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M. BLÜMLINGER
N. OBATA

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PERMUTATIONS PRESERVING CESÀRO MEAN, DENSITIES OF NATURAL NUMBERS AND UNIFORM DISTRIBUTION OF SEQUENCES

by M. BLÜMLINGER & N. OBATA(*)

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1. Permutations defined by invariance properties.

We give different definitions of classes of subgroups and subsemigroups of $\mathrm{Aut}(\mathbb{N})$ and show that they define the same permutations.

Let $\mathbb{N}$ be the set of natural numbers and $\mathrm{Aut}(\mathbb{N})$ the group of all permutations on $\mathbb{N}$. Each $g \in \mathrm{Aut}(\mathbb{N})$ gives rise to a rearrangement of a sequence $a = (a_n)_{n=1}^{\infty}$ in a usual manner, namely, $ga = \left(a_{g^{-1}(n)}\right)_{n=1}^{\infty}$. In general, a subgroup or more generally a subsemigroup of $\mathrm{Aut}(\mathbb{N})$ is called a permutation group, respectively a permutation semigroup.

Let $\mathcal{X}$ be a set of sequences and T a function on $\mathcal{X}$. We set

$$\mathcal{S}(\mathcal{X}) = \{g \in \mathrm{Aut}(\mathbb{N}) : g(\mathcal{X}) \subseteq \mathcal{X}\} \qquad \text{and}$$
$$\mathcal{S}(T) = \{g \in \mathrm{Aut}(\mathbb{N}) : g(\mathcal{X}) \subseteq \mathcal{X};\, T(gx) = Tx \text{ for } x \in \mathcal{X}\}\ .$$

These are subsemigroups of $\mathrm{Aut}(\mathbb{N})$. Furthermore, we define the subgroups

$$\mathcal{G}(\mathcal{X}) = \mathcal{S}(\mathcal{X}) \cap \mathcal{S}(\mathcal{X})^{-1} \quad \text{and} \quad \mathcal{G}(T) = \mathcal{S}(T) \cap \mathcal{S}(T)^{-1}$$

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of Aut($\mathbb{N}$). In particular, we consider the following :

(i) $\mathcal{D}$ be the space of real bounded sequences which are Cesàro summable and let L be the Cesàro mean :

$$L(a) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_n, \qquad a = (a_n)_{n=1}^{\infty} \in \mathcal{D}.$$

Then $\mathcal{D}$ becomes a closed subspace of $l^\infty(\mathbb{N})$ equipped with the usual norm $\|\cdot\|$. Let $\mathcal{S}(L)$ and $\mathcal{G}(L)$ be respectively the semigroup and the group of permutations under which the functional L is invariant on $\mathcal{D}$.

(ii) For a subset $S \subseteq \mathbb{N}$ we put

$$\delta(S) = \lim_{N\to\infty} \frac{1}{N} |S \cap I_N|, \qquad I_N = \{1, 2, ..., N\},$$

if the limit exists and we call $\delta(S)$ the density of S. Let $\mathcal{F}_\alpha$ be the class of subsets of $\mathbb{N}$ having density α and $\mathcal{F}$ be the class of subsets admitting some density $\alpha \in [0,1]$. Then δ is defined on $\mathcal{F}$ and its restriction to $\mathcal{F}_\alpha$ will be denoted by δ_α. We now can define $\mathcal{S}(\delta_\alpha)$, $\mathcal{G}(\delta_\alpha)$ and $\mathcal{S}(\delta)$, $\mathcal{G}(\delta)$.

(iii) Let X be a compact metric space and μ a Borel probability measure on it. A sequence $(x_n)_{n=1}^{\infty}$ in X is called μ-uniformly distributed if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \delta_{x_n}$$

converges weakly to μ, or equivalently if

$$\lim_{N\to\infty} \frac{1}{N} \{n : x_n \in E,\, n \le N\}| = \mu(E)$$

for all μ-continuity sets E in X. We let $\mathcal{U}_{X,\mu}$ be the set of μ-uniformly distributed sequences and define $\mathcal{S}(\mathcal{U}_{X,\mu})$ and $\mathcal{G}(\mathcal{U}_{X,\mu})$. It is known that $\mathcal{U}_{X,\mu}$ is not empty, i.e. for compact metric spaces there always exist μ-uniformly distributed sequences, cf. [KN].

THEOREM 1. — *Let X be a compact metric space equipped with a Borel probability measure μ which is not concentrated in a single point. Then for $\alpha \in (0,1)$, we have*

$$\mathcal{G}(L) = \mathcal{G}(\delta) = \mathcal{G}(\delta_\alpha) = \mathcal{G}(\mathcal{U}_{X,\mu}) \subsetneq \mathcal{S}(L) = \mathcal{S}(\delta) = \mathcal{S}(\delta_\alpha) = \mathcal{S}(\mathcal{U}_{X,\mu})\ .$$

Proof. — By the definition of the maximal subgroups it is sufficient to show that the semigroups coincide and that one group is properly contained in the pertinent semigroup. The inclusions $\mathcal{S}(L) \subseteq \mathcal{S}(\delta) \subseteq \mathcal{S}(\delta_\alpha)$ are obvious. We show $\mathcal{S}(L) \supseteq \mathcal{S}(\delta)$ in Lemma 1, $\mathcal{S}(\delta) \supseteq \mathcal{S}(\delta_\alpha)$ in Lemma 2,

$\mathcal{S}(\delta_{\alpha_0}) \supseteq \mathcal{S}(\mathcal{U}_{X,\mu}) \supseteq \mathcal{S}(\delta)$ for some $\alpha_0 \in (0,1)$ depending on (X,μ) in Lemma 3. $\mathcal{G}(\delta) \subsetneq \mathcal{S}(\delta)$ was proved by J. Coquet (cf. [C], Chapter IV.1).

It follows from results of F.W. Levi that any subgroup of Aut($\mathbb{N}$) being characterized by leaving the limit of a certain class of convergent series invariant has to be Aut($\mathbb{N}$). Therefore no similar characterization of these groups via convergent series can be possible. See [S] for the pertinent literature. However, J. Coquet [C] obtained a characterization of well distributed sequences that is partially similar to our result.

Remark. — The example

$$g : \begin{cases} 3k \to 2k & k = 1,2,\dots \\ 3k+1 \to 4k+1 & k = 0,1,\dots \\ 3k+2 \to 4k+3 & k = 0,1,\dots \end{cases}$$

shows that $\mathcal{S}(\delta_0) = \mathcal{S}(\delta_1) \supsetneq \mathcal{S}(\delta)$.

LEMMA 1. — $\mathcal{S}(\delta) \subseteq \mathcal{S}(L)$.

Proof. — Suppose $g \in \mathcal{S}(\delta)$ and $a = (a_n)_{n=1}^{\infty} \in \mathcal{D}$. It is sufficient to prove that $L(ga) = L(a)$ under the assumption $0 \le a_n \le 1$. We consider a sequence of independent random variables X_n with values in $\{0,1\}$ and expectation value $E(X_n) = a_n$. Applying the strong law of large numbers, we obtain

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (X_{g^{-1}(n)} - a_{g^{-1}(n)}) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (X_n - a_n) = 0 \qquad \text{(a.s.)}\ .$$

Therefore the random sequence $X = (X_n)_{n=1}^{\infty}$ is in $\mathcal{D}$ (a.s.) and as X_n takes values in $\{0,1\}$ we have $L(X) = L(gX)$. Hence $L(a) = L(ga)$ and $g \in \mathcal{S}(L)$.

LEMMA 2. — *$\mathcal{S}(\delta_\alpha) \subseteq \mathcal{S}(\delta)$ for $0 < \alpha < 1$.*

The proof is divided into three steps and the following result ([O2], Proposition 1.3) is useful.

PROPOSITION 1. — *For $A \in \mathcal{F}$ and $0 \le \lambda \le \delta(A)$ there exists a subset $B \subseteq A$ such that $\delta(B) = \lambda$.*

Proof of Lemma 2

(i) First we show that $\mathcal{S}(\delta_{\alpha/2}) \subseteq \mathcal{S}(\delta_\alpha)$ for $0 < \alpha < 1$. Assume $g \in \mathcal{S}(\delta_{\alpha/2})$ and $A \in \mathcal{F}_\alpha$. It follows from Proposition 1 that there exists a partition of $A = A_1 \cup A_2$ with $A_1, A_2 \in \mathcal{F}_{\alpha/2}$. Using finite additivity of the density we get

$$\delta(A) = \delta(A_1) + \delta(A_2) = \delta(g(A_1)) + \delta(g(A_2)) = \delta(g(A_1) \cup g(A_2)) = \delta(g(A)).$$

This proves $g(A) \in \mathcal{F}_\alpha$.

(ii) We next show that $\mathcal{S}(\delta_{2\alpha}) \subseteq \mathcal{S}(\delta_\alpha)$ for $0 < \alpha \leq 1/3$. Suppose $g \in \mathcal{S}(\delta_{2\alpha})$ and $A \in \mathcal{F}_\alpha$. By Proposition 1 we may find $A_1, A_2 \in \mathcal{F}_\alpha$ such that A, A_1 and A_2 are mutually disjoint. Then

$$\delta(A \cup A_1) = \delta(A \cup A_2) = \delta(A_1 \cup A_2) = 2\alpha\ .$$

Since $g \in \mathcal{S}(\delta_{2\alpha})$ we have

$$\delta(g(A) \cup g(A_1)) = \delta(g(A) \cup g(A_2)) = \delta(g(A_1) \cup g(A_2)) = 2\alpha\ .$$

From the obvious identity :

$$\frac{2}{N}|g(A) \cap I_N| = \frac{1}{N}|g(A_1 \cup A) \cap I_N| + \frac{1}{N}|g(A_2 \cup A) \cap I_N| - \frac{1}{N}|g(A_1 \cup A_2) \cap I_N|\ ,$$

it follows that

$$\lim_{N\to\infty} \frac{2}{N}|g(A) \cap I_N| = 2\alpha + 2\alpha - 2\alpha = 2\alpha\ .$$

This proves that $g(A) \in \mathcal{F}_\alpha$.

(iii) Since $\mathcal{S}(\delta_\alpha) = \mathcal{S}(\delta_{1-\alpha})$, Lemma 2 will follow if we prove $\mathcal{S}(\delta_\alpha) \subseteq \mathcal{S}(\delta)$ for $0 < \alpha \leq \frac{1}{2}$. From (i) and (ii) we have already obtained $\mathcal{S}(\delta_\alpha) = \mathcal{S}(\delta_{\alpha/2^n})$, $0 < \alpha \leq 2/3$, $\quad n = 1, 2, \ldots$. Now suppose $g \in \mathcal{S}(\delta_\alpha)$ and $A \in \mathcal{F}$. For any $n \geq 1$ let $l_n \geq 0$ be the integer uniquely determined by

$$\frac{l_n}{2^n}\alpha \leq \delta(A) < \frac{l_n + 1}{2^n}\alpha\ .$$

By Proposition 1 we may choose $A_j \in \mathcal{F}$, $\quad 1 \leq j \leq l_n + 1$, with $\delta(A_j) = \dfrac{\alpha}{2^n}$ such that

$$\begin{aligned} A =& A_1 \cup A_2 \cup \ldots \cup A_{l_n} \cup A' \qquad \text{(disjoint union)} \\ A_{l_n+1} =& A' \cup B', \qquad B' \subseteq \mathbb{N} - A\ . \end{aligned}$$

Then, obviously,

$$g(A_1) \cup \ldots \cup g(A_{l_n}) \subseteq g(A) \subseteq g(A_1) \cup \ldots \cup g(A_{l_n}) \cup g(A_{l_n+1}).$$

Since $g \in \mathcal{S}(\delta_\alpha) = \mathcal{S}(\delta_{\alpha/2^n})$, we have

$$\frac{l_n}{2^n}\alpha \leq \liminf_{N\to\infty} \frac{1}{N}|g(A) \cap I_N| \leq \limsup_{N\to\infty} \frac{1}{N}|g(A) \cap I_N| \leq \frac{l_n + 1}{2^n}\alpha\ .$$

This implies that $g(A) \in \mathcal{F}$ and $\delta(g(A)) = \delta(A)$.

LEMMA 3. — *For X, μ as in Theorem 1 there exists $\alpha_0 \in (0,1)$ such that*

$$\mathcal{S}(\delta_{\alpha_0}) \supseteq \mathcal{S}(\mathcal{U}_{X,\mu}) \supseteq \mathcal{S}(\delta)$$

Proof. — The proof is similar to [C], II.2. There exists a μ-continuity set E in X with $\alpha_0 = \mu(E) \in (0,1)$ (cf. [KN], Lemma 3.4, Chapter 3). Now assume $g \in \mathcal{S}(\mathcal{U}_{X,\mu})$ and $A \in \mathcal{F}_{\alpha_0}$. Let (y_n) be a sequence in E which is $\mu/\mu(E)$-uniformly distributed in the compact metric space $\overline{E}$ and let (z_n) be $\mu/(1-\mu(E))$-uniformly distributed in $\overline{X \setminus E}$. Since $\mu(\partial E) = 0$ we may (after removing all elements in the sequence which are in ∂E) assume that $y_i \in E$ and $z_i \in X \setminus E$. We define a sequence

$$x_n = \begin{cases} y_i, & i = |A \cap I_n|, \qquad \text{for } n \in A \\ z_i, & i = n - |A \cap I_n|, \quad \text{for } n \notin A. \end{cases}$$

For a μ-continuity set F in E we have

$$\frac{1}{N}|\{n : x_n \in F,\, n \leq N\}| = \frac{|A \cap I_N|}{N} \frac{1}{|A \cap I_N|}|\{i : y_i \in F,\, i \leq |A \cap I_N|\}| \,.$$

Since

$$\lim_{N\to\infty} \frac{1}{N}|A \cap I_N| = \mu(E)$$

and

$$\lim_{N\to\infty} \frac{1}{|A \cap I_N|}|\{i : y_i \in F,\, i \leq |A \cap I_n|\}| = \frac{\mu(F)}{\mu(E)} \,,$$

it follows that

$$\lim_{N\to\infty} \frac{1}{N}|\{n : x_n \in F,\, n \leq N\}| = \mu(F) \,.$$

The same is true for μ-continuity sets in $X \setminus E$ and it follows that the same holds for all μ-continuity sets in X. Hence (x_n) is μ-uniformly distributed in X. Since $g \in \mathcal{S}(\mathcal{U}_{X,\mu})$ the sequence $(x_{g^{-1}(n)})$ is μ-uniformly distributed. This implies

$$\lim_{N\to\infty} \frac{1}{N}|\{n : g^{-1}(n) \in E,\, n \leq N\}| = \alpha_0 \,,$$

so g leaves $\mathcal{F}_{\alpha_0}$ invariant and $g \in \mathcal{S}(\delta_{\alpha_0})$.

Suppose next $g \in \mathcal{S}(\delta)$. Then for any μ-uniformly distributed sequence (x_n) and any μ-continuity set E of X we have

$$\mu(E) = \lim_{N\to\infty} \frac{1}{N}|\{n : x_n \in E,\, n \leq N\}|$$
$$= \lim_{N\to\infty} \frac{1}{N}|\{n : x_{g^{-1}(n)} \in E,\, n \leq N\}|$$

which shows that $g \in \mathcal{S}(\mathcal{U}_{X,\mu})$.

2. Cesàro mean and the Lévy group.

The Lévy group is a subgroup of $\mathcal{G}(L)$ defined by

$$\mathcal{G} = \left\{ g \in \mathrm{Aut}(\mathbb{N}) : \lim_{N\to\infty} \frac{1}{N} |\{1 \le n \le N : g(n) > N\}| = 0 \right\}.$$

See [O1], [O2], [R] for some properties and applications.

Since the Lévy group is a proper subgroup of $\mathcal{G}(L)$, see [O1], Proposition 4.1, the Cesàro mean is invariant under the Lévy group. In this section we shall prove that the Cesàro mean is characterized by its invariance under the Lévy group, namely.

THEOREM 2. — *A $\mathcal{G}$-invariant continuous linear functional on $\mathcal{D}$ is a constant multiple of the Cesàro mean. A $\mathcal{G}$-invariant positive normalized functional on $l^\infty(\mathbb{N})$ is a Banach limit.*

A Banach limit Δ is by definition a continuous linear functional on $l^\infty(\mathbb{N})$ such that(i)$\Delta(\mathbf{1}) = 1$; (ii) $\Delta(a) \ge 0$ for $a = (a_n)_{n=1}^\infty \in l^\infty(\mathbb{N})$ with $a_n \ge 0$; (iii) $\Delta(\tau a) = a$. Here $\mathbf{1} = (1, 1, \ldots)$ and $(\tau a)_n = a_{n+1}$.

First of all we recall typical permutations belonging to the Lévy group (cf. [O1]).

LEMMA 4. — *Let $0 = N_0 < N_1 < N_2 < \ldots$ be an increasing sequence of integers such that $\lim_{k\to\infty} N_k/N_{k-1} = 1$. Then a permutation $g \in \mathrm{Aut}(\mathbb{N})$ which leaves every subset $\{N_{k-1}+1, \ldots, N_k\}$ invariant belongs to $\mathcal{G}$.*

Let M denote a $\mathcal{G}$-invariant continuous linear functional on $\mathcal{D}$. The proof of Theorem 2 will be completed with Lemma 10.

LEMMA 5. — *Assume that $a = (a_n)_{n=1}^\infty \in \mathcal{D}$ satisfies $a_n \in \{0, 1\}$, $0 < L(a) < 1$ and $L(a) \in \mathbb{Q}$. Then $M(a) = M(\mathbf{1})L(a)$, where $\mathbf{1} = (1, 1, \ldots) \in \mathcal{D}$.*

Proof. — Put $\{n \in \mathbb{N} : a_n = 0\} = \{m_1, m_2, \ldots\}$ and $\{n \in \mathbb{N} : a_n = 1\} = \{n_1, n_2, \ldots\}$ with $n_i < n_j$, $m_i < m_j$ for $i < j$ and assume $L(a) = r/s$ with $1 \le r < s$. Define a permutation $g \in \mathrm{Aut}(\mathbb{N})$ as follows : to define $g(n_j)$ we write $j = kr + k'$ with $k = 0, 1, 2, \ldots$ and $1 \le k' \le r$ and put $g(n_j) = ks + k'$. For $j = k(s-r) + k'$ with $k = 0, 1, 2, \ldots$ and $1 \le k' \le s - r$ we define $g(m_j) = ks + r + k'$. Thus $g(a)$ is a sequence with period s with the first r elements of each block being 1 and the following $s - r$ elements being 0.

We prove that $g \in \mathcal{G}$.

For a given $N \in \mathbb{N}$ let $j(N)$ be an integer uniquely determined by the condition $n_{j(N)} < N \leq n_{j(N)+1}$. Putting $N = ks + l$ with $0 \leq l < s$, we have

$$\frac{1}{N}|\{n : 1 \leq n \leq N,\, g(n) > N\}| = \frac{1}{N}|kr + \min(r,l) - n_{j(N)}|$$
$$\leq \frac{1}{N}\min(r,l) + \left|\frac{kr}{ks+l} - \frac{n_{j(N)}}{N}\right| .$$

On the other hand,

$$\frac{n_{j(N)}}{N} = \frac{1}{N}|\{n : 1 \leq n \leq N,\, a_n = 1\}| = \frac{1}{N}\sum_{n=1}^{N} a_n \to L(a), \quad N \to \infty ,$$

and therefore,

$$\lim_{N\to\infty} \frac{1}{N}|\{n : 1 \leq n \leq N,\, g(n) > N\}| \leq 0 + \left|\frac{r}{s} - L(a)\right| = 0 .$$

This shows that $g \in \mathcal{G}$, and by assumption $M(a) = M(ga)$.

We next put $B_i = \{i, s+i, 2s+i, \ldots\}$, $1 \leq i \leq s$, and define $b^{(i)} = (b_n^{(i)})_{n=1}^{\infty}$ by

$$b_n^{(i)} = \begin{cases} 1 & \text{if } n \in B_i \\ 0 & \text{otherwise.} \end{cases}$$

For $1 < i \leq s$ define a permutation $g_i \in \mathrm{Aut}(\mathbb{N})$ by products of cycles : $g_i = \prod_{k=0}^{\infty}(ks+1, ks+i)$. It follows from Lemma 4 that $g_i \in \mathcal{G}$. Note also that $g_i b^{(1)} = b^{(i)}$. Since M is $\mathcal{G}$-invariant,

$$M(\mathbf{1}) = M(b^{(1)} + \ldots + b^{(s)}) = M(b^{(1)}) + \ldots + M(b^{(s)})$$
$$= M(b^{(1)}) + M(g_2 b^{(1)}) + \ldots + M(g_s b^{(1)})$$
$$= sM(b^{(1)}) .$$

Hence $M(b^{(1)}) = \ldots = M(b^{(s)}) = \frac{1}{s}M(\mathbf{1})$. Consequently,

$$M(a) = M(ga) = M(b^{(1)} + \ldots + b^{(r)}) = \frac{r}{s}M(\mathbf{1}) = M(\mathbf{1})L(a) .$$

This completes the proof of Lemma 5.

Let $[0,1]^{\mathbb{N}}$ and $\{0,1\}^{\mathbb{N}}$ denote the space of sequences $a = (a_n)_{n=1}^{\infty}$ with values in $[0,1]$respectively $\{0,1\}$. For a fixed $s \in \mathbb{N}$ we define a (non-linear) mapping $\Psi_s : [0,1]^{\mathbb{N}} \mapsto \{0,1\}^{\mathbb{N}}, \Psi_s(a) = b$, as follows : Define a sequence $(l_k)_{k=0}^{\infty}$ inductively by

$$l_0 + \ldots + l_k = \left[\sum_{n=1}^{(k+1)s} a_n\right] ,$$

where $[\cdot]$ denotes the integer part. Given $n \in \mathbb{N}$ we write $n = ks + l$ with $1 \le l \le s$ and put

$$b_n = b_{ks+l} = \begin{cases} 1 & \text{for } l \le l_k \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 6. — *If $a \in [0,1]^{\mathbb{N}} \cap \mathcal{D}$ then $\Psi_s(a) \in \{0,1\}^{\mathbb{N}} \cap \mathcal{D}$ and $L(\Psi_s(a)) = L(a)$.*

Proof. — We retain the above notation. For $N = ks + l$ with $1 \le l \le s$ we have

$$\sum_{n=1}^{N} b_n = l_0 + l_1 + \cdots + l_{k-1} + \min(l, l_k) .$$

Hence

$$\frac{1}{N}\left[\sum_{n=1}^{ks} a_n\right] \le \frac{1}{N}\sum_{n=1}^{N} b_n \le \frac{1}{N}\left[\sum_{n=1}^{(k+1)s} a_n\right]$$

and therefore

$$\frac{1}{N}\sum_{n=1}^{N} a_n - \frac{l\|a\|+1}{N} \le \frac{1}{N}\sum_{n=1}^{N} b_n \le \frac{1}{N}\sum_{n=1}^{N} a_n + \frac{(s-l)\|a\|}{N} .$$

Since $a \in \mathcal{D}$, we conclude that $b = \Psi_s(a) \in \mathcal{D}$ and $L(\Psi_s(a)) = L(a)$.

For a fixed $s \in \mathbb{N}$ we define a continuous linear operator

$$\Lambda_s : l^\infty(\mathbb{N}) \to l^\infty(\mathbb{N}), \qquad \Lambda_s(a) = b$$

as follows : Given $n \in \mathbb{N}$ we write $n = ks + l$ with $1 \le l \le s$ and put

$$b_n = b_{ks+l} = \frac{1}{s}\sum_{l=1}^{s} a_{ks+l}.$$

LEMMA 7. — *$\Lambda_s(\mathcal{D}) \subset \mathcal{D}$ and $L(\Lambda_s(a)) = L(a)$ for all $a \in \mathcal{D}$.*

Proof. — Let $N = ks + l$ with $1 \le l \le s$ and put $b = \Lambda_s(a)$. Then

$$\frac{1}{N}\sum_{n=1}^{N} b_n = \frac{1}{N}\left(\sum_{n=1}^{ks} a_n + \frac{l}{s}\sum_{m=1}^{s} a_{ks+m}\right) .$$

Hence

$$\left|\frac{1}{N}\sum_{n=1}^{N} a_n - \frac{1}{N}\sum_{m=1}^{N} b_n\right| = \frac{1}{N}\left|\sum_{m=1}^{l} a_{ks+m} + \frac{l}{s}\sum_{m=1}^{s} a_{ks+m}\right|$$
$$\le \frac{1}{N}(l\|a\| + l\|a\|) = \frac{2l\|a\|}{N} \to 0 \text{ as } N \to \infty .$$

Consequently $b = \Lambda_s(a) \in \mathcal{D}$ and $L(\Lambda_s(a)) = L(a)$.

LEMMA 8. — *$M(\Lambda_s(a)) = M(a)$ for all $a \in \mathcal{D}$.*

Proof. — Let $\mathfrak{S}_s$ denote the permutation group of $\{1, 2, \dots, s\}$ and $\tilde{g} \in \mathfrak{S}_s$. Then define $g \in \mathrm{Aut}(\mathbb{N})$ by

$$g(ks + l) = ks + \tilde{g}(l), \quad 1 \le l \le s, \quad k = 0, 1, \dots$$

By Lemma 4 we have $g \in \mathcal{G}$. Since $\Lambda_s(a) = \Lambda_s(ga)$,

$$\Lambda_s(a) = \frac{1}{|\mathfrak{S}_s|} \sum_{\tilde{g} \in \mathfrak{S}_s} \Lambda_s(ga) = \Lambda_s \left(\frac{1}{|\mathfrak{S}_s|} \sum_{\tilde{g} \in \mathfrak{S}_s} ga \right) = \frac{1}{|\mathfrak{S}_s|} \sum_{\tilde{g} \in \mathfrak{S}_s} ga.$$

The Lemma follows from

$$M(\Lambda_s(a)) = \frac{1}{|\mathfrak{S}_s|} \sum_{\tilde{g} \in \mathfrak{S}_s} M(ga) = \frac{1}{|\mathfrak{S}_s|} \sum_{\tilde{g} \in \mathfrak{S}_s} M(a) = M(a) .$$

LEMMA 9. — *$M(a) = M(\mathbf{1})L(a)$ for all $a \in \mathcal{D}$.*

Proof. — For $a = (0, 0, \dots)$ the equality is obvious. Suppose $a \neq (0, 0, \dots)$ and consider $a' = (a'_n)_{n=1}^{\infty} \in \mathcal{D}$ defined by

$$a'_n = \frac{a_n + 2\|a\|}{4(L(a) + 2\|a\|)} .$$

Since $|a_n| \le \|a\|$ and $|L(a)| \le \|a\|$ we see that $a' \in [0,1]^{\mathbb{N}} \cap \mathcal{D}$, $0 < L(a') < 1$, $L(a') = 1/4 \in \mathbb{Q}$. Lemma 6 implies $\Psi_s(a') \in \{0,1\}^{\mathbb{N}} \cap \mathcal{D}$ and $L(\Psi_s(a')) = L(a')$. Now applying Lemma 5, we see that

$$M(\Psi_s(a')) = M(\mathbf{1})L(\Psi_s(a')) = M(\mathbf{1})L(a').$$

On the other hand, it is easily verified that $\|\Lambda_s(\Psi_s(a')) - \Lambda_s(a')\| \le 1/s$. In view of Lemma 8

$$\begin{aligned} |M(a') - M(\mathbf{1})L(a')| &= |M(a') - M(\Psi_s(a'))| \\ &= |M(\Lambda_s(a')) - M(\Lambda_s(\Psi_s(a'))| \\ &\le \|M\| \, \|\Lambda_s(a') - \Lambda_s(\Psi_s(a'))\| \le \frac{1}{s}\|M\| \to 0, \ s \to \infty. \end{aligned}$$

This proves that $M(a') = M(\mathbf{1})L(a')$. Now $M(a) = M(\mathbf{1})L(a)$ is an immediate consequence.

LEMMA 10. — *If M is a $\mathcal{G}$-invariant positive normalized functional on $l^{\infty}(\mathbb{N})$ it is a Banach limit.*

Proof. — As in the proof of Lemma 9 we may show that $M(\Lambda_s(a)) = M(a)$ for all $a \in l^\infty(\mathbb{N})$. Let τ be the translation operator defined by $(\tau a)_n = a_{n+1}$. Then

$$\begin{aligned} |M(a) - M(\tau a)| &= |M(\Lambda_s(a)) - M(\Lambda_s(\tau a))| \\ &\leq M(\mathbf{1})\|\Lambda_s(a) - \Lambda_s(\tau a)\| \leq M(\mathbf{1})\|a\|\frac{2}{s}, \qquad \forall s \in \mathbb{N}, \end{aligned}$$

which shows that M is translation invariant, i.e. M is a Banach limit. □

3. $\mathcal{G}$-invariant extension of Cesàro mean.

It follows from Theorem 2 that every $\mathcal{G}$-invariant continuous normalized linear functional on $l^\infty(\mathbb{N})$ is an extension of Cesàro mean. We now prove that such an extension exists from $\mathcal{D}$ to $l^\infty(\mathbb{N})$.

If $\mathcal{G}$ were an amenable group, this would be an immediate consequence of [P], 2.32. The following argument of H. Rindler (private communication) shows that $\mathcal{G}$ is not amenable :

Let $\mathcal{G}_0$ be the subgroup of $\mathcal{G}$ which permutes the elements of the intervals $J_n = [n^2+1, n^2+n]$, $n \in \mathbb{N}$ and leaves all other $i \in \mathbb{N}$ fixed, i.e.

$$g(i) : \begin{cases} = i, & \notin \bigcup_{n=1}^{\infty} J_n \\ \in J_n, & i \in J_n. \end{cases}$$

The free group with two generators F_2 can be embedded homomorphically in the product $\prod_{n=1}^{\infty} \mathfrak{S}_n$ of finite permutation groups ($\mathfrak{S}_n$ being the permutation group of an n-element set) cf. [P], p. 121, p. 425, which is isomorphic to $\mathcal{G}_0$. Therefore $\mathcal{G}$ contains a subgroup isomorphic to F_2 which implies that $\mathcal{G}$ is not amenable cf. [P], Prop. 0.16.

We recall the notion of a paradoxical decomposition: Let G be a permutation group acting on $\mathbb{N}$. A partition $\mathbb{N} = \bigcup_{i=1}^{m} A_i \cup \bigcup_{j=1}^{n} B_j$ is called G-paradoxical if there exist $g_1, \ldots, g_m, \tilde{g}_1, \ldots, \tilde{g}_n \in G$ such that $\mathbb{N} = \bigcup_{i=1}^{m} g_i(A_i) = \bigcup_{j=1}^{n} \tilde{g}_j(B_j)$. Obviously $\mathbb{N}$ admits an Aut($\mathbb{N}$)-paradoxical decomposition. We have however :

PROPOSITION 2. — *There exists no $\mathcal{G}$-paradoxical decomposition of* $\mathbb{N}$.

Proof. — Let A_i, B_j, $i = 1,\dots,m$, $j = 1,2,\dots,n$ be a pairwise disjoint decomposition of $\mathbb{N}$. We may assume without loss of generality that

$$\liminf_{N\to\infty} \frac{1}{N} \left| \bigcup_{i=1}^{m} A_i \cap I_N \right| \le \frac{1}{2} .$$

Since $g_i \in \mathcal{G}$ for $i = 1,\dots,m$,

$$\lim_{N\to\infty} \frac{1}{N} |\{n > N \,:\, g_i(n) \le N\}| = 0 .$$

We then obtain

$$\begin{aligned} &\liminf_{N\to\infty} \frac{1}{N} \left| \bigcup_{i=1}^{m} g_i(A_i) \cap I_N \right| \\ &\le \liminf_{N\to\infty} \frac{1}{N} \left(\left| \bigcup_{i=1}^{m} A_i \cap I_N \right| + \sum_{i=1}^{m} |\{n > N \,:\, g_i(n) \le N\}| \right) \le \frac{1}{2} . \end{aligned}$$

Therefore $\bigcup_{i=1}^{n} g_i(A_i)$ cannot be $\mathbb{N}$, i.e. there is no $\mathcal{G}$-paradoxical decomposition of $\mathbb{N}$.

From Proposition 2 and Tarski's theorem, which states that $l^\infty(\mathbb{N})$ admits a $\mathcal{G}$-invariant mean if and only if $\mathbb{N}$ has no $\mathcal{G}$-paradoxical decomposition (cf. [P], 3.15), we obtain

THEOREM 3. — *There exists a positive continuous linear functional on $l^\infty(\mathbb{N})$ which is invariant under the Lévy group.*

Remark. — The above assertion can be proved in a different manner. With the help of [K], Theorem 2.2, we can show that there exists a $\mathcal{G}$-invariant extension $\tilde{L} \in (l^\infty(\mathbb{N}))^*$ of Cesàro mean. Let λ be the corresponding finitely additive set function defined on the subsets of $\mathbb{N}$. Then the total variation $|\lambda|$ is $\mathcal{G}$-invariant, and Theorem 3 follows.

There arises the natural question whether the Cesàro mean $L \in \mathcal{D}^*$ is characterized by its invariance under a smaller permutation group of $\mathbb{N}$. Let $\mathfrak{S}_\infty$ denote the subgroup of $\mathcal{G}$ whose elements leave all but finitely many elements fixed.

PROPOSITION 3. — $\dim\{M \in \mathcal{D}^* \,:\, M$ *is* $\mathfrak{S}_\infty$ *-invariant* $\} = \infty$.

Proof. — We fix a partition of $\mathbb{N} = \bigcup_{p=1}^{\infty} A_p$ such that $A_p \in \mathcal{F}$, $\delta(A_p) > 0$ for all $p \ge 1$ and denote by $\mathcal{C}(p)$ the space of sequences

$a = (a_n)_{n=1}^{\infty} \in l^{\infty}(\mathbb{N})$ such that $\lim\limits_{\substack{n \in A_p \\ n \to \infty}} a_n$ exists. We put

$$f_p(a) = \lim_{\substack{n \in A_p \\ n \to \infty}} a_n, \quad a = (a_n)_{n=1}^{\infty} \in \mathcal{C}(p) .$$

Then f_p is $\mathfrak{S}_{\infty}$-invariant. We shall prove that f_p admits a $\mathfrak{S}_{\infty}$-invariant extension $\tilde{f}_p \in (l^{\infty}(\mathbb{N}))^*$. By a theorem of Klee (cf. [K], Theorem 2.2) the following condition is necessary and sufficient for the existence of such a $\tilde{f}_p$:

$$f_p(x) \le \left\| x + \sum_{i=1}^{n} (g_i - e) y_i \right\| \quad \text{for } x \in \mathcal{C}(p), \quad y_i \in l^{\infty}(\mathbb{N}) \quad \text{and } g_i \in \mathfrak{S}_{\infty} .$$

Suppose that there exist $a \in \mathcal{C}(p)$, $g_1, \dots, g_k \in \mathfrak{S}_{\infty}$ and $b^{(1)}, \dots, b^{(k)} \in l^{\infty}(\mathbb{N})$ such that

$$f_p(a) - \left\| a + \sum_{j=1}^{k} (g_i - e) b^{(k)} \right\| = \epsilon > 0 .$$

We clearly have

$$\left| a_n + \sum_{j=1}^{k} \left(b^{(j)}_{g_j^{-1}(n)} - b^{(j)}_n \right) \right| \le f_p(a) - \epsilon$$

for all $n \in \mathbb{N}$. Since $g_1, \dots, g_k \in \mathfrak{S}_{\infty}$, we obtain by taking the limit $\lim\limits_{\substack{n \in A_p \\ n \to \infty}}$ the inequality $|f_p(a)| \le f_p(a) - \epsilon$, which is a contradiction. Now we put $M_p = \tilde{f}_p | \mathcal{D}$. Then $M_p \in \mathcal{D}^*$ is $\mathfrak{S}_{\infty}$- invariant. For the proof of the assertion we have to show that $\{M_p : p \ge 1\}$ is independent. This is easily verified using a particular sequence $a^{(p)} = (a_n^{(p)})_{n=1}^{\infty}$ defined by

$$a_n^{(p)} = \begin{cases} 1 & \text{if } n \in A_p \\ 0 & \text{otherwise,} \end{cases}$$

and noting that $M_p(a^{(p)}) = \delta(A_p) > 0$ and $M_q(a^{(p)}) = 0$ if $q \ne p$.

We have been concerned so far with one-sided sequences $a = (a_n)_{n=1}^{\infty}$. It is easy to extend the above results to the case of two-sided sequences $a = (a_n)_{n=-\infty}^{+\infty}$. The Cesàro mean then is defined as

$$L_{\mathbb{Z}}(a) = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{|n| \le N} a_n$$

and $\mathcal{D}_{\mathbb{Z}}$ denotes the set of all sequences $a = (a_n)_{n=-\infty}^{\infty} \in l^{\infty}(\mathbb{Z})$ which admit the Cesàro mean. Then $L_{\mathbb{Z}} \in \mathcal{D}_{\mathbb{Z}}^*$. The Lévy group is defined by

$$\mathcal{G}_{\mathbb{Z}} = \left\{ g \in \mathrm{Aut}(\mathbb{Z}) : \lim_{n \to \infty} \frac{1}{2N+1} |\{|n| \le N : |g(n)| > N\}| = 0 \right\} .$$

Our tool is a bijective map $\varphi : \mathbb{N} \mapsto \mathbb{Z}$ defined by

$$\varphi : \begin{cases} 2n \to n \\ 2n-1 \to -n+1, \end{cases} \qquad n = 1, 2, \ldots$$

Then φ induces an isometric linear isomorphism $\tilde{\varphi} : l^\infty(\mathbb{Z}) \mapsto l^\infty(\mathbb{N})$:

$$(\tilde{\varphi} a)_n = a_{\varphi(n)}, \quad a = (a_n)_{n=-\infty}^{+\infty} \in l^\infty(\mathbb{Z}).$$

It is straightforward that $\tilde{\varphi}(\mathcal{D}_\mathbb{Z}) = \mathcal{D}$, $L \circ \tilde{\varphi} = L_\mathbb{Z}$ on $\mathcal{D}_\mathbb{Z}$ and $\varphi \mathcal{G} \varphi^{-1} = \mathcal{G}_\mathbb{Z}$. Hence the following result is a direct consequence of Theorem 2.

COROLLARY 1. — *Any $\mathcal{G}_\mathbb{Z}$-invariant continuous linear functional on $\mathcal{D}_\mathbb{Z}$ is a constant multiple of Cesàro mean $L_\mathbb{Z}$.*

Let $\mathfrak{S}_\mathbb{Z}$ denote the group of finite permutations of $\mathbb{Z}$. From Proposition 3 we have

COROLLARY 2. — $\dim\{M \in \mathcal{D}_\mathbb{Z}^* : M$ *is* $\mathfrak{S}_\mathbb{Z}$ *-invariant* $\} = \infty$.

Let $\tau : \mathcal{D}_Z \to \mathcal{D}_Z^*$ be the translation operator defined by $(\tau a)_n = a_{n+1}, n \in \mathbb{Z}$. Obviously the groupe generated by *tau* is a subgroup of $\mathcal{G}_Z$. *Remark.* — Let $\tau : \mathcal{D}_\mathbb{Z} \mapsto \mathcal{D}_\mathbb{Z}$ be the translation operator defined by $(\tau a)_n = a_{n+1}, n \in \mathbb{Z}$. Obviously the group generated by τ is a subgroup of $\mathcal{G}_\mathbb{Z}$. The following example shows that continuous translation invariant functionals on $l^\infty(\mathbb{Z})$ need not be extensions of $L_\mathbb{Z}$:

Let $\mathcal{E}$ be the space of all sequences $a = (a_n)_{n=-\infty}^{+\infty} \in l^\infty(\mathbb{Z})$ which admit the limit

$$f(a) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_n .$$

Then, $f \in \mathcal{E}^*$ and, using [K], Theorem 2.2, we see that f admits a translation-invariant extension $\tilde{f} \in (l^\infty(\mathbb{Z}))^*$. Put $M = \tilde{f}|_{\mathcal{D}_\mathbb{Z}}$. Obviously, $M \in \mathcal{D}_\mathbb{Z}^*$ and M is translation invariant. We shall prove that M is not a constant multiple of Cesàro mean. Let $a = (a_n)_{n=-\infty}^{+\infty} \in l^\infty(\mathbb{Z})$ be a sequence such that $\lim_{n\to\pm\infty} a_n = c_\pm$ with $c_+ \neq c_-$. Obviously, $L_\mathbb{Z}(a) = \frac{1}{2}(c_+ + c_-)$. On the other hand, $M(a) = f(a) = c_+$. Hence M is not a constant multiple of Cesàro mean.

The cardinality of continuous translation invariant functionals on $l^\infty(\mathbb{Z})$ is $2^{2^{\aleph_0}}$ cf. [P], Chapter 0.11.

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M. BLÜMLINGER,
Institut für Analysis
Technische Mathematik
Technische Universität Wien
Wiedener Hauptstrasse 8-10/114
1040 WIEN (Austria)
&
N. OBATA,
Dept. of Mathematics
School of Science
Nagoya University
Chikusa-ku
464 NAGOYA (Japan).