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REMARKS
ON THE LICHNEROWICZ-POISSON COHOMOLOGY

by Izu VAISMAN

The Lichnerowicz-Poisson (LP) cohomology of a Poisson manifold was defined in [L], and it provides a good framework to express deformation and quantization obstructions [L], [VK], [H], [V2]. The LP cohomology spaces are, generally, very large, and their structure is known only in some particular cases [VK], [X]. The homological algebraic place of these spaces was clarified in [H]. In the present note, we make a number of further remarks on the LP cohomology, most of them related with a certain natural spectral sequence which shows that, in the case of a regular Poisson manifold, the LP cohomology is connected with the cohomology of the sheaves of germs of foliated (i.e., projectable) forms of the symplectic foliation of the manifold (e.g., [V1]).

1. General remarks.

Let $M^m$ be a Poisson manifold with the Poisson bivector $\Pi$, and put $\mathcal{V}^0(M) \overset{\text{def}}{=} \mathcal{C}^\infty(M)$, $\mathcal{V}(M) \overset{\text{def}}{=} \mathcal{V}^1(M) = \mathcal{V}^1(M)$ the space of $\mathcal{C}^\infty$ vector fields of $M$, $\mathcal{V}^k(M) \overset{\text{def}}{=} \mathcal{V}^k(M)$ the space of $k$-vector fields (i.e., antisymmetric $k$-contravariant tensor fields of $M$), $\mathcal{V}^*(M) \overset{\text{def}}{=} \mathcal{V}^*(M) = \mathcal{V}^*(M)$ the space of Pfaff forms of $M$, and, finally $\mathcal{L}(M) = \bigoplus_{k=0}^{m} \mathcal{V}^k(M) = \mathcal{V}^k(M)$ the contravariant Grassmann algebra of $M$. The bivector $\Pi$ has an associated morphism $\#: T^*M \to TM$, defined by $\beta(\alpha^*) = \Pi(\alpha, \beta)$, $\forall \alpha, \beta \in T^*M$, and it yields the Poisson bracket of functions $\{f, g\} = \Pi(df, dg)$, as well as Hamiltonian vector fields $X_f$, $\forall f \in \mathcal{V}^0(M)$, given by $X_f g = \{f, g\}$. These fields define a generalized foliation with symplectic leaves called the symplectic foliation of $(M, \Pi)$ (i.e., $\{X_f\}$ generate the tangent spaces of $M$.

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the leaves). It is important to remember that the Poisson bracket induces a bracket of Pfaff forms which is the unique natural extension of the formula $\{d\mathcal{f}, d\mathcal{g}\} = d\{\mathcal{f}, \mathcal{g}\}$, and is given by

$$\{\alpha, \beta\} = L_\alpha \# \beta - L_\beta \# \alpha - d(\Pi(\alpha, \beta)).$$

The basic Poisson condition $[\Pi, \Pi] = 0$, where $[ , ]$ denotes the Schouten-Nijenhuis bracket, ensures that $(\mathcal{V}^{-\infty}(M), \{ , , \})$ and $(\mathcal{V}^{-\ast}(M), \{ , , \})$ are Lie algebras. The same condition also shows that the operator $\sigma Q = -[\Pi, Q]$ is a coboundary on $\mathcal{L}(M)$ (i.e., $\sigma^2 = 0$), and the cohomology of the cochain complex $(\mathcal{L}, \sigma)$ is, by definition, the LP cohomology of $(M, \Pi)$. Its spaces will be denoted by $H^k_{LP}(M, \Pi)$. It is also important to remind that, for $Q = \mathcal{V}^{-k}(M)$, one has [BV]

$$\sigma Q(\alpha_0, \ldots, \alpha_k) = \sum_{i=1}^k \alpha_i^p(\mathcal{Q}(\alpha_0, \ldots, \hat{\alpha}_i, \ldots, \alpha_k))$$
$$+ \sum_{i<j=0}^k (-1)^{i+j} \mathcal{Q}(\{\alpha_i, \alpha_j\}, \alpha_0, \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots, \alpha_k),$$

where $\alpha_i \in \mathcal{V}^{\ast}(M)$, and $\hat{\cdot}$ denotes the absence of an argument.

Now, the definitions given above have some easy consequences such as

a) $[X], [VK], H^0_{LP}(M, \Pi) = \{f \in C^\infty(M) / \forall g \in C^\infty(M), X_g f = 0\}$. (Since $\sigma f = -X_f$.)

b) $[X], [VK]. H^1_{LP}(M, \Pi) = \mathcal{V}^{-1}(M)/\mathcal{V}^{-1}(M)$, where

$$\mathcal{V}^{-1}(M) = \{X \in \mathcal{V}(M) / L_X \Pi = 0\}, \quad \mathcal{V}^{-1}(M) = \{X_g / f \in \mathcal{V}^{-0}(M)\}.$$

(Since $\sigma X = -L_X \Pi [L]$.)

c) $[L], \sigma \Pi = 0$, and $\Pi$ defines a fundamental class $[\Pi] \in H^2_{LP}(M, \Pi)$.

d) The LP cohomology satisfies the Mayer-Vietoris exact sequence property i.e., if $U, V$ are open subsets of $M$, there is an exact sequence of the form

$$\cdots \rightarrow H^k_{LP}(U \cup V, \Pi) \rightarrow H^k_{LP}(U, \Pi) \oplus H^k_{LP}(V, \Pi)$$
$$\rightarrow H^k_{LP}(U \cap V, \Pi) \rightarrow H^{k+1}_{LP}(U \cup V, \Pi) \rightarrow \cdots$$

The definition of the arrows and the proof of the exactness are the same as for the de Rham cohomology (e.g., [BT]).
e) [L], [K]. Natural homomorphisms $\rho : H^k(M, \mathbb{R}) \to H^k_{LP}(M, \Pi)$, which are isomorphisms in the symplectic case, exist. Namely, $\rho$ is defined by the extension of $\#$ to $k$-forms $\lambda$ by

$$\lambda^\#(\alpha_1, \ldots, \alpha_k) = (-1)^k \lambda(\alpha_1^\#, \ldots, \alpha_k^\#), \quad \text{(1.4)}$$

since (1.2) shows that $\sigma(\lambda^\#) = (-1)^k (d\lambda)^\#$.

Because of (e), it is natural to ask for a covariant interpretation of the whole LP cohomology via a Riemannian metric, and such an interpretation can be obtained by using Koszul's generating operators of the Schouten-Nijenhuis bracket. If we change signs such as to agree with [L], Koszul's formula for $[A, B]$ where $A \in \mathfrak{X}^1(M)$, $B \in \mathfrak{X}^1(M)$ is

$$[A, B] = D^A B - (D^A) A - (-1)^{|A|} A \wedge (D^B), \quad \text{(1.5)}$$

where $V$ is a torsionless linear connection on $M$, and $D_V$ is defined by the coordinatewise formula

$$(D_V A)^{h_2, \ldots, h_i} = \nabla_{h_2} A^{h_3, \ldots, h_i}. \quad \text{(1.6)}$$

If $V$ is the Riemannian connection of a metric $g$, (1.6) means $D_V = -\#_g \delta_g - \#_g^{-1}$, where $\#_g : T^*M \to TM$ is the well known musical isomorphism, and $\delta_g$ is the codifferential of $(M, g)$. Now, if we denote $\pi = \#_g^{-1} \Pi$, $B = \#_g \lambda$, and take $A = \Pi$ in (1.5), we obtain $\sigma(\#_g \lambda) = \#_g \delta_\pi$, where, if $e (i)$ denotes the exterior (interior) multiplication by a form, one has

$$\delta_n = \delta_g e(\pi) - e(\pi) \delta_g - e(\delta_g \pi). \quad \text{(1.7)}$$

Hence, $H^k_{LP}(M, \Pi)$ are isomorphic to the cohomology spaces of the Grassmann complex $\Lambda M$ endowed with the coboundary $\delta_n$.

Of course, $\pi$ must satisfy the condition $\delta_n \pi = 0$, which is equivalent to $[\Pi, \Pi] = 0$ i.e., we must have

$$\delta_g (\pi \wedge \pi) = 2 \pi \wedge (\delta_g \pi), \quad \text{(1.8)}$$

and this is a new characterization of a Poisson structure which may have some usefulness. For instance, it shows that the parallel 2-forms of a Riemannian manifold (if any) and the harmonic 2-forms of a compact Riemannian symmetric space (where the exterior product of two harmonic forms is again a harmonic form) define Poisson structures. Formulas (1.7), (1.8) may also be used if we are looking for compatible
Poisson structures on a given symplectic manifold $M$ with symplectic form $\omega$ i.e., Poisson bivectors $\Pi$ such that $[\omega^{-1}, \Pi] = 0$ (e.g., [G]). After the choice of a metric $g$ on $M$, this problem amounts to solving the equations

$$\delta_g^{-1}(\omega^{-1}) \pi = 0, \quad \delta_g(\pi \wedge \pi) = 2\pi \wedge \delta_g \pi,$$

where also, if we ask $g$ to be almost Hermitian $\omega$-compatible, then $\#^{-1}(\omega^{-1}) = \omega$. For instance, (1.9) shows that, if $M$ is a compact Hermitian symmetric space, and $\omega$ is its Kähler form, then any harmonic form of $M$ defines an $\omega$-compatible Poisson structure. On the other hand, we shall notice that, in case $M$ is compact and oriented, $\delta_\omega$ has the formal adjoint

$$d_\omega = i(\pi)d - d(i(\pi)) - i(\delta_g \pi),$$

and we may expect to be able to apply the abstract Hodge decomposition theorem of [LT]. (From the expression of the Schouten-Nijenhuis bracket [L], it follows easily that the complex

$$\ldots \to \varphi^{-k}(M) \xrightarrow{\cdot \omega} \varphi^{-k+1}(M) \to \ldots$$

is elliptic along the leaves of the symplectic foliation of $(M, \Pi)$.)

Finally, we make a remark which will be important for the next sections of this paper. Namely, that there is a Serre-Hochschild spectral sequence associated with the LP cohomology. Let $\varphi^\bullet (M) = \ker \# = \text{the space of conormal 1-forms}$ of the symplectic foliation of $(M, \Pi)$. Since the bracket (1.1) satisfies $\{\alpha, \beta\} \# = [\alpha^\#, \beta^\#][BV]$, $\varphi^\bullet (M)$ is an abelian ideal of $(\varphi^\bullet (M), \{,\})$, and we may define the filtration degree of $Q \in \varphi^{-k}(M)$ to be $h$ if $Q(\alpha_1, \ldots, \alpha_k) = 0$ as soon as $h \geq k - h + 1$ of the arguments are conormal. This yields a differential filtration of the LP complex $\mathcal{L}(M)$, where $S^k_\delta (M) = \text{the space of } k\text{-vector fields of filtration degree } h \text{ is equal to the locally finite span of}$

$$\{f_0 X_{f_1} \wedge \ldots \wedge X_{f_h} \wedge Y_1 \wedge \ldots \wedge Y_{k-h} | f_i \in \varphi^0(M), Y_j \in \varphi^{-1}(M)\}. $$

Now, the spectral sequence which we have in mind, and which we shall denote by $E_2^p(M, \Pi)$, is the one associated with this filtration i.e., the Serre-Hochschild sequence of the pair of Lie algebras $(\varphi^\bullet (M), \varphi^\bullet_\# (M), \{,\})$. This sequence converges to $H_\mathcal{L}(M, \Pi)$, and one has (e.g., [F])

$$E_2^{pq}(M, \Pi) = H^p(V^\bullet(M)/\varphi^\bullet_\#(M), H^q(\varphi^\bullet_\#(M); C^\infty(M)).$$
2. The regular case.

In the remaining part of this paper we assume that $\Pi$ is of the constant rank $2n$, and $m = 2n + s$. This is the regularity condition, and then the symplectic foliation of $(M, \Pi)$, hereafter to be denoted by $\mathcal{F}$, is regular. Hence, we can and shall define a transversal distribution $\mathcal{F}'$, and $TM = \mathcal{F}' \oplus T\mathcal{F}$, $T^*M = \mathcal{F}^* \oplus T^*\mathcal{F}$ induce a bigrading of the covariant and contravariant tensors of $M$. A tensor whose transversal degree is $p$ and whose leafwise degree is $q$ is said to be of the type $(p,q)$. We shall denote by $\mathcal{V}^{p,q}(M)$ and $\Lambda^{p,q}(M)$ the spaces of $k$-vector fields and $k$-forms ($k = p + q$) of the type $(p,q)$ of $M$, respectively. For instance, it is easy to understand that $\ker \#$ (i.e., $\mathcal{V}^*_0(M)$) is just $\mathcal{F}^* = \text{the space of the 1-forms of type (1,0)},$ and that type $\Pi = (0,2)$. (E.g., see [V1] for details on the bigrading of differential forms.)

Now, if $Q \in \mathcal{V}^{k}(M)$ is of type $(p,q)(p+q=k)$, and if we use bihomogeneous arguments $\alpha_i$ in (1.2), we see that $\sigma = \sigma' + \sigma''$ where type $\sigma' = (-1,2)$, type $\sigma'' = (0,1)$, and, for arguments $\alpha$ of type $(1,0)$ and $\beta$ of type $(0,1)$, one has

\begin{align*}
\sigma'(Q)(\alpha_0, \ldots, \alpha_{p-2}, \beta_0, \ldots, \beta_{q+1}) &= \sum_{i<j=0}^{q+1} (-1)^i + j Q(\beta_i, \beta_j), \\
\sigma''(Q)(\alpha_0, \ldots, \alpha_{p-1}, \beta_0, \ldots, \beta_q) &= \sum_{i=0}^{q} (-1)^{p+i} \beta_i \# (Q(\alpha_0, \ldots, \alpha_{p-1}, \\
\beta_0, \ldots, \beta_j, \ldots, \beta_q) &= \sum_{i<j=0}^{q} (-1)^{p+i + j} Q(\alpha_0, \ldots, \alpha_{p-1}, \\
\{\beta_i, \beta_j\}''(X) &= (L_X \pi)(\beta_1, \beta_2).
\end{align*}

Remember that type $\alpha = (1,0)$ means $\alpha \in \mathcal{V}^{*,0}(M)$, and that the latter is an ideal of $\mathcal{V}^*(M)$. On the other hand, we denoted by $\{, \}, \{, \}$ the type $(1,0)$ and $(0,1)$ components of $\{, \}$. Particularly, if type $X = (1,0)$, we get easily

(2.3) $\{\beta_1, \beta_2\}''(X) = (L_X \pi)(\beta_1, \beta_2)$. 
In this section we use the type decomposition of \(\sigma\) in order to indicate a recurrent computational process of the LP cohomology which, in fact, is similar to the one used in [VK] for the case where \(\mathcal{S}\) is a fibration. Take \(Q \in \mathcal{V}^k(M)\), and decompose it as

\[
Q = Q^{k,0} + Q^{k-1,1} + \ldots + Q^{0,k},
\]

where the indices denote the type of the components. Then, \(\sigma Q = 0\) means

\[
\sigma'' Q^{i,k-i} + \sigma' Q^{i+1,k-i-1} = 0 \quad (i = 0, \ldots, k).
\]

For \(i = k\), (2.5) gives \(\sigma'' Q^{k,0} = 0\), and, on the other hand, \((Q + \tilde{Q})^{k,0} = Q^{k,0}, \forall \tilde{Q} \in \mathcal{V}^{k-1}(M)\). Therefore, there exist homomorphisms

\[
p_{k,0} : H^k_{LP}(M, \Pi) \to \mathcal{V}^0(M),
\]

where \(\mathcal{V}^0(M)\) is the space of \(\sigma''\)-closed \(k\)-vectors of type \((k,0)\), and, furthermore, (2.5) shows that im \(p_{k,0}\) consists of \(k\)-vectors \(Q^{k,0} \in \mathcal{V}^0(M)\) which satisfy the following sequence of existence conditions of \(k\)-vectors \(Q^{k-1,1}, \ldots, Q^{0,k}\) such that

\[
\begin{align*}
(c_1) \quad & \sigma' Q^{k,0} = \sigma''\text{-exact} \overset{\text{def}}{=} - \sigma'' Q^{k-1,1}, \\
(c_2) \quad & \sigma' Q^{k-1,1} = \sigma''\text{-exact} \overset{\text{def}}{=} - \sigma'' Q^{k-2,2}, \\
& \vdots \\
(c_k) \quad & \sigma' Q^{1,k-1} = \sigma''\text{-exact} \overset{\text{def}}{=} - \sigma'' Q^{0,k}.
\end{align*}
\]

In this case we shall say that \(\sigma' Q^{k,0}\) satisfies \(k\) times the \(\sigma''\)-exactness condition, and we shall denote by \(\mathcal{V}^{k,0}_0(M)\) the space of such \(Q^{k,0}\). If we also denote \(\ker p_{k,0} = H^k_{LP}(M, \Pi) = \text{the space of } k\text{-dimensional LP cohomology classes whose cocycles are (2.4) with } Q^{k,0} = 0\), we obtain the result of the first recurrence step

\[
H^k_{LP}(M, \Pi) \approx H^k_{LP}(M, \Pi) \oplus \mathcal{V}^{k,0}_0(M).
\]

Now, in the next step we have to compute \(H^k_{LP}(M, \Pi)\), and for this purpose we take the subcomplex \(H^k_0(M)\) of \(H^k(M)\) consisting of multivectors \(Q\) with a vanishing \((.,0)\) component, and denote by \(H^k_0(M)\) its cohomology spaces. Then \(H^k_{LP}(M, \Pi)\) is the image of \(H^k_0(M)\) with respect to the inclusion \(H^k_0(M) \subseteq H^k(M)\). It is clear that the complex \(H^k(M)/0_0(M)\) has coboundary zero, therefore, \(H^k_0(M) = (H^k_0(M)/0_0(M)) = \mathcal{V}^{k,0}(M)\). This gives us the exact sequence
\[ \mathcal{L}^{k-1,0}(M) \xrightarrow{\sigma} H^k(\mathcal{L}(M)) \xrightarrow{i^*} H^k(\mathcal{L}(M)), \]
and we get

\[ (2.8) \quad 0^*H^k_{LP}(M,\Pi) \approx H^k(\mathcal{L}(M))/\sigma(\mathcal{L}^{k-1,0}(M)). \]

Hence, the second step will have to consist of an analysis of \( H^k(\mathcal{L}(M)) \),
which can be made in the same way as in step 1, and resulting in a
formula similar to (2.7), and so on.

For \( k = 1 \), we get easily

\[ (2.9) \quad 0^*H^1_{LP}(M,\Pi) = \{ X \in \mathcal{L}^{0,1}(M)/\sigma''X = 0 \}/\sigma''(\mathcal{L}^0(M)). \]

For \( k = 2 \), we have first

\[ (2.10) \quad H^2(\mathcal{L}(M)) = \{ Q^{1,1} + Q^{0,2}/\sigma''Q^{1,1} = 0, \sigma''Q^{0,2} + \sigma'Q^{1,1} = 0 \}/\{\sigma''X^{0,1}\}, \]
and the analysis which gave (2.7) now yields

\[ (2.11) \quad H^2(\mathcal{L}(M)) = H^2(\mathcal{L}^{0,*}(M)) \oplus H^{1,1}(0_{(1)}(M)), \]

where \( \mathcal{L}^{0,*}(M) = \oplus_k \mathcal{L}^{0,k}(M) \), and \( ''H \) is its cohomology with respect
to \( \sigma'' \), and

\[ (2.12) \quad \mathcal{L}^{1,1}_{0(1)}(M) = \{ Q^{1,1}/\sigma''Q^{1,1} = 0 \text{ and } \sigma'Q^{1,1} = \sigma''\text{-exact} \}. \]

(We shall see in Section 3 that, if the foliation \( \mathcal{F} \) is either transversally
Riemannian or transversally symplectic, then

\[ ''H^i(\mathcal{L}^{0,*}(M)) \approx H^i(M,\Phi^0(\mathcal{F})), \]

where \( \Phi^0(\mathcal{F}) \) is the sheaf of germs of functions which are constant
along the leaves of \( \mathcal{F} \).) Summing up the results we get

\[ (2.13) \quad H^2_{LP}(M,\Pi) \approx (''H^2(\mathcal{L}^{0,*}(M)) \oplus ((\mathcal{L}^{1,1}_{0(1)}(M))/\sigma(\mathcal{L}^{1,0}(M))) \oplus \mathcal{L}^{2,0}_{0(2)}(M)), \]

Etc.

3. The spectral sequence.

In this section we continue to refer to a regular Poisson manifold
\( (M,\Pi) \), and use the notation introduced in Section 2, while we are
focussing on the spectral sequence \( E^2_{LP}(M,\Pi) \) defined at the end of
Section 1. We have:
PROPOSITION 3.1. – The first terms of the LP Serre-Hochschild spectral sequence of a regular Poisson manifold \((M, \Pi)\) are given by

\[
E_0^{pq}(M, \Pi) = E_\infty^{pq}(M, \Pi) = \mathcal{V}^{-q,p}(M),
\]

\[
E_\infty^{pq}(M, \Pi) = H^p(\oplus \mathcal{V}^{-q,*}, \sigma^*).
\]

The reader can prove this by noticing that the \(h\)-filtering subcomplex of \(\mathcal{L}(M)\) as defined in Section 1 is equal to \(S_h(M) = \oplus \oplus \mathcal{V}^{-p,i}(M)\), and then following the usual definition of \(E_\infty^{pq}\). Here, we just prefer to observe that \(\mathcal{L}(M) = \oplus \mathcal{W}^{-i,j}(M), \sigma = \Sigma d_{hk}\), where \(\mathcal{W}^{-i,j}(M) = \mathcal{V}^{-i,j}(M)\), and the terms of \(\sigma\) are \(d_{01} = 0, d_{10} = \sigma'\), \(d_{2-1} = \sigma'\), is a double semipositive cochain complex in the sense of [VI], p. 76-77, and then (3.1) follows from this reference.

Now, let \(G\) be a metric of the vector bundle \(\mathcal{S}^*\) of Section 2, and let \(\tilde{#} \overset{\text{def}}{=} \#_G \oplus \# : \mathcal{S}'^* \oplus T^* \mathcal{S} \rightarrow \mathcal{S}'^* \oplus T \mathcal{S}\) be the corresponding musical isomorphism also extended to \(A^k(M) \rightarrow \mathcal{V}^{-k}(M)\). Then, if \(\lambda\) is a differential form of type \((p,q)\), \(\lambda^\#\) is a multivector of the same type, and we have

\[
\left(\tilde{#}^{-1} \sigma'' \lambda^\#\right)(X_0, \ldots, X_{p-1}, Y_0, \ldots, Y_q) = (-1)^{q+1}(\sigma'' \lambda^\#)(\#_G^{-1}X_0, \ldots, \#_G^{-1}X_{p-1}, \#_G^{-1}Y_0, \ldots, \#_G^{-1}Y_q).
\]

In this relation, and in the sequel, we agree that type \(X = (1,0)\) and type \(Y = (0,1)\). Furthermore, in order to compute \(\sigma'' \lambda^\#\) by (2.2) we establish first

\[
\{#^{-1}Y_i, #^{-1}Y_j\}^\# = \{#^{-1}Y_i, #^{-1}Y_j\}^\# = [Y_i, Y_j]
\]

(remember that \(\{\alpha, \beta\}^\# = [\alpha^\#, \beta^\#]\) [BV], and using (1.1))

\[
\{#_G^{-1}X_i, #^{-1}Y_j\}(X) = -(L_{Y_j}G^*)(X_i, X) - G^*([Y_i, X], X),
\]

where \(G^*\) is the dual metric of \(G\) on \(\mathcal{S}'\). If these formulas are used, and the result is compared with the formula of the \(\mathcal{S}\)-leafwise exterior differential \(d_f\) [V1], p. 184, one gets

\[
\left(\tilde{#}^{-1} \sigma'' \lambda^\#\right)(X_0, \ldots, X_{p-1}, Y_0, \ldots, Y_q)
\]

\[
= - (d_f \lambda)(X_0, \ldots, X_{p-1}, Y_0, \ldots, Y_q)
+ \sum_{i=0}^{p-1} \sum_{j=0}^{q} (-1)^{p+i+j} \lambda^\#:\left[(L_{Y_j}G^*)(X_i, \cdot)\right]^\#, X_0, \ldots, \hat{X}_i, \ldots, X_{p-1}, Y_0, \ldots, \hat{Y}_j, \ldots, Y_q.
\]
Remark. — The same result holds if $G$ is a symplectic structure on $\mathcal{F}^*$. 

This computation leads to

**Proposition 3.2.** — If the symplectic foliation $\mathcal{F}$ of the regular Poisson manifold $(M, \Pi)$ is either transversally Riemannian or transversally symplectic, one has

\[(3.4) \quad E^p_q(M, \Pi) = E^p_q(\mathcal{F}) = H^p(M, \Phi^q(\mathcal{F}))\]

where $E^p_q(\mathcal{F})$ is the spectral sequence of the foliation $\mathcal{F}$ (e.g., [KT]), and $\Phi^q(\mathcal{F})$ is the sheaf of germs of $\mathcal{F}$-foliated $q$-forms of $M$ (e.g., [VI]). Particularly, (3.4) holds if $\mathcal{F}$ is a fibration.

Indeed, under the hypotheses, $L_\gamma G = 0$ in (3.3), and in view of (3.1) we get an isomorphism $E^p_q(M, \Pi) = H^p(\oplus \Lambda^{q,*}(M), d_\gamma)$. But then (3.4) is known [VI], p. 216, 222, 77. (Remember that an $\mathcal{F}$-foliated $q$-form is a $q$-form which, locally, is the pull-back of a form of a local transversal manifold of the foliation $\mathcal{F}$.)

Now, let us define an interesting special class of Poisson manifolds. A vector field $V$ of $M$ is $\mathcal{F}$-foliated if it sends leaves to leaves or, equivalently, $\forall Y \in T\mathcal{F}, [V, Y] \in T\mathcal{F}$. For instance, this happens if $V$ is an infinitesimal automorphism of $\Pi$ i.e., $L_\gamma \Pi = 0$, a condition which is easily seen to be equivalent to each of the following two conditions, where $f, g \in C^\infty(M)$,

\[(3.5) \quad V\{f, g\} = [V, X_f](g) - [V, X_g](f),\]
\[(3.6) \quad [V, X_f] = X_{V(f)}.

A regular Poisson structure $\Pi$ of $M$ will be called transversally constant if $\mathcal{F}$ has a transversal distribution $\mathcal{F}'$ such that every local foliate vector field $V \in \mathcal{F}'$ is a local infinitesimal automorphism of $\Pi$. For instance, if $M = S \times N$, and $\Pi$ is defined by a symplectic structure of $S$, the distribution $\mathcal{F}' = TN$ has this property. Particularly, the existence of the local canonical coordinates of $\Pi$ in the sense of [L] p. 256-257, shows that every regular Poisson manifold is locally transversally constant. Another example is the Dirac bracket defined as follows. Let $(M, \omega)$ be a symplectic manifold endowed with a foliation $\mathcal{F}$ such that $\omega$ induces symplectic structures of its leaves. These induced structures yield a Poisson bivector $\Pi$ such that $\mathcal{F}(\Pi) = \mathcal{F}$, and $\{ \ , \ \}_\Pi$ is the Dirac bracket of $(M, \omega, \mathcal{F})$. It follows that every $\mathcal{F}$-foliate vector
field $V$ which is $\omega$-orthogonal to $\mathcal{F}$ is an infinitesimal automorphism of $\Pi$. Indeed, for such $V$, (3.5) is equivalent to $(L_\gamma \omega)(X_f, X_\beta) = 0$, and this is an easy consequence of $d\omega = 0$. Using this definition, we have

**Proposition 3.3.** If $\Pi$ is transversally constant, $\sigma' = 0$, and

$$H^k_{D}(M, \Pi) = \bigoplus_{q=0}^{q} E_{2}^{k-q, q}(M, \Pi).$$

**Proof.** Of course, the proposition refers to $\sigma'$ of (2.1) taken with respect to the distribution $\mathcal{F}'$ involved in the definition of a transversally constant Poisson structure. Let us use the notation of (2.1), and evaluate there $(\beta_1, \beta_2)'(X_p)(p \in M, X_p \in \mathcal{F}')$. This may be done by extending $X_p$ to a local foliate $(1,0)$-vector field $X$, and using (2.3). Since $\Pi$ is transversally constant, $L_\gamma \Pi = 0$ and we get $\sigma' = 0$. Then, (3.7) follows from (3.1). Q.e.d.

We shall finish by giving various corollaries of Propositions 3.1, 3.2, 3.3.

**Corollary 3.1.** If $(M, \Pi)$ is a transversally constant Poisson manifold whose symplectic foliation is either transversally Riemannian or transversally symplectic, one has

$$H^k_{D}(M, \Pi) = \bigoplus_{q=0}^{q} E_{4,2}^{k-q, q}(\mathcal{F}) = \bigoplus_{q=0}^{q} H^q(M, \Phi^{k-q}(\mathcal{F})).$$

**Corollary 3.2.** Let $\Pi$ be a Dirac bracket of a symplectic manifold $(M, \omega)$ endowed with a leafwise symplectic foliation $\mathcal{F}$, and its $\omega$-orthogonal distribution $\mathcal{F}'$. Assume that the bihomogeneous components of $\omega$ with respect to the decomposition $TM = \mathcal{F}' \oplus T\mathcal{F}$ are closed. Then, again, formula (3.8) holds good.

**Proof.** Being a Dirac bracket, $\Pi$ is transversally constant. On the other hand, if $\omega = \omega_{(2,0)} + \omega_{(0,2)}$, the hypothesis $d\omega_{(2,0)} = 0$ implies $(L_\gamma \omega_{(2,0)})(X_1, X_2) = 0$ for $(Y \in T\mathcal{F}, X_1, X_2 \in \mathcal{F}')$, and we see that $\omega_{(2,0)}$ defines a transversal symplectic structure of $\mathcal{F}$.

**Corollary 3.3 [X].** Let $\Pi$ be the Poisson structure defined on $M = S \times N$ by a fixed symplectic structure of $S$, and assume that $S$ has finite Betti numbers. Then one has

$$H^k_{D}(M, \Pi) = \bigoplus_{q=0}^{k} [H^q(S, \mathbb{R}) \otimes \Lambda^{k-q}(N)].$$
This result follows from (3.8) and from

**Proposition 3.4.** Let \( \mathcal{F} \) be the foliation of \( M = F \times N \) by the leaves \( F \times \{x\} \) \((x \in N)\), and assume that \( F \) has finite Betti numbers. Then

\[
H^q(M, \Phi^p(\mathcal{F})) = H^q(F, \mathbb{R}) \otimes \Lambda^p(N).
\]

**Proof.** For \( q = 0 \) the result was proven in [E] by a spectral sequence argument. Generally, we have the following straightforward argument. By the foliated de Rham theorem [VI], p. 216, we have

\[
H^q(M, \Phi^p(\mathcal{F})) = \ker [d_f: \Lambda^{p,q}(M) \to \Lambda^{p,q+1}(M)] / \text{im} [d_f: \Lambda^{p,q-1}(M) \to \Lambda^{p,q}(M)].
\]

In our case, \( \Lambda^{p,q}(M) \) is isomorphic to the space \( \Lambda^q(F, \Lambda^p(N)) \) of \( \Lambda^p(N) \)-valued \( q \)-forms on \( F \) by the mapping which sends \( \lambda \in \Lambda^{p,q}(M) \) to \( \hat{\lambda} \in \Lambda^p(F, \Lambda^p(N)) \) defined by

\[
(\hat{\lambda}_{y}(Y_1, \ldots, Y_q))_x(X_1, \ldots, X_p) = (-1)^p \lambda_{(x,y)}(X_1, \ldots, X_p, Y_1, \ldots, Y_q),
\]

\( y \in F, \ x \in N, \ Y_i \in T_y F, \ X_j \in T_x N \). Moreover, this isomorphism sends \( d_f \) to the exterior differential of \( \Lambda^p N \)-valued forms. Hence (3.11) becomes

\[
H^q(M, \Phi^p(\mathcal{F})) = H^q(F, \Lambda^p(N)) = H^q(F, \mathbb{R}) \otimes \Lambda^p(N),
\]

where the last equality follows from the hypothesis on \( F \). Q.e.d.

**Remark.** If \( M = S \times N \) of Corollary 3.3 is given a Poisson structure \( \Pi \) which has the symplectic foliation \( S \times \{x\} \) \((x \in N)\), but where each leaf has a different symplectic structure (e.g., the structure studied in [X]), \( \Pi \) is no more transversally constant, but we may use Propositions 3.2. and 3.4, and get

\[
E^{pq}_2(M, \Pi) = H^p(S, \mathbb{R}) \otimes \Lambda^q(N).
\]

**Corollary 3.4.** Let \((M, \Pi)\) be an arbitrary regular Poisson manifold. Then every \( x \in M \) has a connected open neighbourhood \( Y \) such that

\[
H^k_{\text{cU}}(U, \Pi_{/U}) = \Gamma(\Phi^k(\mathcal{F}_{/U})),
\]

i.e., the space of the \( \mathcal{F} \)-foliated \( k \)-forms over \( U \).
Indeed, we may take $U = S \times N$ where $S$ is contractible, and such that the product coordinates are canonical for $\Pi$ in the sense of [L], p. 257. Then Corollary 3.3 holds on $U$, and we get (3.13). We shall say that such a neighbourhood $U$ is LP-simple.

**Corollary 3.5** (The LP Poincaré Lemma [L]). — Let $(M, \Pi)$ be a regular Poisson manifold, and $x \in M$. Then, there exists an open neighbourhood $U$ of $x$ in $M$ such that, if $Q \in \mathcal{V}^{k}(U)$ and $\sigma Q = 0$, one has $Q = A + \sigma B$ for some $B \in \mathcal{V}^{k-1}(U)$ and a $k$-vector field $A$ over $U$ which is projectable to a $k$-vector field of a local transversal submanifold of $G$ in $U$.

**Proof.** — Take $U$ LP-simple, and with $\Pi$-canonical coordinates. The latter define a bigrading, and we may write $Q = \sum_{p=0}^{k} (\lambda^{p,k-p}) \#$, where $\lambda$ are differential forms, and $\#$ is like in (3.2). The use of the canonical coordinates makes $\Pi/\mu$ transversally constant and transversally Riemannian hence, by Proposition 3.3 and formula (3.3), $\sigma = \sigma^{n}$, and $\sigma Q = 0$ is equivalent to $d_{f}^{k} = 0$ ($k = 0, \ldots, p$). But $d_{f}$ satisfies a local Poincaré lemma [VI], p. 215, hence, there are local forms $\mu$ such that $\lambda^{p,k-p} = d_{f}^{k} \mu^{p,k-p}$ for $k - p > 0$, while $\lambda^{k,0}$ is a foliate form. The conclusion follows by using again (3.3).  Q.e.d

**BIBLIOGRAPHY**


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