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SMOOTHING OF REAL ALGEBRAIC HYPERSURFACES BY RIGID ISOTOPIES

by Alexander NABUTOVSKY

1. MAIN RESULTS

We consider in this paper compact smooth hypersurfaces of Euclidean spaces which can be represented as the zero set of some polynomial p of degree d such that $\mathrm{grad}\, p$ and p do not vanish simultaneously. We will call such hypersurfaces *d-hypersurfaces.* Two d-hypersurfaces are called *rigidly isotopic* if there exists a connecting isotopy which passes only through d-hypersurfaces. This notion was generalized in [N1], [N2]. Namely, two d-hypersurfaces are called D-rigidly isotopic ($D \geq d$) if there exists a connecting isotopy which passes only through $(D-i)$-hypersurfaces, where $i \geq 0$.

Rigid isotopies were studied (mostly in projective case) in works of Rokhlin, Viro, Kharlamov, Marin, Finashin and others (*cf.* surveys [R], [Vi] and references there). The main goal there was the complete description of rigid isotopy classes of algebraic curves and surfaces for small values of the degree d. Here we prove a result of a different type. Informally, it can be described as follows.

We consider two functionals which characterize a "badness" of embedding of a smooth hypersurface in the Euclidean space. Informally speaking, these functionals assume large values if the hypersurface has a large principal curvature at some point or if it comes close to itself. Suppose a

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d-hypersurface Σ^n in $\mathbf{R}^{n+1}$ is given and we are looking for a d-hypersurface Σ^n_* rigidly isotopic to Σ^n and embedded in $\mathbf{R}^{n+1}$ as "nicely" as possible. This means that we try to minimize the values of both functionals mentioned above on Σ^n_*. We prove here a priori upper bounds on the values of the functionals at Σ^n_* in terms of d for an arbitrary fixed n. Thus, we prove a quantitative version of the following almost obvious statement : although Σ^n can be embedded arbitrary badly, we can guarantee the existence of a d-hypersurface rigidly isotopic to it which is embedded nicely enough.

More precisely, let $r(\Sigma^n)$ denote the distance from Σ^n to its central set Center(Σ^n) (which is defined as the union of the focal point set and the set of points $y \in \mathbf{R}^{n+1}$ such that the minimum of the distance between y and a point x of Σ^n, $\min_{x\in\Sigma^n} |x-y|$, is attained at more than one point x). We refer to [LF1], [LF2], [M], [MW], [VEL] for some properties of central sets. Define *the crumpleness* of Σ^n, $\kappa(\Sigma^n)$, as the ratio $\mathrm{diam}_{\mathbf{R}^{n+1}}(\Sigma^n)/r(\Sigma^n)$. Some properties of the crumpleness functional are discussed in [N2]. We introduce also a functional $\mu(\Sigma^n) = \max_{x\in\Sigma^n,\, i\in\{1,\dots,n\}} (|k_i(x)|)(\mathrm{Vol}(\Sigma^n))^{\frac{1}{n}}$, where $k_i(x)$ are principal curvatures of Σ^n at x. Note that κ and μ are invariant under transformations $x \to \alpha x + b$ of $\mathbf{R}^n$.

THEOREM 1.1. — *For any n there exist constants $c(n)$ and $\alpha(n)$ such that for any compact d-hypersurface $\Sigma^n \subset \mathbf{R}^{n+1}$ there exists a d-hypersurface Σ^n_* rigidly isotopic to it such that*

$$\kappa(\Sigma^n_*) \leq \exp(c(n)d^{\alpha(n)d^{n+1}}), \tag{1.1}$$

$$\mu(\Sigma^n_*) \leq \exp(c(n)d^{\alpha(n)d^{n+1}}). \tag{1.2}$$

Remark. — From (1.1), (1.2) one can derive estimates on another global geometric characteristics of Σ^n_* invariant under rescaling. For example, let $i(\Sigma^n)$ be the injectivity radius of Σ^n_* in the inner metric. Then (1.1), (1.2) and the inequality (10) in [BZ], Corollary 34.1.9, imply that

$$\mathrm{vol}(\Sigma^n_*)^{\frac{1}{n}}/i(\Sigma^n_*) \leq \exp(\nu(n)\exp(d^{\alpha(n)d^{n+1}})),$$

(perhaps one can derive a better estimate).

Let us describe the idea of the proof of Theorem 1.1. First, we show that it is sufficient to find a hypersurface Σ^n_* isotopic to Σ^n, which is a zero set of a polynomial q of degree d such that

$$\max_{x\in\mathrm{Conv}(\Sigma^n_*)} \|\,\mathrm{Hess}\, q(x)\|/\min_{x\in\Sigma^n_*} |\,\mathrm{grad}\, q(x)| \leq \exp(c_0(n)d^{\alpha_0(n)d^{n+1}}), \tag{1.3}$$

where $c_0(n)$ and $\alpha_0(n)$ are some constants depending only on n and $\mathrm{Conv}(\Sigma^n_*)$ denotes the convex hull of Σ^n_* in $\mathbf{R}^{n+1}$.

Consider in the space of coefficients the discriminant variety which corresponds to polynomials p such that $\mathrm{grad}\, p$ vanishes at a point of the zero set of p. It is contained in a possibly larger algebraic variety which is the zero set of a polynomial T. This polynomial is defined as the sum of squares of resultants of a system of $n+2$ homogeneous forms of $n+2$ variables (these forms are the projectivisations of polynomials p, $\partial p/\partial x_i$, $i = 1, \ldots, n+1$, *cf.* [VW]). The key moment in the proof is that the polynomial T is a polynomial with integer coefficients which can be estimated. This enables us to prove that every connected component of the complement to the zero set of T contains a ball of a large enough radius having a non-empty intersection with the unit ball. Note that if two vectors of coefficients belong to the same connected component of the complement of the zero set of T then the corresponding zero sets are rigidly isotopic (any path connecting two vectors inside the component corresponds to a connecting rigid isotopy). Thus, any d-hypersurface is rigidly isotopic to a zero set of some polynomial q of degree d with rational coefficients such that numerators and denominators of all the coefficients can be bounded by some bounds depending only on n and d. It is obvious from the finiteness of the set of such polynomials that there exist an upper bound depending only on n and d on $\max_{x \in \mathrm{Conv}(\Sigma^n_*)} \|\mathrm{Hess}\, q(x)\| / \min_{x \in \Sigma^n_*} |\mathrm{grad}\, q(x)|$, where Σ^n_* is the zero set of q. Such a bound follows from a result of Vorobjov [V] (see also [GV]). (The proof of this result uses Lazard's work on the u-resultant [L]). This bound coincides with (1.3). This completes the proof.

As an application we prove the following result.

THEOREM 1.2. — *There exist such constants c, β that any two isotopic compact plane non-singular algebraic curves of degree d are $\lfloor \exp(cd^{\beta d^2}) \rfloor$-rigidly isotopic.*

Note that it was proven in [N1], [N2] that for any n there exists a function $A_n(d)$ with the following property : any two isotopic d-hypersurfaces in $\mathbf{R}^{n+1}$ are $A_n(d)$-rigidly isotopic. However, for any $n \geq 5$ all such functions are non-recursive. (Moreover, they will be still non-recursive if we will consider only hypersurfaces isotopic to S^n). So, there is no analogue of Theorem 1.2 in the case of arbitrary dimension.

We would like to mention here also an application of Theorem 1.1 to a question posed in [ABB], p. 156. Namely, Theorem 1.1 can be used to majorize the minimal number of simplices of a C^∞-smooth triangulation of a d-hypersurface in $\mathbb{R}^{n+1}$ in terms of d and n only. Indeed, the minimal number of simplices of a C^∞-triangulation of a submanifold is invariant under isotopies of the submanifold. Hence Theorem 1.1 permits to reduce the case of an arbitrary triangulated d-hypersurface Σ^n to the case of a d-hypersurface Σ^n_* satisfying inequalities (1.1), (1.2). But for such hypersurfaces it is easy to derive an explicit upper bound on the minimal number of simplices of a C^∞-triangulation.

Indeed, one cas use Whitney's proof of the triangulability of smooth manifolds ([Wh], ch. IV). On the first stage of the proof a subdivision L_0 of $\mathbb{R}^{n+1}$ into cubes of a small enough side length h is chosen. Then the regular subdivision L of L_0 into simplices is considered. One can slightly move the vertices of L forming a new triangulation L^* of $\mathbb{R}^{n+1}$ which will be "in general position" with respect to Σ^n_*. In particular, all vertices of L^* will be at a certain positive distance from Σ^n_*. Since all one-dimensional simplices of L^* have small enough length, one can ensure that any one-dimensional simplex of L^* intersecting Σ^n_* intersects it in a unique point, at an angle not too small. The intersections of Σ^n_* with the simplices of L^* are approximately convex cells and the desired simplicial complex K (homeomorphic to Σ^n_*) is like the regular subdivision of this set of cells. It is clear from this proof of Whitney that the number of simplices of $K \leq \mathrm{const}_1(n)(\mathrm{diam}_{\mathbb{R}^{n+1}}(\Sigma^n_*)/h)^{n+1}$, where $\mathrm{const}_1(n)$ depends only on n. The length h is defined by formula (5) on p. 129 of [Wh]. One can see from this formula that $h \geq \mathrm{const}_2(n)\min\{\delta_0, \xi_0, \xi_1\}$, where $\mathrm{const}_2(n)$ depends only on n and δ_0, ξ_0, ξ_1 are defined, respectively, in texts of Theorem 10.A, Lemma 8a and Lemma 8b of Chapter IV of [Wh]. Since Σ^n_* is a C^∞-smooth hypersurface, δ_0 coincides with $r(\Sigma^n_*)$. Not entering into details, note that using a quantitative version of the implicit function theorem (*cf.* [N3], Proposition 3.2, or [AMR], Supplement 2.5A) and the inequality $r(\Sigma^n_*) \leq 1/\max_{x\in\Sigma^n_*, i\in\{1,\dots,n\}} |k_i(x)| (= 1/\max_{x\in\Sigma^n_*} \|W(x)\|)$, one can prove that $\min\{\xi_0, \xi_1\} \geq \mathrm{const}_3(n) r(\Sigma^n_*)$. Here, as before, $k_i(x)$ are principal curvatures of Σ^n_* at a point x and $W(x)$ denotes the second fundamental form of Σ^n_* at x. This argument implies a polynomial in $\kappa(\Sigma^n_*)$ upper bound for the number of simplices of the triangulation. This method together with Theorem 1.1 leads to a doubly exponential in d upper bound for the number of simplices. Thus, we come to the following corollary of Theorem 1.1 :

COROLLARY. — *For any n there exists a constant $\gamma(n)$ with the* *following property. Any d-hypersurface in* $\mathbb{R}^{n+1}$ *admits a* C^∞*-smooth triangulation into not more than* $\lfloor \exp(d^{\gamma(n)d^{n+1}}) \rfloor$ *simplices.*

Estimates provided by Theorem 1.1 are doubly exponential in d. This is related to the fact that there are $\binom{n+d}{n}$ monomials in a polynomial of degree d of n variables. If, however, we consider d-hypersurfaces Σ^n representable as zero sets of a polynomial p of degree d ($\operatorname{grad} p \neq 0$ on Σ^n) such that only $\ell(n)$ monomials of p have non-zero coefficients, then the estimates can be made only exponential in d. (Here $\ell(n)$ is an arbitrary function of n not depending on d). Moreover, it is sufficient to require only that the remaining $\binom{n+d}{n} - \ell(n)$ coefficients of p will be integer numbers from an fixed interval $[-M, M]$.

THEOREM 1.3. — *For any positive integer n there exist constants $c(n)$, $\alpha(n)$ with the following property. Let some integer numbers ℓ and $M \geq 2$ be given. Suppose that $p : \mathbb{R}^{n+1} \to \mathbb{R}$ is a polynomial such that $\binom{n+d+1}{n+1} - \ell$ of its coefficients are integer numbers from the interval $[-M, M]$. Let Σ^n be the (non-empty) zero set of p and* $\operatorname{grad} p(x) \neq 0$ *for any $x \in \Sigma^n$. There exists a d-hypersurface Σ^n_* rigidly isotopic to Σ^n such that*

$$\kappa(\Sigma^n_*) \leq \exp(c(n)(d^{\ell+1} \ln M)^{\alpha(n)}),$$
$$\mu(\Sigma^n_*) \leq \exp(c(n)(d^{\ell+1} \ln M)^{\alpha(n)}).$$

As a corollary from Theorem 1.3 one can prove the following generalisation of Theorem 1.2.

THEOREM 1.4. — *There exist constants c, β with the following property. Suppose that some integer numbers ℓ and $M \geq 2$ are given. Let $p_1, p_2 : \mathbb{R}^2 \to \mathbb{R}$ be polynomials such that for $i \in \{1, 2\}$:*

1) *All coefficients of p_i but ℓ are integer numbers from the interval $[-M, M]$;*

2) *The zero set of p_i is a compact smooth curve Σ^1_i and* $\operatorname{grad} p_i(x) \neq 0$ *for any $x \in \Sigma^1_i$;*

and

3) *Σ^1_1 is isotopic to Σ^1_2.*

Then Σ^1_1 is $\lfloor \exp(c(d^{\ell+1} \ln M)^{\beta}) \rfloor$-rigidly isotopic to Σ^1_2.

Note that the method of the proof of Theorem 1.1 can be easily generalized for the case of greater codimensions and a multi-codimensional analogue of Theorem 1.1 can be formulated. We believe that the proposed method using the fact that coefficients of the resultant are integers and Vorobjov's estimates of sizes of compact zero sets of multivariate polynomials with bounded integer coefficients can be widely applied in the following situation. Suppose that we want to prove the existence of an algebraic submanifold with some prescribed topological properties by a desingularization by a small variation of parameters (using, for example, Sard's theorem to prove the existence of such variation). Upper bounds for a geometric complexity of some such submanifold (for example, for principal curvatures) can be obtained by this method. We would like also to mention that the result of Vorobjov was used in a somewhat similar manner to a problem from the geometry of point configurations in the recent article [GPS].

Note also that Theorems 1.1–1.4 can be similarly stated and proven in the projective case.

2. Geometry of algebraic hypersurfaces.

Here we relate defined in Section 1 geometric characteristics of the zero set of a polynomial to its analytic characteristics.

Let a compact hypersurface Σ^n be the zero set of a polynomial $p : \mathbf{R}^{n+1} \to \mathbf{R}$ such that $\operatorname{grad} p(x) \neq 0$ for $x \in \Sigma^n$. Denote by $a(p)$ the ratio

$$\max_{x\in \mathrm{Conv}(\Sigma^n)} \|\operatorname{Hess} p(x)\| / \min_{x\in\Sigma^n} |\operatorname{grad} p(x)|.$$

LEMMA 2.1. — $r(\Sigma^n) \geq \dfrac{1}{2a(p)}$.

Proof. — Suppose $r(\Sigma^n) = \mathrm{dist}(x_0, y_0)$, where $x_0 \in \Sigma^n$, $y_0 \in \mathrm{Center}(\Sigma^n)$. There are two possibilities.

1) y_0 is a focal point of Σ^n. Then $r(\Sigma^n) = 1/|k(x_0)|$, where $|k(x_0)|$ is the maximal absolute value of a principal curvature at x_0. But the principal curvatures at x_0 are the eigenvalues of the matrix of the second fundamental form of Σ^n at x_0. It is well-known (*cf.* [T]) that the linear operator corresponding to the second fundamental form at x_0 can be expressed

as $-\dfrac{1}{|\operatorname{grad} p(x_0)|} \operatorname{Hess} p(x_0)|_{T\Sigma^n(x_0)}$, where $T\Sigma^n(x_0)$ denotes the tangent hyperplane to Σ^n at x_0. This implies immediately the lemma in the first case.

2) There are two different points $x_1, x_2 \in \Sigma^n$ such that

$$|y_0 - x_1| = |y_0 - x_2| = \operatorname{dist}(y_0, \Sigma^n) = r(\Sigma^n).$$

This means that

$$y_0 = x_1 + r(\Sigma^n)n(x_1) = x_2 + r(\Sigma^n)n(x_2).$$

Here $n(x_1), n(x_2)$ are unit normal vectors to Σ^n at x_1 and at x_2, correspondingly. Note that in the intersection of the open $1/(2a(p))$-neighborhood $N^{n+1}_{1/(2a(p))}$ of Σ^n with $\operatorname{Conv}(\Sigma^n)$

$$|\operatorname{grad} p(x)| \geq \frac{1}{2} \min_{x \in \Sigma^n} |\operatorname{grad} p(x)|.$$

Extend the map $n : \Sigma^n \to S^n$ on $N^{n+1}_{1/(2a(p))} \cap \operatorname{Conv}(\Sigma^n)$ letting $n(x) = \dfrac{\operatorname{grad} p(x)}{|\operatorname{grad} p(x)|}$. Differentiating $n(x)$ we get easily that $\|Dn(x)\| \leq 2a(p)$. Hence, if $|x_1 - x_2| < 1/a(p)$ then the segment $[x_1, x_2]$ is contained in $N^{n+1}_{1/2a(p)} \cap \operatorname{Conv}(\Sigma^n)$ and

$$|x_1 - x_2| = r(\Sigma^n)|n(x_2) - n(x_1)| \leq 2a(p)r(\Sigma^n)|x_1 - x_2|.$$

So,

$$r(\Sigma^n) \geq \frac{1}{2a(p)}.$$

If $|x_1 - x_2| \geq 1/a(p)$ then also

$$1/a(p) \leq |x_1 - x_2| = r(\Sigma^n)|n(x_2) - n(x_1)| \leq 2r(\Sigma^n).$$

$\square$

For a set $X \subset \mathbf{R}^{n+1}$ denote by X_ρ the set of all points $x \in \mathbf{R}^{n+1}$ such that $\operatorname{dist}(x, X) \leq \rho$. For a hypersurface M^n let M_- denote the union of the bounded connected components of the complement $\mathbf{R}^{n+1} \backslash M^n$.

LEMMA 2.2. — *For any n there exists $\theta(n) > 0$ with the following property. Let $\rho \leq r(\Sigma^n)$. Then*

$$\operatorname{Vol}_{n+1}\left(\Sigma^n_\rho \cap \Sigma_-\right) \geq \rho\theta(n) \operatorname{Vol}_n(\Sigma^n).$$

Proof. — Using the transformation $x \to x/\rho$ we can reduce the proof of lemma to the proof of the particular case of $\rho = 1$. Denote for any $\lambda \in [0,1]$ the hypersurface $\{x + \lambda n(x) | x \in \Sigma^n,\ n(x)$ is the unit normal vector to Σ^n at x directed towards $\Sigma_-\}$ by $\Sigma^n(\lambda)$. By the Cavaliery principle

$$\begin{aligned} \mathrm{Vol}_{n+1}\left(\Sigma_1^n \cap \Sigma_-\right) &= \int_0^1 \mathrm{Vol}_n(\Sigma^n(\lambda))d\lambda \\ &= \int_{\Sigma^n} \int_0^1 \det(I - \lambda W(x))d\lambda d\,\mathrm{Vol}_n(\Sigma^n), \end{aligned} \tag{2.1}$$

where $W(x)$ denotes the matrix of the second fundamental form of Σ^n at $x \in \Sigma^n$. Consider $\det(I - \lambda W(x))$ as a polynomial of λ. The condition $\rho \leq r(\Sigma^n)$ implies that this polynomial assumes positive values for $\lambda \in [0,1)$. This polynomial is equal to one at zero and its degree is less or equal to n. Let $U(n)$ be the set of all polynomials q of degree n such that $q(0) = 1$ and $q(\lambda) \geq 0$ for all $\lambda \in [0,1]$. It is easy to see that $\theta_0(n) \equiv \inf_{q \in U(n)} \|q\|_{L^1[0,1]} > 0$. Indeed, let $U_{\geq 1}(n)$ denote the set of polynomials q of degree n such that the norm of the vector of coefficients of q is greater or equal to one. Let $U_1(n)$ denote the set of polynomials q of degree n such that the norm of the vector of coefficients of q is equal to one. Obviously, $U(n) \subset U_{\geq 1}(n)$. So,

$$\inf_{q \in U(n)} \|q\|_{L^1[0,1]} \geq \inf_{q \in U_{\geq 1}(n)} \|q\|_{L^1[0,1]} = \inf_{q \in U_1(n)} \|q\|_{L^1[0,1]} > 0.$$

This proves the positivity of $\theta_0(n)$. Now it follows from (2.1) that

$$\mathrm{Vol}_{n+1}\left(\Sigma_1^n \cap \Sigma_-\right) \geq \theta_0(n)\,\mathrm{Vol}_n(\Sigma^n)$$

(recall that we consider only the case $\rho = 1$). Thus, Lemma 2.2 holds with $\theta(n) = \theta_0(n)$. □

Denote the volume of a unit n-dimensional ball by v_n. The following lemma relates introduced in Section 1 functionals κ and μ.

LEMMA 2.3. — *For any smooth compact hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ the following inequality takes place :*

$$\mu^n(\Sigma^n) \leq \frac{v_{n+1}}{\theta(n)} \kappa^{n+1}(\Sigma^n).$$

Proof. — By its definition Σ_- is contained in a ball of radius $\mathrm{diam}_{\mathbb{R}^{n+1}} \Sigma^n$. Let $\rho = r(\Sigma^n)$. By Lemma 2.2

$$v_{n+1}\left(\mathrm{diam}_{\mathbb{R}^{n+1}} \Sigma^n\right)^{n+1} \geq \mathrm{Vol}_{n+1}\left(\Sigma_\rho^n \cap \Sigma_-\right) \geq \theta(n)\rho\,\mathrm{Vol}_n(\Sigma^n).$$

Now Lemma 2.3 follows from the definitions of μ and κ. □

Remark. — Actually the proof of Lemma 2.3 gives even more. Namely, we majorized $(\mathrm{Vol}(\Sigma^n))^{1/n}/r(\Sigma^n)$ in terms of $\kappa(\Sigma^n)$. This result can be generalized for codimensions greater than one. One can also majorize $\kappa(\Sigma^n)$ in terms of $(\mathrm{Vol}(\Sigma^n))^{1/n}/r(\Sigma^n)$. (The exact formulations and proofs of the two last statements will appear elsewhere.) Thus, the normalizations of $1/r(\Sigma^n)$ by factors $\mathrm{diam}_{\mathbf{R}^{n+1}}(\Sigma^n)$ and $\mathrm{Vol}(\Sigma^n)^{1/n}$ are in some sense equivalent. However, it is impossible to majorize $\mu(\Sigma^n)$ in terms of $\mathrm{diam}_{\mathbf{R}^{n+1}}(\Sigma^n) \max_{x\in\Sigma^n, i\in\{1,2,\ldots,n\}} |k_i(x)|$. Indeed, there exist simple examples showing that it is impossible to majorize the volume of Σ^n in terms of $\mathrm{diam}_{\mathbf{R}^{n+1}}(\Sigma^n)$ and of $\max_{x\in\Sigma^n, i\in\{1,2,\ldots,n\}} |k_i(x)|$, even if $n=1$. (This fact was pointed out to the author by Professor M. Gromov).

3. Zero sets of polynomials with integer coefficients.

An important ingredient in the proof of Theorem 1.1 is the following observation. Suppose that a polynomial $p:\mathbb{R}^n\to\mathbb{R}$ of degree d has integer coefficients and absolute values of these coefficients are bounded by some number N. Let the zero set Z of this polynomial be a compact non-empty set. Then Z belongs to the ball of radius $r(N,d,n)$ centered at the origin of $\mathbb{R}^n$ for some $r(N,d,n)$ depending only on N, d and n. This follows immediately from the finiteness of the set of considered polynomials. The following result of Vorobjov [V] provides an upper bound for $r(N,d,n)$.

THEOREM 3.1 [V]. — *There exist constants* $\mathrm{const}_1, \mathrm{const}_2, \mathrm{const}_3$ *such that for any* $d\geq 2$, $N\geq 2$, n *the following inequality holds :*

$$r(N,d,n)\leq \exp\left[\mathrm{const}_1 \binom{nd}{n}^{\mathrm{const}_2} (\ln N)^{\mathrm{const}_3}\right].$$

Proof. — This theorem is a reformulation of the second statement of Theorem 3 in [V].

The next lemma is a simple corollary of Theorem 3.1. It provides a positive lower bound for $\min_{\Sigma^n} |\operatorname{grad} p|$.

LEMMA 3.2. — *Let* $\Sigma^n = \{x\in\mathbb{R}^{n+1} | p(x)=0\}$ *be a compact hypersurface and* $\operatorname{grad} p(x)\neq 0$ *for any* $x\in\Sigma^n$. *Suppose that all coefficients of* p *are integer and their absolute values do not exceed some given number*

$N \geq 2$. Then for some constants const_4, const_5 *not depending of N, d, p, n the following inequality holds :*

$$\min_{x\in\Sigma^n} |\operatorname{grad} p(x)| \geq \exp\Big(-\text{const}_4 \Big[\binom{nd}{n}\ln N\Big]^{\text{const}_5}\Big).$$

Proof. — Consider the following system of two algebraic equations :

$$p(x_1,\ldots,x_{n+1}) = 0$$
$$x_{n+2}^2|\operatorname{grad} p(x_1,\ldots,x_{n+1})|^2 - 1 = 0.$$

The statement of Lemma 3.2 is equivalent to the statement that for any solution $(x_1,\ldots,x_{n+1},x_{n+2})$ of this system

$$|x_{n+2}| \leq \exp\Big(\text{const}_4\Big(\binom{nd}{n}\ln N\Big)^{\text{const}_5}\Big)$$

for some constants const_4, const_5. But this system can be replaced by a single equation

$$p^2(x_1,\ldots,x_{n+1}) + \Big(x_{n+2}^2\sum_i\Big(\frac{\partial p}{\partial x_i}(x_1,\ldots,x_{n+1})\Big)^2 - 1\Big)^2 = 0.$$

Note that the set of solutions of this equation is bounded. (The existence of an upper bound on x_{n+2}^2 follows from the fact that $|\operatorname{grad} p|^2$ has a positive minimum on Σ^n.) Now Lemma 3.2 follows from Theorem 3.1.

Remark. — The idea to use a new variable x_{n+2} in the proof is an adaptation of a similar idea from the proof of Corollary 3 in [V].

The proof of the following lemma is similar to the proof of Lemma 3.2.

LEMMA 3.3. — *Let $p : \mathbb{R}^{n+1} \to \mathbb{R}$ be a non-trivial homogeneous polynomial of degree d with integer coefficients from the interval $[N,-N]$. Let $p|_{S^n}$ denote its restriction on the unit hypersphere S^n centered at the origin and p_0^2 be a locally maximal value of $p^2|_{S^n}$. Then for some constants* const_6, const_7 *the following inequality holds :*

$$p_0^2 \geq \exp\Big(-\text{const}_6\Big(\binom{nd}{n}\ln N\Big)^{\text{const}_7}\Big).$$

Proof. — Let the value p_0^2 be assumed at some point $x \in S^n$. Denote p_0^{-1} by x_{n+2}. Then coordinates of the point x and x_{n+2} satisfy the following equation :

$$\Big(\sum_{i=1}^{n+1} x_i^2 - 1\Big)^2 + \sum_{i=1}^{n+1}\Big(\frac{\partial p}{\partial x_i}(x_1,\ldots,x_{n+1}) - 2\lambda x_i\Big)^2 + (x_{n+2}p(x_1,\ldots,x_{n+1}) - 1)^2 = 0. \tag{3.1}$$

Here λ is, of course, a Lagrange multiplier. We consider it as an additional variable and want to apply Theorem 3.1 to the polynomial Q of $n+3$ variables at the left hand side of (3.1). The absolute values of (n+2)-th coordinate of solutions of the equation $Q = 0$ are uniformly bounded because of the finiteness of the set of critical values of $p|_{S^n}$. It is easy also to verify the boundedness of λ on the solution set of $Q = 0$. So, Theorem 3.1 can be applied. It provides an upper bound on x_{n+2}^2 which by the definition of x_{n+2} yields a lower bound on p_0^2. It is easy to check that this is the necessary bound. □

We will need also the following proposition which is a well-known corollary of the elimination theory (cf. [VW], Ch. 11).

PROPOSITION 3.4. — *Consider a space of coefficients of all polynomials $p : \mathbb{R}^{n+1} \to \mathbb{R}$ of degree d (i.e. the Euclidean space $\mathbb{R}^k$, where $k = \binom{n+1+d}{n+1}$). There exists a homogeneous polynomial $T : \mathbb{R}^k \to \mathbb{R}$ with integer coefficients such that :*

1) *If $T(a) \neq 0$ then for a polynomial p having the vector of coefficients a p and* grad p *do not vanish simultaneously.*

2) *The zero set of T is of codimension one in $\mathbb{R}^k$.*

3) *The degree of T is less than* $(\text{const}_8\, d)^{2^n}$. *Absolute values of all coefficients of T are less than* $\exp((\text{const}_9\, d)^{2^n})$. *Here* const_8 *and* const_9 *are some universal constants.*

Proof. — Consider the system of $(n+2)$ equations

$$\begin{aligned} p(x_1,\dots,x_{n+1}) &= 0 \\ \frac{\partial p}{\partial x_1}(x_1,\dots,x_{n+1}) &= 0 \\ &\dots \\ \frac{\partial p}{\partial x_{n+1}}(x_1,\dots,x_{n+1}) &= 0 \end{aligned}$$

where p is the polynomial with indeterminate coefficients a_i $(i = 1,\dots,k)$. Homogenizing it and excluding all variables x_j, $(j = 1,\dots n+1)$ ([VW], ch. 11), we obtain a resultant system of homogeneous polynomials b_ℓ of variables a_i with integer coefficients. The simultaneous vanishing of all the polynomials b_ℓ on a vector of coefficients a_0 of a polynomial p_0 is equivalent to the simultaneous vanishing of homogenizations of p and $\operatorname{grad} p$ on a non-zero complex $n+2$ dimensional vector of variables. Let $T = \sum_{\ell} b_\ell^2$. Now the properties 1) and 2) are evident. To demonstrate the property 3) it is sufficient to follow the computation of the polynomials b_ℓ in [VW].

Remark. — The statement of Proposition 3.4 can be significantly improved. It is well-known that exists a polynomial of degree $n(d-1)^{n-1}$ called the *discriminant* of p satisfying statements 1) and 2) of Proposition 3.4 (*cf.* [VW], ch. 11 and [GZK]). However, coefficients of this polynomial cannot be estimated so straightforwardly as coefficients of the polynomial T. Also, it is not so simple to generalize the notion of discriminant for polynomial maps $\mathbb{R}^{n+k} \to \mathbb{R}^k$ (but see [L]). Nevertheless, if one wants to obtain an estimate of $\alpha(n)$ in Theorem 1.1 then it is better to use the discriminant instead of the polynomial T.

4. Proof of main results.

In this section we present proofs of Theorems 1.1, 1.2.

Proof of Theorem 1.1. — Consider the polynomial T on the space of coefficients of polynomials $p : \mathbb{R}^{n+1} \to \mathbb{R}$ of degree d which was defined by Proposition 3.4. Its follows from Proposition 3.4 that if vectors of coefficients of two polynomials p_1 and p_2 are in the same connected component of the complement to the zero set of T then zero sets of p_1 and p_2 are rigidly isotopic d-hypersurfaces. Without any loss of generality we can assume that Σ^n is the zero set of a polynomial p such that the vector a of coefficients of p satisfies $|a| = 1$ and $T(a) \neq 0$. Denote by W a connected component of the complement of the zero set of T containing a. Our first aim is to prove the existence in W of a point a_* with rational coordinates such that numerators and denominators of coordinates can be bounded by $\exp(c_3(n)d^{c_4(n)d^{n+1}})$, where $c_3(n)$, $c_4(n)$ depend only on n.

Consider the restriction of T on the unit sphere S^{k-1} centered at the origin of the coefficient space $\mathbb{R}^k$. Let $V = W \cap S^{k-1}$ and t_0 denote a locally maximal value of $|T|$ on V. Suppose this value is attained at a point $a_1 \in V$. Lemma 3.3 implies that $t_0 \geq \exp(-c_5(n)d^{c_6(n)d^{n+1}})$, where $c_5(n)$, $c_6(n)$ depend only on n. The dependence on d is doubly exponential here because of the fact that the dimension k depends on d here. It follows easily from Proposition 3.4, part 3) that $|\operatorname{grad} T(a)| \leq \exp(c_7(n)d^{c_8(n)})$ when $|a| \leq 2$

for some $c_7(n)$, $c_8(n)$. Thus, there exists a ball $B_0^k \subset W$ centered at a_1 of radius $\exp(-c_9(n)d^{c_{10}(n)d^{n+1}})$ for some $c_9(n)$, $c_{10}(n)$. Obviously, there exists a point a_* in this ball B_0^k such that all its coordinates are rational numbers with a numerator and a denominator bounded by $\exp(c_{11}(n)d^{c_{10}(n)d^{n+1}})$ for some $c_{11}(n)$. This completes the first stage of the proof.

Let p_* be a polynomial with the vector of coefficients a_* and Σ_*^n be its zero set. Our next aim is to prove that Σ_*^n is the hypersurface existence of which is stated by Theorem 1.1.

By the virtue of Lemma 3.1 we have an upper bound on $\mathrm{diam}(\Sigma_*^n)$. Now Lemmas 2.1, 2.3 imply that it is sufficient to prove an upper bound on the ratio

$$\max_{x\in \mathrm{Conv}(\Sigma_*^n)} \|\mathrm{Hess}\, p_*(x)\| / \min_{x\in\Sigma_*^n} |\mathrm{grad}\, p_*(x)|$$

of the type $\exp(c_{13}(n)d^{c_{14}(n)d^{n+1}})$ to complete the proof of Theorem 1.1. Theorem 3.1 yields upper bounds on all coordinates of any point $x \in \Sigma_*^n$ which imply immediately an upper bound of the necessary type for $\max_{x\in \mathrm{Conv}(\Sigma_*^n)} \|\mathrm{Hess}\, p_*(x)\|$. The necessary lower bound for $\min_{x\in\Sigma_*^n} |\mathrm{grad}\, p_*(x)|$ of the type $\exp(-c_{15}(n)d^{c_{16}(n)d^{n+1}})$ follows from Lemma 3.2. Combining these two bounds together we get the necessary upper bound for the ratio. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. — Let $\Sigma_i^1 \subset \mathbb{R}^2$ $(i \in \{1,2\})$ be two isotopic smooth compact curves of degree d. By Theorem 1.1 there exist two curves of degree d $\Sigma_{1*}^1, \Sigma_{2*}^1$ rigidly isotopic to Σ_1^1 and Σ_2^1, correspondingly, such that $\kappa(\Sigma_{i*}^1) \leq \exp(c_* d^{\omega d^2})$ for some constants c_* and ω. Moreover, the proof of Theorem 1.1 implies that these curves can be chosen to be zero sets of polynomials p_{i*} of degree d such that $\mathrm{grad}\, p_{i*}$ does not vanish on Σ_{i*} having the following additional property :

$$\|\mathrm{Hess}\, p_{i*}(x)\| / |\mathrm{grad}\, p_{i*}(y)| \leq \exp(c^* d^{\delta d^2}) \tag{4.1}$$

for some constants c^*, δ and for any x, y in the $\mathrm{diam}_{\mathbb{R}^2}(\Sigma_{i*}^1)/(2\exp(c_* d^{\omega d^2}))$ -neighborhood of Σ_{i*}^1.

To prove Theorem 1.2 it is sufficient to prove that it is possible to connect Σ_{1*}^1 and Σ_{2*}^1 by an $\lfloor \exp(cd^{\beta d^2}) \rfloor$-rigid isotopy.

In [Mo], ch. 3, a shelling algorithm for shrinking of a two-dimensional polygon in $\mathbb{R}^2$ is described (see also [ABB]). Omitting lengthy but simple details note that a smooth version of this algorithm can be used to construct a smooth isotopy Σ_{t*}^1, $(t \in [1,2])$ between Σ_{1*}^1 and Σ_{2*}^1 such that for any

$t\ \kappa(\Sigma^1_{t*}) \leq \exp(\mathrm{const}_0\ d^{\beta_0 d^2})$, where const_0 and β_0 are some universal constants. As it was noted in [N2] it is easy to deduce from that using the Jackson-Bernstein theorem about polynomial approximation of smooth functions that we can approximate this isotopy by an $\lfloor \exp(\mathrm{const}\, d^{\beta_1 d^2}) \rfloor$-rigid isotopy A_t ($t \in [1,2]$, const and β_1 are some constants) and it is easy to see that the inequality (4.1) makes possible to find an approximating $\lfloor \exp(cd^{\beta d^2}) \rfloor$-rigid isotopy which satisfies $A_i = \Sigma^1_{i*}$ for $i \in \{1,2\}$. This completes the proof of the theorem.

Proof of Theorems 1.3 and 1.4. — The proof of Theorem 1.3 is very similar to the proof of Theorem 1.1. The single difference is on the first stage when we look for a point $a_* \in W$ such that coordinates of a_* are rational numbers with bounded numerators and denominators. In the proof of Theorem 1.1 we looked for a_* in all W and, correspondingly, we needed to prove a lower bound on all locally maximal values of $|T|$ on $W \cap S^{k-1}$.

To prove Theorem 1.3 we fix the values of all $\binom{n+d+1}{n+1} - \ell$ variables of T corresponding to the integer coefficients of p. We assume that their values are equal to the values of corresponding coefficients of p. Now we consider the restriction of T on the ℓ-dimensional set $V_\ell = W \cap S^{k-1} \cap \mathbb{R}^\ell$. Here $\mathbb{R}^\ell$ denotes the space of coefficients of p which are not supposed to be integer in the conditions of Theorem 1.3. Now Lemma 3.3 provides the lower bound $\exp(-\ \mathrm{constant}_1(n) d^{\mathrm{constant}_2(n)(\ell+1)} (\ln M)^{\mathrm{constant}_3(n)})$ on any locally maximal value of $T^2|_{V_\ell}$ for some constants $\mathrm{constant}_1(n)$, $\mathrm{constant}_2(n)$, $\mathrm{constant}_3(n)$. The rest of the proof goes on exactly as the proof of Theorem 1.1.

Theorem 1.4 follows from Theorem 1.3 exactly as Theorem 1.2 follows from Theorem 1.1.

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