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METRIC PROPERTIES OF EIGENFUNCTIONS OF THE LAPLACE OPERATOR ON MANIFOLDS

by Nikolai S. NADIRASHVILI

In this note, we prove two theorems which express a quasi-symmetry relation between the positive and the negative part of the distribution function of an eigenfunction of the Laplace operator on a Riemannian manifold.

1. An estimate of the volume of a domain on which an eigenfunction of the Laplace operator on a Riemannian surface has constant sign.

Let M be a two-dimensional compact real analytic Riemannian manifold, $u_1, u_2, \ldots$-eigenfunctions of the Laplace operator on M, $\Delta u_i = \lambda_i u_i$.

THEOREM. — *There exists a positive constant C which depends on M such that, for every $i = 1, 2, \ldots$,*

$$\mathrm{vol}\{x \in M, u_i(x) > 0\} > C\ .$$

The proof is based on the two following lemmas.

Let f be a bounded function, continuous on $[0,1]$. Let us denote by $N(f)$ the number of changes in the sign of the function f on $[0,1]$.

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LEMMA 1. — *Let f_n be a sequence of non-zero continuous functions, defined on* $\mathbf{R}$*, with values in* $\mathbf{R}$*, with support in* $[0,1]$ *and assume that* $N(f_n)$ *is bounded by some fixed number* N*.*

Then there exists a subsequence n_i of $\mathbf{N}$ *($n_i \to \infty$), real numbers α_{n_i}, such that : $\alpha_{n_i}.f_{n_i}$ converges as $i \to \infty$, for the (weak-)topology of the space of distributions $\mathcal{D}'$ to a non-zero distribution of order less than N.*

Remark. — First, we recall that a distribution is said of order less than N if it is a sum of derivatives of order less than N of Radon measures. Moreover, if P is a polynomial of degree N and μ a Radon measure, the set of all $T \in \mathcal{D}'$ satisfying $PT = \mu$ is an N-dimensionnal affine subspace of the space of all distributions of order less than N.

Proof of Lemma 1 (suggested by Y. Colin de Verdière). — Let $P_n = \prod_{k=1}^{N}(x - x_k)$ be a sequence of polynomials of degree N such that $P_n.f_n$ is ≥ 0. By renormalisation and taking a subsequence, we may assume that $\int_0^1 P_n.f_n = 1$, that $P_n.f_n$ converges to a probability measure μ and that P_n converges to a polynomial P of degree exactly N.

Let $T_0 \in \mathcal{D}'$ be such that : $PT_0 = \mu$.

Let $T_n = f_n - T_0$, then we get :

$$\lim P_n.T_n = 0 \ .$$

We introduce now the following decomposition of the space of distributions :

$$\mathcal{D}' = Z_P \oplus W \ ,$$

where $Z_P = \{T \in \mathcal{D}' | PT = 0\}$, and W is a topological complement of Z_P.

W is a complement to Z_{P_n} if n is big enough. Now we can write uniquely :

$$T_n = z_n + w_n \ ,$$

where $z_n \in Z_{P_n}$ and $w_n \in W$. Now $P_n.w_n \to 0$ and we deduce that $w_n \to 0$, because the multiplication by P_n is uniformly invertible in W.

Now $z_n \in Z_{P_n}$ and Z_{P_n} converges to Z_P.

Two cases are possible :

First case : z_n is bounded and we can extract a convergent subsequence converging to T_1 in Z_P. Then $T_0 + T_1$ is not zero and we get the conclusion.

Second case : z_n is unbounded; then there exists a sequence $\beta_{n_i} \to 0$ such that $\beta_{n_i}.z_{n_i}$ converges to a non-zero distribution T_1 and then :

$$\beta_{n_i}.f_{n_i}$$

converges to T_1.

Let us denote by B the unit disk in $\mathbf{R}^2$, $S = \partial B$, if f is a continuous function on S then $N(f)$ is the number of changes of sign of the function f on S.

LEMMA 2. — *Let u be a harmonic function in B which is continuous in $\overline{B}$, $u\,|_S = f$, $u(0) = 0$. Let $N(f) = k < \infty$. Define*

$$G_u = \{x \in B\ ,\ u(x) > 0\}\ .$$

Then mes $G_u > C$, *where constant* $C > 0$ *is dependent on* k .

Proof. — Let us assume the contrary. This means that there exists a sequence of harmonic functions u_n in B, $u_n|_S = f_n$, $u_n(0) = 0$, $N(f_n) \le k$, mes $G_{u_n} \to 0$, $n \to \infty$. According to lemma 1 there exists a real valued sequence α_m and a subsequence f_{n_m} such that, $\alpha_m f_{n_m} \to \tilde{f} \not\equiv 0$ in the sense of distributions. From the convergence of the distributions $\alpha_m f_{n_m}$ on S it follows that in an arbitrary compact subdomain of B the convergence of functions $\alpha_m u_{n_m}$ is uniform. Let $\alpha_m u_{n_m} \to U$ in B. From [1] it follows that $U \not\equiv 0$ in B if $\tilde{f} \not\equiv 0$ on S. We have $U(0) = 0$. From the assumption mes $G_{u_n} \to 0$, $n \to \infty$, it follows that $U \le 0$ in B. Equality $U(0) = 0$ and inequality $U \le 0$ in B contradicts the maximum principle for harmonic functions.

Proof of the theorem.

1. Let us denote by B_r^x, $x \in M$, r, the geodesic circle on M with centre x and radius r.

There is a constant $C_0 > 0$, such that for every $\varepsilon > 0$ there exists points $x_1 \ldots x_N \in M$, $N > C_0/\varepsilon^2$, such that the circles $B_\varepsilon^{x_2} \ldots B_\varepsilon^{x_N}$ mutually have no intersections.

2. There exists a constant r_0, such that for every $x \in M, 0 < r < r_0$, B_r^x is diffeomorphic to a disk.

3. There is a constant $C_1 > 0$, such that for all $x \in M$, $\lambda > 0$ in the circle $B^x_{1/C_1\sqrt{\lambda}}$ there exists a positive solution of the equation $\Delta u + \lambda u = 0$.

4. Let $x \in M$, $\lambda > 0$, $r \doteq 1/C_1\sqrt{\lambda} < r_0$, u is a solution of the equation $\Delta u + \lambda u = 0$ in B_r^x. Then there exists a diffeomorphism h of the

unit disk B on B_r^x, $h(0) = x$, and a function s in B, $0 < s < \infty$, such that $s.u(h)$ is a harmonic function in B (by a representation theorem in quasiconformal mapping theory, [2]). From the compactness of M it follows that the Jacobian of the mapping h is uniformly bounded.

5. There is a constant $C_2 > 0$, such that for all $x \in M$, $\lambda > 0$ in the circle $B^x_{1/C_2\sqrt{\lambda}}$ every solution $u \not\equiv 0$ of the equation $\Delta u + \lambda u = 0$ changes its sign [3].

6. Let u_i be an eigenfunction, $\Delta u_i = \lambda_i u_i$ on M, γ is a nodal line of the function u_i. For a two-dimensional real analytic manifold the following estimate is true, [4],

$$\text{length } \gamma \le C_3\sqrt{\lambda_i}$$

where constant $C_3 > 0$ is dependent on M.

7. Let u_i be an eigenfunction, $\Delta u_i = \lambda_i u_i$ on M. According to 1 we can choose circles $B_\varepsilon^{x_1} \dots B_\varepsilon^{x_n}$ with $\varepsilon = 2/C_2\sqrt{\lambda_i}$. We have $N > C_0 C_2^2 \lambda_i/4$.

According to 5 there exist points $y_n \in B^{x_n}_{\varepsilon/2}$, $n = 1 \dots N$, such that $u_i(y_n) = 0$.

According to 6 at least $N/2$ points $y_{k_1} \dots y_{k_J}$, $J > N/2$, from the set $\{y_n\}$ have the following property : for all $j = 1 \dots J$ there exist r_j,

$$\frac{1}{2C_1\sqrt{\lambda_i}} < r_j < \frac{1}{C_1\sqrt{\lambda_i}} ,$$

such that restriction of the function u_j on $\partial B_{r_j}^{y_{k_j}}$ has no more than

$$\frac{8C_1C_3}{C_2^2 C_0}$$

zeros.

According to 4 and lemma 2 for all $j = 1 \dots J$

$$\text{mes}\{x \in B_{r_j}^{y_{k_j}} \ , \ u_i(x) > 0\} > C_4\varepsilon^2 .$$

We have $J > C_0/2\varepsilon^2$ and so the theorem is proved.

2. An estimate of the relation between the positive and the negative extrema of an eigenfunction.

Let M be a n-dimensional compact smooth Riemannian manifold, $u_1, u_2, \dots$-eigenfunctions of the Laplace operator on M, $\Delta u_i = \lambda_i u_i$.

THEOREM 2. — *There exists a positive constant C which depends only on n and a positive constant N which depends on M such that, for every $i > N$,*

$$\frac{1}{C} < \frac{\sup\limits_{M} u_i(x)}{|\inf\limits_{M} u_i(x)|} < C .$$

We denote by $B_r \subset \mathbb{R}^n$ the ball centered at 0 of radius r.

In B_r we consider a uniformly elliptic second order operator L defined by

$$L = \sum_{i,j} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}) , \tag{2.1}$$

where a_{ij} is a symmetric positive definite matrix in B_r. If the eigenvalues of the matrix $\|a_{ij}(x)\|$ lie on the segment $[e^{-1}, e]$, $e \geq 1$ we say that the operator L has an ellipticity constant e.

LEMMA 3. — *Let u be a solution of the equation*

$$a(x)Ly + \lambda u = 0$$

in the ball B_1, $1/A < a(x) < A$, $A > 0$, L is an elliptic operator with the ellipticity constant e, λ is a constant such that $|\lambda| < C$. Let us assume that $u(x_0) > 0$ and that there exists $x_0 \in B_{1/2}$ with $u(x_0) = 0$. Then

$$|\inf_{B_1} u| > \delta u(0) ,$$

where the constant $\delta > 0$, $\delta = \delta(n, A, e, C)$.

Proof.

1. We shall prove Lemma 2 under the assumption that $\lambda = 0$. Denote

$$\varphi_1 = \sup\{0, u\,|_{\partial B_1}\}$$
$$\varphi_2 = \inf\{0, u\,|_{\partial B_1}\} .$$

Let u_1, u_2 be the solutions of the following Dirichlet problems :

$$Lu_1 = 0 \quad \text{in } B_1 ,\ u_1\,|_{\partial B_1} = \varphi_1,$$
$$Lu_2 = 0 \quad \text{in } B_2 ,\ u_2\,|_{\partial B_1} = \varphi_2.$$

Then, $u = u_1 + u_2$, $u_1 > 0$ in B_1, $u_1(0) \geq u(0)$. From the Harnack inequality [5] it follows that there exists a constant $\delta > 0$, $\delta = \delta(n, e)$ such that

$$u_1\,|_{B1/2} > \delta u_1(0) .$$

Since $u(x_0) = 0$, $x_0 \in B_{1/2}$, then

$$\inf \varphi_2 < -\delta u_1(0) < -\delta u(0) .$$

2. Let $\lambda \notin 0$. Let us make a cylindric extension of the functions $u(x), a(x)$ and the operator L in the new coordinate x_{n+1}. After this extension we shall keep the notations u, a, L. Denote

$$v = ue^{\sqrt{\lambda}\, x_{n+1}}$$

clearly the function v is a solution of the elliptic equation

$$aLv + \frac{\partial^2 v}{\partial x_{n+1}^2} = 0 .$$

Now the statement of Lemma 3 follows from the assertion 1 to the function v in the unit ball in $\mathbb{R}^{n+1}$.

Proof of Theorem 2.

1. There are constant $C_1 = C_1(M) > 0$, $C_2 = C_2(M) > 0$ such that for all $x \in M$, $\lambda > C_2$ any solution of the equation $\Delta u + \lambda u = 0$ in the ball $B^x_{C_1/\sqrt{\lambda}}$ change its sign.

2. There exists a constant $N > C_2$, $N = N(M)$, such that for all $x \in M$ there exists a diffeomorphism

$$d : B^x_{2C_1/\sqrt{\lambda}} \subset M \to B_1 \subset \mathbb{R}^n$$

such that the equation $\Delta u + \lambda u = 0$ in $B^x_{2C_1/\sqrt{\lambda}}$ viewed in the ball B_1 has the form

$$a(x)Lu + \lambda' u = 0 \tag{2.2}$$

where L is an elliptic operator of the type (2.1), $e = 2$, $A = 2$, $|\lambda| < C = C(n) > 0$. We can obtain such a diffeomorphism d if we introduce in the ball $B^x_{2C_1/\sqrt{\lambda}}$ a normal coordinate system. Applying Lemma 3 to the solution u of the equation (2.2) we obtain the statement of Theorem 2.

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