.

*Proof of Proposition.* — Let $f$ denote the hyperbolic automorphism of the torus $T^2$ given by the $2 \times 2$ matrix $\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^2$. Then the fixed points of $f$ are $[(0,0)]$, $\left[\left(\frac{1}{5}, \frac{2}{5}\right)\right]$, $\left[\left(\frac{2}{5}, \frac{4}{5}\right)\right]$, $\left[\left(\frac{3}{5}, \frac{1}{5}\right)\right]$ and $\left[\left(\frac{4}{5}, \frac{3}{5}\right)\right]$, where $T^2$ is identified with the quotient of $\mathbb{R}^2$ by the integer lattice and the element of $T^2$ represented by $z \in \mathbb{R}^2$ is denoted by $[z]$. Let $K$ denote the set $\left\{\left[\left(\frac{1}{5}, \frac{2}{5}\right)\right], \left[\left(\frac{4}{5}, \frac{3}{5}\right)\right]\right\}$ and let $\alpha$, $\beta$ and $\varepsilon$ denote the generators of $\pi_1(T^2 - K)$ where $\alpha$, $\beta$ and $\varepsilon$ are represented by $([0,1] \times \{0\})/_\sim, (\{0\} \times [0,1])/_\sim$ and a loop winding around $\left[\left(\frac{1}{5}, \frac{2}{5}\right)\right]$, respectively.

Let $S_1$ and $S_2$ denote two copies of $T^2 - \left\{[(t, 2t)]; -\frac{1}{5} \leq t \leq \frac{1}{5}\right\}$. By attaching $S_1$ to $S_2$ along $\left\{[(t, 2t)]; -\frac{1}{5} < t < \frac{1}{5}\right\}$ alternatively, we obtain a double covering $p : \mathring{\Sigma}_2 \to T^2 - K$, where $\mathring{\Sigma}_2$ is a 2-punctured surface with genus 2. Let $\eta : \pi_1(T^2 - K) \to \mathbb{Z}/2\mathbb{Z}$ denote the homomorphism satisfying $\eta(\alpha) = \eta(\beta) = \eta(\varepsilon) = 1$. Then $p_*\pi_1(\mathring{\Sigma}_2) = \operatorname{Ker}\eta$. Since $\eta f_*([\alpha]) = \eta f_*([\beta]) = \eta f_*([\varepsilon]) = 1$, there is a lift $f'$ of $f$.

By collapsing two holes of $\mathring{\Sigma}_2$, $f'$ extends to a homeomorphism $f''$ of the closed orientable surface $\Sigma_2$ with genus 2, which is a pseudo-Anosov diffeomorphism ([1], Exposé 13). We take two lifts of $\left\{\left[\left(t, \frac{1}{2}\right)\right]; 0 \leq t \leq 1\right\}$ and $\left\{\left[\left(\frac{1}{2}, t\right)\right]; 0 \leq t \leq 1\right\}$ as the generators of $H_1(\Sigma_2)$. Since $f$ maps $\left\{\left[\left(t, \frac{1}{2}\right)\right]; 0 \leq t \leq 1\right\}$ (resp. $\left\{\left[\left(\frac{1}{2}, t\right)\right]; 0 \leq t \leq 1\right\}$) on $\left\{\left[\left(5t + \frac{3}{2}, 3t + 1\right)\right]; 0 \leq t \leq 1\right\}$ (resp. $\left\{\left[\left(3t + \frac{5}{2}, 2t + \frac{3}{2}\right)\right]; 0 \leq t \leq 1\right\}$) which intersects $\left\{[(t, 2t)]; -\frac{1}{5} < t < \frac{1}{5}\right\}$ two times, the isomorphism of $H_1(\Sigma_2; \mathbb{Z})$ induced by $f$" is represented by the $4 \times 4$ matrix $\begin{pmatrix} 2 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 3 & 1 & 2 & 2 \\ 2 & 1 & 1 & 1 \end{pmatrix}$, whose eigenvalues are $\frac{7 \pm 3\sqrt{5}}{2}$ and $\frac{-1 \pm \sqrt{-3}}{2}$. Therefore the $\Sigma_2$-bundle over $S^1$ whose monodromy map is $C^0$ isotopic to $f''$ satisfies the conditions of the main theorem. $\square$

## 2. An embedded surface with the (un-)stable foliation.

The purpose of this section is to prove the existence of a finite covering of $\mathcal{F}$ whose restriction to a fiber is $C^0$ isotopic to an (un-)stable foliation of a pseudo-Anosov diffeomorphism (Theorem 2). First we show the following theorem.

THEOREM 1. — *Let $\pi : M \to S^1$ be as in the main theorem. If $\mathcal{F}$ is a transversely oriented and transversely affine foliation of $M$ without compact leaves, then the holonomy representation of $\mathcal{F}$ is equal to $\mathrm{hol}_{\tilde{\mathcal{F}}^s_{\pm}}$ or $\mathrm{hol}_{\tilde{\mathcal{F}}^u_{\pm}}$ up to an inner automorphism of $\mathrm{Aff}\,\mathbf{R}$, where $\mathrm{hol}_{\tilde{\mathcal{F}}^\sigma_{\pm}}$ $(\sigma = s, u)$ is the holonomy representation of the suspension foliation of the pseudo-Anosov diffeomorphism defined in Section 1.*

*Proof.* — We define homomorphisms $u : \mathbf{R} \to \mathrm{Aff}^+\mathbf{R}$ by $u(b) = (x \mapsto x + b)$ and $v : \mathrm{Aff}^+\mathbf{R} \to \mathbf{R}^*_+$ by $v(x \mapsto ax + b) = a$. Then the sequence $0 \to \mathbf{R} \xrightarrow{u} \mathrm{Aff}^+\mathbf{R} \xrightarrow{v} \mathbf{R}^*_+ \to 1$ is an exact sequence ([8]).

Let $\iota : \Sigma \to M$ be the inclusion map of a fiber, and let $f : \Sigma \to \Sigma$ be a monodromy map of $M$ according to $\iota$. (I.e. there is a diffeomorphism $\phi : (\Sigma \times I)/((x,1) \sim (f(x),0)) \to M$ $(I = [0,1])$ such that $\phi|(\Sigma \times \{0\}) = \iota$). Choose a fixed point $y_0$ of $f$, and the base point of $M$ is given by $\iota(y_0)$. Let $\ell$ denote the loop $\phi(\{y_0\} \times I)$ of $M$ oriented by the positive orientation of $\{y_0\} \times I$, let $\beta$ denote the element of $\pi_1(M, \iota(y_0))$ represented by $\ell$. Then $\iota_* f_* \gamma = \beta^{-1}(\iota_*\gamma)\beta$ for any $\gamma \in \pi_1(\Sigma, y_0)$.

For the homomorphism $\log\cdot v \cdot \mathrm{hol}_{\mathcal{F}} \cdot \iota_* : \pi_1(\Sigma, y_0) \to \mathbf{R}$, the following equation holds for any $\gamma \in \pi_1(\Sigma, y_0)$ :

$$\begin{aligned}
&\log\cdot v \cdot \mathrm{hol}_{\mathcal{F}} \cdot \iota_*(f_*\gamma) \\
&= \log\cdot v \cdot \mathrm{hol}_{\mathcal{F}}(\beta^{-1}(\iota_*\gamma)\beta) \\
&= \log\cdot v \cdot \mathrm{hol}_{\mathcal{F}}(\beta) + \log\cdot v \cdot \mathrm{hol}_{\mathcal{F}}(\iota_*\gamma) + \log\cdot v \cdot \mathrm{hol}_{\mathcal{F}}(\beta^{-1}) \\
&= \log\cdot v \cdot \mathrm{hol}_{\mathcal{F}} \cdot \iota_*(\gamma).
\end{aligned}$$

This shows that the cohomology class $[\log\cdot v \cdot \mathrm{hol}_{\mathcal{F}} \cdot \iota_*]$ $(\in H^1(\Sigma; \mathbf{R}))$ is a fixed point of $f^\# : H^1(\Sigma; \mathbf{R}) \to H^1(\Sigma; \mathbf{R})$. Since $f_\# : H_1(\Sigma; \mathbf{Z}) \to H_1(\Sigma; \mathbf{Z})$ has no eigenvalue equal to 1, $[\log \cdot v \cdot \mathrm{hol}_{\mathcal{F}} \cdot \iota_*] = 0$ in $H^1(\Sigma; \mathbf{R})$, and $v \cdot \mathrm{hol}_{\mathcal{F}} \cdot \iota_*(\pi_1(\Sigma, y_0)) = \{1\}$. Thus the following commutative diagram exists :

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(\Sigma, y_0) & \underset{\iota_*}{\longrightarrow} & \pi_1(M, \iota(y_0)) & \underset{\pi_*}{\longrightarrow} & \pi_1(S^1) & \longrightarrow & 1 \\
 & & \Big\downarrow H_N & & \Big\downarrow \mathrm{hol}_{\mathcal{F}} & & \Big\downarrow H_L & & \\
1 & \longrightarrow & \mathbf{R} & \underset{u}{\longrightarrow} & \mathrm{Aff}^+\mathbf{R} & \underset{v}{\longrightarrow} & \mathbf{R}^*_+ & \longrightarrow & 1
\end{array}$$

where the upper sequence is the homotopy exact sequence of the fibration $\pi$. For the cohomology class $[H_N]$ represented by $H_N$, the following equation holds for any element $\gamma \in \pi_1(\Sigma, y_0)$ :

$$\begin{aligned}
&f^{\#}[H_N](\gamma) \\
&\quad = u^{-1}\mathrm{hol}_{\mathcal{F}}\ \iota_*(f_*\gamma) \\
&\quad = u^{-1}\mathrm{hol}_{\mathcal{F}}(\beta^{-1}(\iota_*\gamma)\beta) \\
&\quad = u^{-1}(x \mapsto x + ce)
\end{aligned}$$

where $\mathrm{hol}_{\mathcal{F}}(\beta) = \left(x \mapsto \frac{1}{c}x + d\right)$ and $\mathrm{hol}_{\mathcal{F}}(\iota_*\gamma) = (x \mapsto x + e)$

$$\begin{aligned}
&= cu^{-1}(\mathrm{hol}_{\mathcal{F}}(\iota_*\gamma)) \\
&= c[H_N](\gamma).
\end{aligned}$$

First assume that $[H_N] \neq 0$ in $H^1(\Sigma; \mathbf{R})$. Then $c$ is an eigenvalue of $f^{\#}$ and $[H_N]$ is an eigenvector with respect to $c$. By the conditions of the monodromy matrix, $c$ is equal to $\lambda$ or $\frac{1}{\lambda}$. Since the cohomology class $[\mathrm{Per}_\mu s]$ (resp. $[\mathrm{Per}_\mu u]$) is also an eigenvector of $f^{\#}$ with respect to $\lambda$ (resp. $\frac{1}{\lambda}$), there is a non-zero number $c'$ such that $[H_N] = c'[\mathrm{Per}_\mu s]$ (resp. $[H_N] = c'[\mathrm{Per}_\mu u]$) if $c = \lambda$ (resp. $c = \frac{1}{\lambda}$). Therefore $\mathrm{hol}_{\mathcal{F}}$ is equal to $\mathrm{hol}_{\tilde{\mathcal{F}}^s_\pm}$ or $\mathrm{hol}_{\tilde{\mathcal{F}}^u_\pm}$ up to an inner automorphism of $\mathrm{Aff}\,\mathbf{R}$.

If $[H_N] = 0$, then $\mathrm{hol}_{\mathcal{F}}\pi_1(M, \iota(y_0))$ is an abelian subgroup. Such transversely affine foliations were studied in [12], [17]. Since $\mathcal{F}$ has no compact leaves, $\mathcal{F}$ has no holonomy and $\mathcal{F}$ is defined by a non-singular closed 1-form ([12], Theorem 7, 8). The cohomology class of this closed 1-form is $\pi^*(c''[dt])$ for some non-zero number $c''$ where $[dt]$ is the generator of $H^1(S^1; \mathbf{Z})$. By the theorem ([6]) of Laudenbach-Blank in a weak form, $\mathcal{F}$ is isotopic to a bundle foliation (the referee showed the author the existence of direct proofs). This contradicts the non-existence of compact leaves of $\mathcal{F}$. □

THEOREM 2. — *Let $\pi : M \to S^1$ be an oriented $\Sigma$-bundle over $S^1$ of pseudo-Anosov type. If $\mathcal{F}$ is a transversely oriented and transversely affine foliation of $M$ without compact leaves such that $\chi(T\mathcal{F}) = \pm\chi(T\pi)$ and the holonomy representation of $\mathcal{F}$ is equal to $\mathrm{hol}_{\tilde{\mathcal{F}}^s_\pm}$ (resp. $\mathrm{hol}_{\tilde{\mathcal{F}}^u_\pm}$) up to an inner automorphism of $\mathrm{Aff}\,\mathbf{R}$, then there exists a finite covering $\widehat{p} : \widehat{M} \to M$ and an embedding $\widehat{g} : \Sigma \to \widehat{M}$ isotopic to a fiber of the $\Sigma$-bundle $\widehat{M}$ over $S^1$ such that $\widehat{g}^*\widehat{p}^*\mathcal{F}$ is $C^0$ isotopic to the stable (resp. unstable) foliation of a pseudo-Anosov diffeomorphism which is $C^0$ isotopic to the monodromy map of $\widehat{M}$.*

The holonomy representation $\mathrm{hol}_{\mathcal{F}}$ satisfies that either $v{\cdot}\mathrm{hol}_{\mathcal{F}}(\beta) = \dfrac{1}{\lambda}$ and $[H_N] = c[\mathrm{Per}_\mu s]$ $(c \neq 0)$ or $v{\cdot}\mathrm{hol}_{\mathcal{F}}(\beta) = \lambda$ and $[H_N] = c[\mathrm{Per}_\mu u]$ $(c \neq 0)$. To simplify the following proof of Theorem 2, we assume that $v{\cdot}\mathrm{hol}_{\mathcal{F}}(\beta) = \lambda$ and $[H_N] = c[\mathrm{Per}_\mu u]$.

By the Roussarie's lemma ([11], [9]), there exists an embedding $g : \Sigma \to M$ isotopic to a fiber of $M$ such that $g^*\mathcal{F}$ is a singular foliation with 4-saddle singularities, which are saddle singularities with four separatrices.

Let $f : \Sigma \to \Sigma$ be a monodromy map of $M$ with respect to $g(\Sigma)$. (I.e. there exists a diffeomorphism $\phi : (\Sigma \times I)/((x,1) \sim (f(x),0)) \to M$ satisfying $\phi|(\Sigma\times\{0\}) = g$.) We define the infinite cyclic covering $q : N \to M$ $(N = \Sigma\times\mathbf{R})$ by $q(x,t) = \phi(f^i(x), t-i)$ $(i \le t \le i+1, i \in \mathbf{Z})$. In the following, we give the base point $\overline{x}_0$ of $N$ by $(y_0, 0)$ where $y_0$ is a fixed point of $f$, and the base point $x_0$ of $M$ by $g(y_0)$. The holonomy representation does not depend on the choice of the base points up to inner automorphisms.

Let $r : (\widetilde{M}, \widetilde{x}_0) \to (N, \overline{x}_0)$ be a universal covering of $N$ with the base point and let $p = q \cdot r$. For the transversely affine structure $(\omega, \omega_1)$ of $\mathcal{F}$, there are two functions $h : (\widetilde{M}, \widetilde{x}_0) \to (\mathbf{R}^*_+, 1)$ and $k : (\widetilde{M}, \widetilde{x}_0) \to (\mathbf{R}, 0)$ such that $p^*(\omega, \omega_1) = \left(\dfrac{dk}{h}, \dfrac{dh}{h}\right)$.

In order to prove Theorem 2, we need the following lemmas.

LEMMA 1. — *$q^*\mathcal{F}$ is defined by a non-singular closed 1-form. Especially $g^*\mathcal{F}\Big(= (q|\Sigma \times \{0\})^*\mathcal{F}\Big)$ is defined by a closed 1-form.*

*Proof.* — For each element $\gamma \in \pi_1(N, \overline{x}_0)$, $q_*\gamma \in \pi_1(M, x_0)$ is homotopic to an element of $g_*\pi_1(\Sigma, y_0)$. Hence $\mathrm{hol}_{\mathcal{F}}(q_*\gamma)$ is a translation, and $h \cdot q_*\gamma(x) = h(x)$ $(x \in \widetilde{M})$ by the definition of the holonomy representation. For any elements $z_1$ and $z_2$ ($\in \widetilde{M}$), $h(z_1) = h(z_2)$ if $r(z_1) = r(z_2)$.

We define $s : (N, \overline{x}_0) \to (\mathbb{R}_+^*, 1)$ by $s = h \cdot r^{-1}$. Since $r^*(q^*\omega_1 - \frac{ds}{s}) = p^*\omega_1 - \frac{d(s \cdot r)}{s \cdot r} = 0$, $q^*\omega_1$ is equal to $\frac{ds}{s}$. Hence $d(sq^*\omega) = ds \wedge q^*\omega + sdq^*\omega = 0$. Therefore $q^*\mathcal{F}$ is defined by the non-singular closed 1-form $sq^*\omega$. □

In the following, the non-singular closed 1-form $sq^*\omega$ is denoted by $\Omega$, which defines $q^*\mathcal{F}$.

LEMMA 2. — *There exists a non-singular vector field $X$ of $M$ transverse to both $\mathcal{F}$ and $g(\Sigma)$.*

*Proof.* — Let $s_i$ ($1 \leq i \leq n$) denote the saddle singularities of $\mathcal{F}|g(\Sigma)$. Then there exists a non-singular vector field $X$ of $M$ and pairwise disjoint small neighborhoods $U_i$ of $s_i$ contained in $g(\Sigma)$ such that $X$ is transverse to $\mathcal{F}$ and tangent to $g(\Sigma) - \bigcup_{i=1}^{n} U_i$.

The saddle singularity $s_i$ is called *positive* (resp. *negative*) if the orientation of $X$ at $s_i$ is equal to the positive (resp. negative) orientation of the base space $S^1$. Let $I_+$ (resp. $I_-$) denote the number of positive (resp. negative) saddle singularities. By Thurston's lemma ([15]), the following equations hold :

1) $-I_+ + I_- = \langle \chi(T\mathcal{F}), [g(\Sigma)] \rangle$,

2) $-I_+ - I_- = \chi(\Sigma)$,

where $\chi(T\mathcal{F}) \in H^2(M; \mathbb{Z})$ denotes the Euler class of the tangent bundle of $\mathcal{F}$, and $[g(\Sigma)]$ denotes the element of $H_2(M; \mathbb{Z})$ represented by $g(\Sigma)$. Since $\chi(T\mathcal{F}) = \pm\chi(T\pi)$, either $I_+$ or $I_-$ is equal to 0. Hence the saddle singularities of $\mathcal{F}|g(\Sigma)$ are all negative or all positive. If all the saddle singularities of $\mathcal{F}|g(\Sigma)$ are positive (resp. negative), then we can perturb $X$ toward the positive (resp. negative) direction of the base space $S^1$ in a neighborhood of $g(\Sigma)$ so that $X$ is transverse to both $\mathcal{F}$ and $g(\Sigma)$. □

LEMMA 3. — *There exists an embedding $\Gamma : \Sigma \times \mathbb{R}_+ \to N$ such that $\Gamma(\Sigma \times \{0\}) = \Sigma \times \{0\}$, $\Gamma(\Sigma \times \mathbb{R}_+) \subset \Sigma \times \mathbb{R}_+$ and $\Gamma^*\Omega = \iota_0^*\Omega \pm dt$, where the inclusion map $\iota_t : \Sigma \to N$ ($t \in \mathbb{R}$) is defined by $\iota_t(x) = (x, t)$.*

*Proof.* — Let $\widetilde{X}$ denote the lift of $X$ with respect to $q$. Then there is a non-singular vector field $Y$ of $N$ such that $\Omega(Y) = \pm 1$, $Y = u\widetilde{X}$ for some non-zero function $u$ of $N$, and the orientation of $Y$ at $\Sigma \times \{0\}$ coincides with the positive orientation of $\{*\} \times \mathbf{R}$ ($* \in \Sigma$). The integral manifolds of $Y$ are called the *leaves* of $Y$, which are to be oriented by $Y$.

Let $z$ be an element of $N$. Denote by $L$ the leaf of $Y$ passing through $z$. The point $w$ of $L$ satisfying $\int_z^w \Omega|L = \Omega(Y)t$ ($t \in \mathbf{R}$) is denoted by $\psi(z,t)$. Then $\psi$ is the flow of $Y$ because $\Omega\left(\frac{\partial\psi}{\partial t}\right) = \frac{d}{dt}\left(\int_0^t \Omega\left(\frac{\partial\psi}{\partial t}\right)dt\right) = \frac{d}{dt}(\Omega(Y)t) = \Omega(Y)$. Note that $\psi$ is not always defined in the whole $N \times \mathbf{R}$. However $\psi$ is defined on $(\Sigma\times\{0\})\times\mathbf{R}_+$, which will be shown in the following.

Let $L(x)$ denote the leaf of $Y$ passing through $(x,0) \in \Sigma \times \{0\} \subset N$, and let $L_i(x) = L(x) \cap (\Sigma \times [i, i+1])$ and $L_+(x) = L(x) \cap (\Sigma \times [0,\infty))$.

When $L_+(x)$ is contained in $\Sigma \times [0, n_0)$ for some integer $n_0$ ($> 0$), $\psi$ is defined on $(x,0) \times \mathbf{R}_+$ because $\psi|(\Sigma \times [0,n_0])$ is the flow of the compact manifold $\Sigma \times [0,n_0]$ transverse to the boundary.

Suppose that $L_+(x)$ is not contained in a compact region. Then $L_i(x)$ is not empty for every $i \geq 0$ ($i \in \mathbf{Z}$). Let $\ell$ denote $\min_{y\in\Sigma} \Omega(Y)\left(\int_{L_0(y)} \Omega\right) > 0$. $\ell$ is the shortest time to reach $\Sigma \times \{1\}$ from $\Sigma \times \{0\}$ by the flow $\psi$. We define the covering transformation $\theta : \Sigma \times \mathbf{R} \to \Sigma \times \mathbf{R}$ of $q$ by $\theta(x,t) = (f^{-1}(x), t+1)$. Since $\theta^*\Omega = \theta^*(sq^*\omega) = (s\cdot\theta)(q\theta)^*\omega = \lambda sq^*\omega = \lambda\Omega$, $\theta^*\Omega = \lambda\Omega$. Thus the following inequality holds :

$$\Omega(Y)\int_{L_i(x)} \Omega = \Omega(Y)\int_{\theta^{-i}L_i(x)} (\theta^i)^*\Omega = \Omega(Y)\int_{L_0(\theta^{-i}(x_i,i))} \lambda^i\Omega \geq \lambda^i\ell,$$

where $\{x_i\} = L(x) \cap (\Sigma \times \{i\})$. Hence $\Omega(Y)\int_{L_+(x)} \Omega = \infty$ and $\psi$ is defined on $(x,0) \times \mathbf{R}_+$. Therefore $\psi$ is defined on $(\Sigma \times \{0\}) \times \mathbf{R}_+$.

We define an embedding $\Gamma : \Sigma \times \mathbf{R}_+ \to N$ by $\Gamma(x,t) = \psi((x,0),t)$. Then

$$\begin{aligned}
&\Gamma^*\Omega(v,a) \quad (v \in T_x\Sigma, a \in T_t\mathbf{R}_+ = \mathbf{R}) \\
&= \Gamma^*\Omega\left((\iota_t)_*v + a\left(\frac{\partial}{\partial t}\right)\right) \\
&= \iota_t^*\Gamma^*\Omega(v) + a\Omega\Gamma_*\left(\frac{\partial}{\partial t}\right)
\end{aligned}$$

$$
\begin{aligned}
&= (\Gamma \cdot \iota_t)^* \Omega(v) + a\Omega(Y) \\
&= (\psi_t \cdot \iota_0)^* \Omega(v) \pm a \quad (\psi_t(z) = \psi(z,t),\ z \in N,\ t \in \mathbf{R}) \\
&= \iota_0^* \psi_t^* \Omega(v) \pm a \\
&= \iota_0^* \Omega(v) \pm a \quad \text{(See [3], Chapter VIII, Lemma 1.1.2)} \\
&= (p_1^* \iota_0^* \Omega \pm dt)\Big((\iota_t)_* v + a\Big(\frac{\partial}{\partial t}\Big)\Big) \quad (p_1(x,t) = x) \\
&= (\iota_0^* \Omega \pm dt)(v, a).
\end{aligned}
$$

Therefore $\Gamma^* \Omega = \iota_0^* \Omega \pm dt$. □

LEMMA 4. — *There exists a non-zero number $c$ such that $\int_\gamma \iota_0^* \Omega = c[\mathrm{Per}_\mu u](\gamma)$ for any $\gamma \in \pi_1(\Sigma, y_0)$.*

*Proof.* — For any $\gamma \in \pi_1(\Sigma, y_0)$, $\mathrm{hol}_{\mathcal{F}}(g_* \gamma) = (x \mapsto x + \int_{(\iota_0)_* \gamma} \Omega)$. In fact,

$$
\begin{aligned}
&k \cdot g_* \gamma(\widetilde{x}_0) - k(\widetilde{x}_0) \\
&= \int_{\overline{g_* \gamma}} dk \quad \text{where } \overline{g_* \gamma} \text{ is the lift of } g_* \gamma \text{ with respect to } p \\
&\qquad\qquad \text{whose starting point is } \widetilde{x}_0, \\
&= \int_{\overline{g_* \gamma}} h p^* \omega \\
&= \int_{\overline{g_* \gamma}} r^*(s q^* \omega) \\
&= \int_{r_* \overline{g_* \gamma}} \Omega \\
&= \int_{(\iota_0)_* \gamma} \Omega.
\end{aligned}
$$

Since $\mathrm{hol}_{\mathcal{F}}(g_* \gamma)$ is also equal to $(x \mapsto x + c[\mathrm{Per}_\mu u](\gamma))$ for some non-zero number $c$, $\int_{(\iota_0)_* \gamma} \Omega = c[\mathrm{Per}_\mu u](\gamma)$. □

By changing the differentiable structure of $\Sigma$, there exists a closed 1-form $\widehat{\omega}^\sigma$ $(\sigma = s, u)$ of $\Sigma$ such that $\widehat{\omega}^\sigma$ defines $(\mathcal{G}^\sigma, \mu^\sigma)$ and $\widehat{\omega}\sigma = 0$ at the saddle singularities of $\mathcal{G}^\sigma$. (I.e. there is a homeomorphism $\rho$ of $\Sigma$ isotopic to the identity map such that $\rho^*(\mathcal{G}^\sigma, \mu^\sigma)$ is the measured foliation defined by $\widehat{\omega}^\sigma$.) By Lemma 4, $\int_\gamma \iota_0^* \Omega = c \int_\gamma \widehat{\omega}^u$ for any $\gamma \in \pi_1(\Sigma, y_0)$.

LEMMA 5. — *There exist embeddings $\eta_+, \eta_- : \Sigma \to \Sigma \times \mathbb{R}_+$ satisfying the following conditions :*

1) $c\widehat{\omega}^u = \eta_+^*(\iota_0^*\Omega + dt) = \eta_-^*(\iota_0^*\Omega - dt)$.

2) *$\eta_\pm(\Sigma)$ is transverse to $\{*\} \times \mathbb{R}_+$ for each $* \in \Sigma$, and $\eta_\pm$ is isotopic to $\Sigma \times \{0\}$.*

*Proof.* — By the above argument, $[\iota_0^*\Omega]$ and $[c\widehat{\omega}^u]$ are cohomologous in $H^1(\Sigma;\mathbb{R})$. Hence there is a function $\xi : \Sigma \to \mathbb{R}$ such that $\iota_0^*\Omega - c\widehat{\omega}^u = d\xi$. We define $\eta_+ : \Sigma \to \Sigma \times \mathbb{R}_+$ by $\eta_+(x) = (x, \mathrm{Max}(\xi) - \xi(x))$ and $\eta_- : \Sigma \to \Sigma \times \mathbb{R}_+$ by $\eta_-(x) = (x, \xi(x) - \mathrm{Min}(\xi))$. Then

$$\begin{aligned}
&\eta_\pm^*(p_1^*\iota_0^*\Omega \pm p_2^*dt) \quad (p_1(x,t) = x, \quad p_2(x,t) = t)\\
&= (p_1\eta_\pm)^*\iota_0^*\Omega \pm (p_2\eta_\pm)^*dt\\
&= \iota_0^*\Omega - d\xi\\
&= c\widehat{\omega}^u.
\end{aligned}$$

□

*Proof of Theorem 2.* — There exists a sufficiently large integer $m$ ($> 0$) such that $\Gamma\eta_+(\Sigma)$ and $\Gamma\eta_-(\Sigma)$ are contained in $\Sigma \times [0, m)$. Let $q' : N \to \widehat{M}$ denote the quotient map of $N$ by $\theta^m$. Denote by $\widehat{p} : \widehat{M} \to M$ the finite covering satisfying $q = \widehat{p} \cdot q'$. If $\Gamma^*\Omega = \iota_0^*\Omega + dt$ (resp. $\Gamma^*\Omega = \iota_0^*\Omega - dt$), then we define $\widehat{g} : \Sigma \to \widehat{M}$ by $q'\Gamma\eta_+$ (resp. $q'\Gamma\eta_-$). Then $\widehat{g} : \Sigma \to \widehat{M}$ is an embedding isotopic to the fiber of $\widehat{M}$. Since $\widehat{g}^*\widehat{p}^*\mathcal{F}$ is defined by $(\Gamma\eta_\pm)^*\Omega = \eta_\pm^*(\iota_0^*\Omega \pm dt) = c\widehat{\omega}^u$, $\widehat{g}^*\widehat{p}^*\mathcal{F}$ is $C^0$ isotopic to $\mathcal{G}^u$, which is an unstable foliation of a pseudo-Anosov diffeomorphism which is $C^0$ isotopic to the monodromy map $f^m$ of $\widehat{M}$. □

*Remark.* – The foliation $\mathcal{H}$ obtained by cutting $\widehat{p}^*\mathcal{F}$ along $\widehat{g}(\Sigma)$ is a $C^0$ foliation of $\Sigma \times I$ with a transverse invariant measure with full support such that $\mathcal{H}|(\Sigma \times \{0\})$ is the (un-)stable foliation of a pseudo-Anosov diffeomorphism which is $C^0$ isotopic to $f^m$. If we choose the pseudo-Anosov diffeomorphism as the monodromy map of $\widehat{M}$, then $\mathcal{H}|(\Sigma \times \{0\})$ is equal to $\mathcal{H}|(\Sigma \times \{1\})$. (Here $\mathcal{H}$ is not a foliation at the saddle singularities of $\mathcal{H}|(\Sigma \times \partial I)$ by the ordinary definition of foliations. Such foliations are called pseudo-foliations in [9]. However, in this paper, we call them also foliations.)

## 3. Foliations of $\Sigma \times I$ with transverse invariant measures.

By Theorems 1 and 2 (see also Remark of Section 2), the main theorem obviously follows from the following Theorem 3.

THEOREM 3. — *Let $\Sigma$ be a closed orientable surface with genus greater than 1. Let $f$ be a pseudo-Anosov diffeomorphism with an (un-)stable foliation $(\mathcal{G}^\sigma, \mu^\sigma)$ $(\sigma = s, u)$. Suppose that $\mathcal{H}$ is a transversely orientable $C^0$ foliation of $\Sigma \times I$ $(I = [0,1])$ satisfying the following conditions :*

1) *$\mathcal{H}$ has a transverse invariant measure $\nu$ with full support.*

2) $\mathcal{H}|(\Sigma \times \{0\}) = \mathcal{H}|(\Sigma \times \{1\}) = \mathcal{G}^\sigma$.

*Then $\mathcal{H}$ is $C^0$ isotopic to $\widehat{\mathcal{H}}(\sigma, \alpha, \mathcal{G}^\sigma, \mu^\sigma)|(\Sigma \times I)$ with the boundary fixed for some non-zero number $\alpha$, where $\widehat{\mathcal{H}}(\sigma, \alpha, \mathcal{G}^\sigma, \mu^\sigma)$ is the foliation of $\Sigma \times \mathbb{R}$ defined in Section 1.*

In order to prove Theorem 3, we need some consideration.

First we consider some properties of singular foliations of $\Sigma$. Let $\mathcal{G}$ be a singular foliation of $\Sigma$ (all the singularities of $\mathcal{G}$ are saddle ones). A leaf $L$ of $\mathcal{G}$ is called *ordinary* if $L$ is neither a saddle singularity nor a separatrix, and $\mathcal{G}$ is called *minimal* if all the leaves except for the saddle singularities are dense in $\Sigma$. The next lemma is the generalization of Levitt's pantalon decomposition theorem ([7]) to singular foliations having saddle singularities with many separatrices.

LEMMA 6. — *Let $\mathcal{G}$ be a transversely orientable minimal singular foliation of $\Sigma$. Then there exist disjoint simple closed curves $\gamma_i$ $(1 \le i \le n)$ satisfying the following conditions :*

1) *$\gamma_i$ $(1 \le i \le n)$ is transverse to $\mathcal{G}$. Denote by $S_j$ $(1 \le j \le m)$ the connected components obtained by cutting $\Sigma$ along $\bigcup_{i=1}^{n} \gamma_i$. Then,*

2) *$\mathcal{G}|S_j$ $(1 \le j \le m)$ is a singular foliation transverse to $\partial S_j$ with a unique saddle singularity whose separatrices reach $\partial S_j$.*

3) *All the ordinary leaves of $\mathcal{G}|S_j$ are properly embedded arcs which connect different boundaries of $S_j$, and there are ordinary leaves $\beta_1^j, \beta_2^j, \beta_3^j, \ldots, \beta_{p_j}^j$ which cut $S_j$ into a 2-disk.*

*Proof.* — Suppose that disjoint submanifolds $S_j$ $(1 \leq j \leq q \leq m)$ satisfying the conditions 2) and 3) of Lemma 6 are constructed. Denote by $N$ the closure of $\Sigma - \bigcup_{j=1}^{q} S_j$.

If $\mathcal{G}|N$ has no saddle singularities, then $N$ is the disjoint union of annuli, say $A_i$ $(1 \leq i \leq n)$, and each $\mathcal{G}|A_i$ is the product foliation $\left\{D^1 \times \{*\}; * \in S^1\right\}$. Denote by $\gamma_i$ one of the boundaries of $A_i$. Then $\gamma_i$'s $(1 \leq i \leq n)$ satisfy the conditions of Lemma 6.

Next suppose that $\mathcal{G}|N$ has a saddle singularity $s$. Denote by $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_{2r}$ the separatrices of $s$ in the clockwise order. Since the singular foliation $\mathcal{G}$ is minimal, $\sigma_{2k}$ $(k = 1, 2, 3, \ldots, r)$ intersects $\partial N$. Hence there exist pairwise disjoint closed transversals $\rho_k$ $(k = 1, 2, 3, \ldots, r)$ contained in the interior of $N$ and intersecting $\sigma_{2k} - \{s\}$. Let $z_k$ denote the point of $\sigma_{2k} \cap (\bigcup_{l=1}^{r} \rho_\ell)$ nearest to $s$ along $\sigma_{2k}$. The closed transversal $\rho_\ell$ containing $z_k$ is denoted by $\rho'_k$ and the restriction of $\sigma_{2k}$ to $[s, z_k]$ is denoted by $w_k$. Then there exists a sufficiently small closed neighborhood $S_{q+1}$ $(\subset \operatorname{int} N)$ of $\bigcup_{k=1}^{r} (w_k \cup \rho'_k)$ whose boundary is transverse to $\mathcal{G}$. The singular foliation $\mathcal{G}|S_{q+1}$ satisfies the conditions 2) and 3) of Lemma 6. By induction on the number of the saddle singularities of $\mathcal{G}|\bigcup_{j=1}^{q} S_j$, Lemma 6 holds. □

Next we prove the following lemmas about foliations obtained by cutting $\mathcal{H}$ along $\bigcup_{i=1}^{n} (\gamma_i \times I)$.

Let $S$ be an orientable surface with boundary. A transversely orientable $C^0$ foliation $\mathcal{U}$ of $S \times I$ having a transverse invariant measure $\nu$ with full support is called a *unit foliation* if it satisfies the following conditions :

1) $(\mathcal{U}, \nu)|(S \times \{0\})$ is a measured foliation of $S$ transverse to $\partial S$ satisfying the conditions 2) and 3) of Lemma 6.

2) $(\mathcal{U}, \nu)|(S \times \{1\}) = (\mathcal{U}, \nu)|(S \times \{0\})$.

3) $\mathcal{U}$ is transverse to $\partial S \times I$.

LEMMA 7. — *Let $(\mathcal{U}, \nu)$ be a unit foliation. Then $\mathcal{U}|(\partial S \times I)$ has no vertical leaves, where a leaf of $\mathcal{U}|(\partial S \times I)$ is called vertical if it is isotopic to $\{*\} \times I$ with $\{*\} \times \partial I$ fixed.*

*Proof.* — If $\mathcal{U}|(\partial S \times I)$ has a vertical leaf, then all the leaves of the component of $\mathcal{U}|(\partial S \times I)$ containing the vertical leaf are vertical because $\mathcal{U}$ has the transverse invariant measure $\nu$.

Let $\ell$ be a vertical leaf of $\mathcal{U}|(\partial S \times I)$ such that $\partial \ell$ is not contained in any separatrix of $\mathcal{U}|(S \times \partial I)$. Let $x_0$ (resp. $x_1$) denote the endpoint of $\ell$ contained in $\partial S \times \{0\}$ (resp. $\partial S \times \{1\}$). Denote by $\beta_{x_0}$ (resp. $\beta_{x_1}$) the ordinary leaf of $\mathcal{U}|(S \times \partial I)$ containing $x_0$ (resp. $x_1$), and denote by $y_0$ (resp. $y_1$) the other endpoint of $\beta_{x_0}$ (resp. $\beta_{x_1}$). Since $\mathcal{U}|(\partial S \times I)$ has no holonomy, $\mathcal{U}|(\partial S \times I)$ contains no interior compact leaves. Hence there exists a properly embedded arc $\alpha$ ($\subset \partial S \times I$) connecting $y_0$ and $y_1$ and isotopic to $\{*\} \times I$ ($* \in \partial S$) with $\{*\} \times \partial I$ fixed such that $\alpha$ is either transverse or tangent to $\mathcal{U}|(\partial S \times I)$.

If $\alpha$ is transverse to $\mathcal{U}|(\partial S \times I)$, then there exists a null-homotopic closed transversal near $\ell \cup \beta_{x_0} \cup \alpha \cup \beta_{x_1}$. Since this contradicts the existence of the transverse invariant measure $\nu$ with full support, $\alpha$ is tangent to $\mathcal{U}|(\partial S \times I)$.

By Roussarie's theorem ([11], see also [9] for foliations with saddle singularities in the boundary), a null-homotopic simple closed curve $\ell \cup \beta_{x_0} \cup \alpha \cup \beta_{x_1}$ bounds a leaf of $\mathcal{U}$ homeomorphic to the 2-disk $D^2$. By Reeb's global stability theorem, there exists an immersion $\psi : D^2 \times [-1, 1] \to S \times I$ satisfying the following conditions 1), 2) and 3) :

1) $\psi(D^2 \times \{t\})(t \in (-1, 1))$ is a leaf of $\mathcal{U}$.

2) $\psi|(D^2 \times (-1, 1))$ is an embedding.

3) Both $\psi(\partial D^2 \times \{1\})$ and $\psi(\partial D^2 \times \{-1\})$ contain two saddle singularities of $\mathcal{U}|(S \times \partial I)$.

By considering the transverse orientation of $\mathcal{U}|(S \times \{0\})$ in the neighborhood of the saddle singularity of $\mathcal{U}|(S \times \{0\})$, there exists a number $t_0 \in (-1, 1)$ sufficiently near 1 or $-1$ such that $\psi(D^2 \times \{t_0\})$ contains a properly embedded short arc crossing the saddle singularity of $\mathcal{U}|(S \times \{0\})$ (Fig. 3). However this contradicts the non-existence of saddle connections of $\mathcal{U}|(S \times \{0\})$.

Thus $\mathcal{U}|(\partial S \times I)$ has no vertical leaves. □

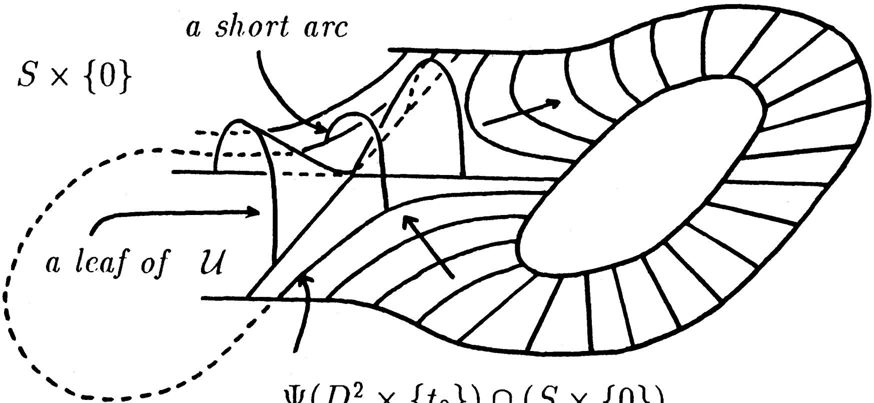


Figure 3

*Remark.* — The original proof of Roussarie's theorem demands that the foliations are of class $C^r$ ($r \geq 2$). However it has already been known that his theorem is true for $C^0$ foliations (see [3], [5], [13]).

A unit foliation $(\mathcal{U}, \nu)$ is called *normalized* if $\mathcal{U}|(\partial S \times I)$ is transverse to $\{*\} \times I$ for any $* \in \partial S$.

LEMMA 8. — *Let $(\mathcal{U}, \nu)$ be a normalized unit foliation. For any $x, y \in \partial S$, $\nu(\{x\} \times I) = \nu(\{y\} \times I)$ and the orientation of $\{x\} \times I$ induced by the transverse orientation of $\mathcal{U}$ coincides with that of $\{y\} \times I$.*

*Proof.* — If $x$ and $y$ are contained in the same connected component of $\partial S$, then $\nu(\{x\} \times I) = \nu(\{y\} \times I)$ and the orientation of $\{x\} \times I$ induced by the transverse orientation of $\mathcal{U}$ coincides with that of $\{y\} \times I$.

Let $\mathcal{G}$ denote $\mathcal{U}|(S \times \{0\})$. Suppose that an ordinary leaf $\beta$ of $\mathcal{G}$ connects $x$ and $y$ ($x, y \in \partial S$). Since $\{x\} \times I$ is homotopic to $(\beta \times \{1\}) \cup (\{y\} \times I) \cup (\beta \times \{0\})$, $\nu(\{x\} \times I)$ is equal to $\nu(\{y\} \times I)$. If the orientation of $\{x\} \times I$ induced by the transverse orientation of $\mathcal{U}$ is opposite to that of $\{y\} \times I$, then there is a null-homotopic closed transversal, which contradicts the existence of the transverse invariant measure $\nu$ with full support.

Let $\gamma$ and $\gamma'$ be connected components of $\partial S$. Denote by $\sigma$ and $\sigma'$ the separatrices of $\mathcal{G}$ intersecting $\gamma$ and $\gamma'$, respectively. Then there exists a series of separatrices $\sigma = \sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_k = \sigma'$ where $\sigma_i$ is adjacent to $\sigma_{i+1}$ for each $i$. Since there is an ordinary leaf of $\mathcal{G}$ near $\sigma_i \cup \sigma_{i+1}$ for each

$i$, $\nu(\{x\} \times I)$ $(x \in \gamma)$ is equal to $\nu(\{y\} \times I)$ $(y \in \gamma')$, and the orientation of $\{x\} \times I$ coincides with that of $\{y\} \times I$. □

LEMMA 9. — *Let $(\mathcal{U}_1, \nu_1)$ and $(\mathcal{U}_2, \nu_2)$ be normalized unit foliations of $S \times I$ satisfying $(\mathcal{U}_1, \nu_1)|\partial(S \times I) = (\mathcal{U}_2, \nu_2)|\partial(S \times I)$, then there exists a homeomorphism $h : S \times I \to S \times I$ such that $h|\partial(S \times I) = \mathrm{id}$ and $h(\mathcal{U}_1, \nu_1) = (\mathcal{U}_2, \nu_2)$.*

*Proof.* — Let $\mathcal{G}$ denote $\mathcal{U}_1|(S \times \{0\})$, and let $\beta_j$ $(1 \le j \le p)$ be the ordinary leaves of $\mathcal{G}$ which cut $S$ into a 2-disk. By Roussarie's theorem ([11]), there are pairwise disjoint properly embedded disks $D_j$ (resp. $D'_j$) transverse to $\mathcal{U}_1$ (resp. $\mathcal{U}_2$) and bounded by $\partial(\beta_j \times I)$. Since $\mathcal{U}_1|D_j$ and $\mathcal{U}_2|D'_j$ are foliations whose leaves are properly embedded arcs, there is a homeomorphism $h : \partial(S \times I) \cup \Big(\bigcup_{j=1}^{p} D_j\Big) \to \partial(S \times I) \cup \Big(\bigcup_{j=1}^{p} D'_j\Big)$ such that $h(\mathcal{U}_1, \nu_1) = (\mathcal{U}_2, \nu_2)$.

Let $\widehat{\mathcal{U}}_1$ (resp. $\widehat{\mathcal{U}}_2$) denote the foliation of $D^3$ obtained by cutting $\mathcal{U}_1$ (resp. $\mathcal{U}_2$) along $\bigcup_{j=1}^{p} D_j$ (resp. $\bigcup_{j=1}^{p} D'_j$) (Fig. 4). $\widehat{\mathcal{U}}_i$ $(i = 1, 2)$ has $2p$ collapsing leaves homeomorphic to $I$ and two saddle singularities in the boundary. The leaves of $\widehat{\mathcal{U}}_i$ near the collapsing leaves are all homeomorphic to $D^2$. By Poincaré-Bendixson's theorem, the ordinary leaves of $\partial\widehat{\mathcal{U}}_i$ are all homeomorphic to $S^1$ and the union of the leaves of $\partial\widehat{\mathcal{U}}_i$ containing a saddle singularity is a bouquet. Hence the leaves of $\widehat{\mathcal{U}}_i$ containing no saddle singularities of $\partial\widehat{\mathcal{U}}_i$ are homeomorphic to the 2-disks, and the union of the leaves of $\widehat{\mathcal{U}}_i$ containing the saddle singularity is the union of 2-disks whose intersection point is the saddle singularity. Therefore $h$ extends to a homeomorphism of $S \times I$ which satisfies the conditions of Lemma 9. □

*Proof of Theorem 3.* — Let $\gamma_i$ $(1 \le i \le n)$ denote the disjoint simple closed curves transverse to $\mathcal{G}^\sigma$ constructed by Lemma 6, and let $S_j$ $(1 \le j \le m)$ denote the connected components obtained by cutting $\Sigma$ along $\bigcup_{i=1}^{n} \gamma_i$. Since $\mathcal{H}$ has the transverse invariant measure $\nu$ with full support, $\mathcal{H}$ has no interior compact leaves. By Roussarie's theorem ([11]), $\gamma_i \times I$ can be taken by an isotopy of $\Sigma \times I$ with $\Sigma \times \partial I$ fixed so that $\gamma_i \times I$ is transverse to $\mathcal{H}$. Since all the leaves of $\mathcal{H}|(\gamma_i \times I)$ are properly embedded arcs, $\nu(\gamma_i \times \{0\})$ is equal to $\nu(\gamma_i \times \{1\})$. By the unique ergodicity of the (un-)stable foliation

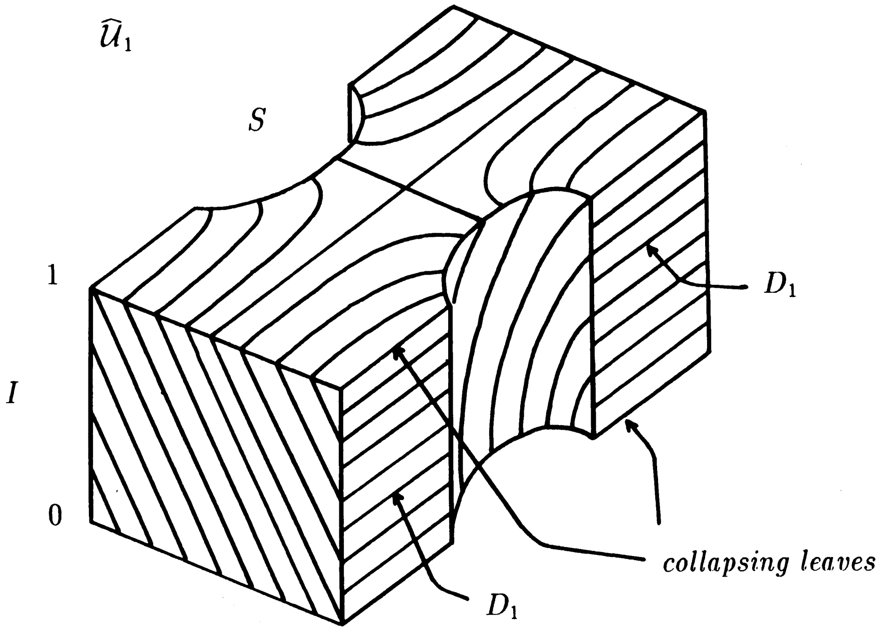


Figure 4

of the pseudo-Anosov diffeomorphism ([1]), $\nu|(\Sigma \times \{0\}) = \nu|(\Sigma \times \{1\})$. Therefore $(\mathcal{H}|(S_j \times I), \nu|(S_j \times I))$ is a unit foliation.

By Lemma 7, $\mathcal{H}|(S_j \times I)$ has no vertical leaves. We change $\Sigma \times I$ again by an isotopy with $\Sigma \times \partial I$ fixed so that $\{*\} \times I$ is transverse to $\mathcal{H}$ for any $* \in \bigcup_{i=1}^{n} \gamma_i$. Then $(\mathcal{H}|(S_j \times I), \nu|(S_j \times I))$ is a normalized unit foliation.

We take the transverse orientation of $\mathcal{H}$ so that the transverse orientation of $\mathcal{H}|(\Sigma \times \{0\})$ coincides with that of $\mathcal{G}^\sigma$. Since all the leaves of $\mathcal{H}|(\gamma_i \times I)$ are properly embedded arcs, the transverse orientation of $\mathcal{H}|(\Sigma \times \{1\})$ also coincides with that of $\mathcal{G}^\sigma$.

By Lemma 8, the orientations of $\{*\} \times I$ $(* \in \partial S_j)$ induced by the transverse orientation of $\mathcal{H}$ are either all positive or all negative. For each $\gamma_i$ and $\gamma_j$, there is an arc in a leaf of $\mathcal{G}^\sigma$ connecting $\gamma_i$ with $\gamma_j$ by the minimality of $\mathcal{G}^\sigma$. Thus the orientations of $\{*\} \times I$ $(* \in \bigcup_{i=1}^{n} \gamma_i)$ are either all positive or all negative. If they are positive (resp. negative), then we put $\delta(\mathcal{H}) = 1$ (resp. $\delta(\mathcal{H}) = -1$).

Denote by $c$ the positive number satisfying $c\nu|(\Sigma \times \partial I) = \mu^\sigma$. In the following, the transverse invariant measure of $\mathcal{H}$ is given by $c\nu$.

Let $\alpha$ denote the positive number satisfying $c\nu(\{*\}\times I) = \alpha\int_0^1 \lambda^{-\varepsilon(\sigma)t}dt$ ($* \in \gamma_i$). The foliation $\widehat{\mathcal{H}}(\sigma, \alpha\delta(\mathcal{H}), \mathcal{G}^\sigma, \mu^\sigma)$ of $\Sigma \times \mathbf{R}$ (defined by $\lambda^{\varepsilon(\sigma)t}\omega^\sigma + \alpha\delta(\mathcal{H})dt$ in $(\Sigma - K) \times \mathbf{R}$) has a transverse invariant measure $\widehat{\nu} = \left|\int(\omega^\sigma + \alpha\delta(\mathcal{H})\lambda^{-\varepsilon(\sigma)t}dt)\right|$. The transverse orientation of $\widehat{\mathcal{H}}(\sigma, \alpha\delta(\mathcal{H}), \mathcal{G}^\sigma, \mu^\sigma)$ is given by the positive orientation of $\lambda^{\varepsilon(\sigma)t}\omega^\sigma + \alpha\delta(\mathcal{H})dt$.

In the following, we construct a homeomorphism $h'' : \Sigma \times I \to \Sigma \times I$ satisfying $h''(\mathcal{H}, c\nu) = (\widehat{\mathcal{H}}(\sigma, \alpha\delta(\mathcal{H}), \mathcal{G}^\sigma, \mu^\sigma)|(\Sigma \times I), \widehat{\nu}|(\Sigma \times I))$.

First we define the homeomorphism $h : \Sigma \times \partial I \to \Sigma \times \partial I$ by the identity map. The transversely oriented measured foliations of $S^1 \times I$ transverse to both $S^1 \times \partial I$ and $\{*\} \times I$ (for any $* \in S^1$), are determined by the lengths of $S^1 \times \{0\}$ and $\{*\} \times I$, and the orientations of $S^1 \times \partial I$ and $\{*\} \times I$ ($* \in S^1$) ([1]). Hence $h$ extends to $h' : (\Sigma \times \partial I) \cup (\bigcup_{i=1}^n \gamma_i \times I) \to (\Sigma \times \partial I) \cup (\bigcup_{i=1}^n \gamma_i \times I)$ such that $h'(\mathcal{H}, c\nu) = (\widehat{\mathcal{H}}(\sigma, \alpha\delta(\mathcal{H}), \mathcal{G}^\sigma, \mu^\sigma), \widehat{\nu})$ and $h'(\{*\} \times I) = \{*\} \times I$ for any $* \in \bigcup_{i=1}^n \gamma_i$. By Lemma 9, $h'$ extends to $h'' : \Sigma \times I \to \Sigma \times I$ which brings $\mathcal{H}$ to $\widehat{\mathcal{H}}(\sigma, \alpha\delta(\mathcal{H}), \mathcal{G}^\sigma, \mu^\sigma)|(\Sigma \times I)$. Therefore $\mathcal{H}$ is $C^0$ isotopic to $\widehat{\mathcal{H}}(\sigma, \alpha\delta(\mathcal{H}), \mathcal{G}^\sigma, \mu^\sigma)|(\Sigma \times I)$ with the boundary fixed. □

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