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The asymptotics of the Ray-Singer analytic torsion of the symmetric powers of a positive vector bundle


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Let $X$ be a compact complex manifold, equipped with a smooth Hermitian metric. Let $(E,||\cdot||_E), (\xi,||\cdot||_\xi)$ be holomorphic Hermitian vector bundles on $X$. Assume that $(E,||\cdot||_E)$ is positive, i.e. if $L^E$ is the curvature of the holomorphic Hermitian connection on $(E,||\cdot||_E)$, for any $U \in TX\setminus\{0\}, \varepsilon\in E\setminus\{0\}$, then $\langle L^E(U,\bar{\varepsilon})e,\bar{\varepsilon}\rangle > 0$. By [K] Theorem III 6.19, $E$ is an ample vector bundle on $X$.

For $p \in \mathbb{N}$, let $S^p(E)$ be the $p^{\text{th}}$ symmetric tensor power of $E$. Then by a result of Le Potier [LP], [K] Theorem III.6.25, for $p$ large enough, and $q > 0$, $H^q(S^p(E)\otimes\xi) = 0$. Let $\tau_p$ be the Ray-Singer analytic torsion of the Dolbeault complex $\Omega^{0,\cdot}(S^p(E)\otimes\xi)$ [RS]. The purpose of this paper is to establish an asymptotic formula for $\log(\tau_p)$ as $p \to +\infty$. This extends an earlier result by ourselves [BV] Theorem 8, in the case where $E$ is a positive line bundle.

The general strategy is the same as in [BV]. Namely if $\Box^X_{\tau}^q$ denotes the Hodge Laplacian acting on $\Omega^{0,\cdot}(S^p(E)\otimes\xi)$, we first establish in Theorem 1 an asymptotic formula for $\text{Tr} \left[ \exp \left( -\frac{t}{p} \Box^X_{\tau}^q \right) \right]$ as $p \to +\infty$.

In Theorem 8 we prove that if $\lambda^q_p$ is the lowest eigenvalue of $\Box^X_{\tau}^q$, if $q > 0$, as $p \to +\infty$, $\lambda^q_p$ grows at least like $p$. The combination of these two results leads us in Theorem 11 to an asymptotic formula for $\log(\tau_p)$ very much like in [BV].

To establish these intermediary results, we use a trick due to Getzler [Ge] in a different context. In [Ge], Getzler extended a result of Bismut [B2] on the asymptotics of certain heat equation operators, which
is valid for line bundles, to vector bundles, associated with representations of the structure group with weight \( p\lambda \) as \( p \to +\infty \). Here if \( \mu \) is the dual of the universal line bundle on \( \mathbb{P}(E^*) \), we consider \( S^p(E) \) as the direct image of \( \mu^{\otimes p} \) by the map \( \pi: \mathbb{P}(E^*) \to X \). We then use Getzler's trick to lift our initial problem to a corresponding problem on the line bundle \( \mu^{\otimes p} \) on \( \mathbb{P}(E^*) \), to which the techniques of \([BV]\) can be applied.

Our paper is organized as follows. In § 1, we introduce our main assumptions and notation. In § 2, we calculate the asymptotics of \( \text{Tr} \left[ \exp \left( -\frac{t}{p} \Box_{\mu} X^q \right) \right] \) as \( p \to \infty \). Finally in § 3, we establish our main result on the asymptotics of \( \log (\tau_p) \) as \( p \to +\infty \).

As was pointed out by the referee, the results contained in this paper can be extended to other irreducible representations of \( E \), which are associated with the weights \( pa \) (where \( a \) is a given weight) when \( p \) tends to \( +\infty \). The corresponding vector bundles can be expressed as direct images of the \( p^{\text{th}} \) power of a certain line bundle over the corresponding flag manifold. Arguments of Demailly [De] Lemma 3.7, can then be used to prove the positivity of this Hermitian line bundle when \((E, || \cdot ||_E)\) is positive, and the trick of Getzler [Ge] together with the techniques used in our paper still apply. This extension of our main result is left to the reader.

1. Assumptions and notation.

Let \( X \) be a compact complex manifold of complex dimension \( \ell \). Let \( TX \) be the complex holomorphic tangent space.

Let \( E \) be a holomorphic vector bundle on \( X \), of complex dimension \( k \). Let \( E^* \) be the dual of \( E \). For \( p \in \mathbb{N} \), \( S^p(E) \) denotes the \( p^{\text{th}} \) symmetric tensor power of \( E \).

Let \( \mathbb{P}(E^*) \) denote the projectivization of \( E^* \), and let \( \pi \) be the projection \( \mathbb{P}(E^*) \to X \). Let \( \mu \) be the dual of the universal line bundle on \( \mathbb{P}(E^*) \).

Let \( || \cdot ||_E \) be a smooth Hermitian metric on \( E \). Let \( || \cdot ||_{SP(E)}, || \cdot ||_{E^*}, || \cdot ||_\mu \) be the Hermitian metrics on \( S^p(E), E^*, \mu \) induced by the metric \( || \cdot ||_E \).
Let $\nabla^E, \nabla^{E\ast}$ be the holomorphic Hermitian connections on $(E, || ||_E), (E^\ast, || ||_{E^\ast})$, and let $L^E, L^{E\ast}$ be the corresponding curvatures. Let $\nabla^\mu$ be the holomorphic Hermitian connection on $(\mu, || ||_\mu)$ and let $r$ be its curvature.

We first calculate $r$. Let $T^\ast \mathbb{P}(E\ast)$ be the relative tangent bundle to the fibres of $\pi : \mathbb{P}(E\ast) \to X$. The connection $\nabla^{E\ast}$ induces a horizontal subbundle $T^\ast \mathbb{P}(E\ast)$ of $T\mathbb{P}(E\ast)$.

Let $r^V$ be the restriction of $r$ to $T^\ast \mathbb{P}(E\ast)$. $r^V$ is explicitly known by a formula given in [GrH] p. 30, and defines the Fubini-Study metric along the fibres of $\mathbb{P}(E\ast)$. $r^V$ extends into a $(1,1)$ form on $\mathbb{P}(E\ast)$ such that if $U \in T^H \mathbb{P}(E\ast)$, $i_0 r^V = 0$.

Set
\[
(1) \quad r^H = -\pi^* \frac{\langle L^{E\ast} y, y \rangle}{|y|^2}; \quad y \in E\ast \setminus \{0\}.
\]

Then $r^H$ is a $(1,1)$ form on $\mathbb{P}(E\ast)$. Also by [K] p. 90,

\[
(2) \quad r = r^V + r^H
\]

$r^H$ is then the restriction of $r$ to $T^H \mathbb{P}(E\ast) \times T^H \mathbb{P}(E\ast)$.

For $p \geq 1$, the connection $\nabla^E$ induces on $S^p(E)$ the holomorphic Hermitian connection $\nabla^{S^p(E)}$ on $(S^p(E), || ||_{S^p(E)})$.

Let $(\xi, || \xi ||)$ be a holomorphic Hermitian vector bundle on $X$. Let $\nabla^\xi$ be the corresponding holomorphic Hermitian connection.

Let $|| ||_\tau_X$ be a Hermitian metric on $X$.

We then equip $\Lambda(T^{*(0,1)}X) \otimes S^p(E) \otimes \xi$ with the tensor product of the metrics induced by $|| ||_\tau_X$ on $\Lambda(T^{*(0,1)}X)$, of the metric $|| ||_{S^p(E)}$ and of the metric $|| ||_\xi$.

For $0 \leq q \leq \ell$, let $\Omega^{(0,q)}(S^p(E) \otimes \xi)$ be the set of $C^\infty$ sections of $\Lambda^q(T^{*(0,1)}X) \otimes S^p(E) \otimes \xi$ over $X$. Set $\Omega^{(0,\ast)}(S^p(E) \otimes \xi) = \bigoplus_{q=0}^{\ell} \Omega^{(0,q)}(S^p(E) \otimes \xi)$.

Let $dx$ be the volume form on $X$ associated with the metric $|| ||_\tau_X$. We equip $\Omega^{(0,\ast)}(S^p(E) \otimes \xi)$ with the $L_2$ Hermitian product
\[
(3) \quad \alpha, \alpha' \in \Omega^{(0,\ast)}(S^p(E) \otimes \xi) \to \langle \alpha, \alpha' \rangle = \int_X \langle \alpha, \alpha' \rangle(x) \frac{dx}{(2\pi)^{\dim X}}.
\]
Let $\tilde{\partial}_p^X$ be the Dolbeault operator acting on $\Omega^{(0,1)}(S^p(E) \otimes \xi)$, and let $\tilde{\partial}_p^{X*}$ be the formal adjoint of $\tilde{\partial}_p^X$ with respect to the Hermitian product (3). Set

$$\square_q^X = (\tilde{\partial}_p^X + \tilde{\partial}_p^{X*})^2.$$ 

For $0 \leq q \leq \ell$, let $\square_{p,q}^X$ be the restriction of $\square_q^X$ to $\Omega^{(0,q)}(S^p(E) \otimes \xi)$.

### 2. The asymptotics of the trace of certain heat kernels as $p \to + \infty$

If $U \in T_RX$, let $U^H$ be its horizontal lift in $T^H_R \mathbb{P}(E)$, so that $U^H \in T^H_R \mathbb{P}(E)$, $\pi_* U^H = U$.

Let $w_1, \ldots, w_{\ell}$ be an orthonormal base of $TX$, let $w^1, \ldots, w^\ell$ be the corresponding dual base of $T^*X$. If $z \in \mathbb{P}(E*)$, $r_z$ acts as a derivation $r^H_{d,z}$ of $\Lambda^{(0,1)}_\mathbb{R}(T^{*^{(0,1)}}X)$ by the formula

$$r^H_{d,z} = -\sum r_z(w_i^H, \bar{w}_j^H) \bar{w}_j \wedge i_{w_i}.$$ 

We identify $r^H_z$ with the self-adjoint matrix $r^H_z \in \text{End}_{x_*} (TX)$ such that if $U, V \in T_{x_*}X$,

$$r^H_z(U^H, \bar{V}^H) = \langle U^H, r^H_z \bar{V}^H \rangle.$$ 

For $0 \leq q \leq \ell$, let $r^H_{d,q}$ be the restriction of $r^H_z$ to $\Lambda^{q}(T^{*^{(0,1)}}X)$. As $t \to 0$, we have the asymptotic expansion

$$\int_{\mathbb{P}(E*)} \text{Tr} [e^{tr^H_{d,q}}] \det (1 - e^{-it^H}) \exp \left( \frac{-r}{2i\pi} \right) = \sum_{j=0}^m a^q_j t^j + o(t^m).$$

Also for $p \in \mathbb{N}$, $0 \leq q \leq \ell$, $t > 0$, let $\text{Tr} [\exp (-t \square_{p,q}^X)]$ be the trace of the operator $\exp (-t \square_{p,q}^X)$. For any $m \in \mathbb{N}$, as $t \to 0$

$$p^{-(\dim X + \dim E - 1)} \text{Tr} \left[ \exp \left( \frac{-t}{p} \square_{p,q}^X \right) \right] = \sum_{j=-\ell}^m a^q_{p,j} t^j + o(t^m).$$

**THEOREM 1.** — For any $t > 0$, $0 \leq q \leq \ell$, the following identity holds

$$\lim_{p \to + \infty} p^{-(\dim X + \dim E - 1)} \text{Tr} \left[ \exp \left( \frac{-t}{p} \square_{p,q}^X \right) \right] = rk(\xi) \int_{\mathbb{P}(E^*)} \frac{\text{Tr}[e^{r^H_{d,q}}]}{\det (1 - e^{-it^H})} \exp \left( \frac{-r}{2i\pi} \right).$$
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and the convergence in (8) is uniform as \( t \) varies in compact sets of \( \mathbb{R}^* \). For any \( j \geq - \dim X, 0 \leq q \leq \dim X \), as \( p \to + \infty \)

\[
(9) \quad a_{p,j}^q = rk(\xi) a_j^q + O \left( \frac{1}{\sqrt{p}} \right).
\]

In (7), for any \( m \in \mathbb{N} \), \( o(t^m) \) is uniform with respect to \( p \in \mathbb{N} \).

Proof. – By [GrH] p. 165 and [K] Theorem III.4.10, for any \( x \in X, p \in \mathbb{N} \)

\[
(10) \quad H^q(\mathbb{P}_x(E^*), \mu_{\mathbb{P}_x(E^*)}^p) = S^p(E)_x \quad \text{if} \quad q = 0
\]

\[
H^q(\mathbb{P}_x(E^*), \mu_{\mathbb{P}_x(E^*)}^p) = 0 \quad \text{if} \quad q > 0.
\]

To prove (8), we will use (10) together with a procedure used by Getzler [Ge] in a similar situation, to transform the initial problem into a corresponding problem on \( \mathbb{P}(E^*) \) associated with \( \mu^{\otimes p} \), to which we can apply results of Bismut [B2] and Bismut-Vasserot [BV].

Let \( U(E^*) \) be the bundle of orthonormal frames in \( E^* \). We identify \( U(E^*) \) with the set of linear isometries from \( \mathbb{C}^k \) into \( E^* \). Clearly

\[
\mathbb{P}(E^*) = U(E^*) \times_{U(k)} \mathbb{P}(\mathbb{C}^k).
\]

The connection \( \nabla^{E^*} \) on \( U(E^*) \) induces a connection on the fibration \( \pi: \mathbb{P}(E^*) \to X \). The associated horizontal subbundle of \( T\mathbb{P}(E^*) \) is exactly the vector bundle \( T^h\mathbb{P}(E^*) \) considered in §1.

We then have the identification of \( C^\infty \) vector bundles

\[
T\mathbb{P}(E^*) \cong T^h\mathbb{P}(E^*) \oplus T^v\mathbb{P}(E^*)
\]

\[
(11) \quad T^h\mathbb{P}(E^*) \cong \pi^* TX.
\]

From (11), we deduce the identification of \( C^\infty \) vector bundles

\[
\Lambda(T^*(0,1)\mathbb{P}(E^*)) \cong \pi^*(\Lambda(T^*(0,1)X)) \otimes \Lambda(T^v*(0,1)\mathbb{P}(E^*)�).
\]

We equip \( T^v\mathbb{P}(E^*) \) with the Fubini-Study metric \( || \cdot ||_{T^v\mathbb{P}(E^*)} \). Let \( || \cdot ||_{T\mathbb{P}(E^*)} \) be the metric on \( T\mathbb{P}(E^*) \cong T^h\mathbb{P}(E^*) \oplus T^v\mathbb{P}(E^*) \) which is the orthogonal sum of \( \pi^* || \cdot ||_{TX} \) and \( || \cdot ||_{T^v\mathbb{P}(E^*)} \). For \( p \in \mathbb{N} \), set

\[
(13) \quad \xi_p = \mu^{\otimes p} \otimes \pi^* \xi.
\]
Let $\overline{\partial}_p^V$ be the $\overline{\partial}$ operator along the fibres of $\mathcal{P}(E^*)$ acting on smooth sections of $\pi^*(\Lambda(T^{*(0,1)}X)) \otimes \Lambda(T^{*(0,1)}\mathcal{P}(E^*)) \otimes \xi_p$, and let $\overline{\partial}_p^{V^*}$ be its formal adjoint with respect to the considered metrics.

Let $V^{U\mathcal{P}(E^*)}$, $\overline{V}^{p}$ be the holomorphic Hermitian connections on $T^\mathcal{P}(E^*)$, $\xi_p$ respectively. These connections induce a natural connection on $\Lambda(T^{*(0,1)}\mathcal{P}(E^*)) \otimes \xi_p$, which we note $V^{\Lambda(T^{*(0,1)}\mathcal{P}(E^*)) \otimes \xi_p}$.

**Definition 2.** If $\alpha$ is a smooth section of $\Lambda(T^{*(0,1)}\mathcal{P}(E^*)) \otimes \xi_p$ over $\mathcal{P}(E^*)$, if $U \in T_RX$, set

$$v_{p,U} \alpha = V^{\Lambda(T^{*(0,1)}\mathcal{P}(E^*)) \otimes \xi_p} \alpha.$$ 

We extend $\hat{V}_p$ to a differential operator acting on smooth sections of $\pi^*(\Lambda(T^{*(0,1)}X)) \otimes \Lambda(T^{*(0,1)}\mathcal{P}(E^*)) \otimes \xi_p$, with the convention that if $\omega$ is a smooth section of $\Lambda(T^{*(0,1)}X)$, and if $\alpha$ is a smooth section of $\Lambda(T^{*(0,1)}\mathcal{P}(E^*)) \otimes \xi_p$, then

$$\hat{V}_p(\omega \alpha) = \pi^*(d\omega) \alpha + (-1)^{\deg \omega} \omega \wedge \hat{V}_p \alpha.$$ 

Let $\check{V}_p$, $\check{V}_p^*$ be the holomorphic and antiholomorphic parts of $\hat{V}_p$, so that

$$\check{V}_p = \check{V}_p^* + \check{V}_p^*.$$ 

For $0 \leq q' \leq k + \ell - 1$, let $\Omega^{(0,q')}(\xi_p)$ be the set of smooth sections of $\Lambda(T^{*(0,1)}\mathcal{P}(E^*)) \otimes \xi_p$ over $\mathcal{P}(E^*)$. Set $\Omega^{(0,q')}(\xi_p) = \bigoplus_{q' = 0}^{k + \ell - 1} \Omega^{(0,q')}(\xi_p)$. Let $\partial_p^{\mathcal{P}(E^*)}$ be the classical Dolbeault operator acting on $\Omega^{(0,q')}(\xi_p)$.

Using the identification (12), it is clear that $\hat{V}_p^*$, $\partial_p^V$, $\partial_p^{V^*}$ act on $\Omega^{(0,q')}(\xi_p)$.

If $A$, $B$ are operators acting on the $\mathbb{Z}$-graded vector space $\Omega^{(0,q')}(\xi_p)$, $[A,B]$ denotes the supercommutator of $A$ and $B$ in the sense of [Q].

**Proposition 3.** The following identities of operators acting on $\Omega^{(0,q')}(\xi_p)$ hold

$$(\hat{V}_p^*)^2 = 0$$

$$[\hat{V}_p^*, \partial_p^V] = [\hat{V}_p^*, \partial_p^{V^*}] = 0$$

$$\partial_p^{\mathcal{P}(E^*)} = \hat{V}_p^* + \partial_p^V.$$
Proof. — Assume first that $\xi = C$. Let $F$ be the vector space of smooth sections of $\Lambda (T^{\nu, (1, 0)} \mathbb{P}(E^*)) \otimes \mu^{\otimes p}$ over $\mathbb{P}(E^*)$. As in [B1, Section 1f), we view $F$ as an infinite dimensional vector bundle over $X$. If $x \in X$, the fibre $F_x$ is simply the vector space of smooth sections $\Lambda (T^{\nu, (1, 0)} \mathbb{P}(E^*)) \otimes \mu^{\otimes p}$ on the fibre $\mathbb{P}(E^*)_x$.

Clearly $F$ is a $U(k)$-equivariant vector bundle on $X$, in the sense it comes from a representation space for $U(k)$. $\nabla^F$ is then a connection on the vector bundle $F$, which is inherited from the original connection $\nabla^E$ on $U(E)$. Since $(\nabla^E)^2 = 0$, then $(\nabla^F)^2 = 0$. It is now trivial to prove in full generality the equation $(\nabla^F)^2 = 0$.

Since $U(k)$ acts on $\mathbb{P}(\mathbb{C}^*)$ by holomorphic isometries which lift unitarily to the dual of the universal bundle on $\mathbb{P}(\mathbb{C}^*)$, we get

$$\nabla_p \circ \nabla^E = 0; \quad [\nabla_p, \nabla^E] = 0.$$  \hfill (18)

In particular, the second equation in (17) holds.

If $B \in \text{End}(E^*)$ is skew-adjoint, let $By$ be the holomorphic Killing vector field on $\mathbb{P}(E^*)$ induced by the corresponding vector field on $E^*$. The vector field $By$ lies in $T^v \mathbb{P}(E^*)$. Then $L^E_y$ is a $(1, 1)$ form on $X$ taking values in vector fields in $T^v \mathbb{P}(E^*)$. $L^E_y$ lifts to a $(1, 1)$ form on $\mathbb{P}(E^*)$.

Assume first that $\nu = 0$, and $\xi = C$. Let $d^\nu$ be the de Rham operator along the fibres of $\mathbb{P}(E^*)$, and let $d$ be the de Rham operator on $\mathbb{P}(E^*)$. Similarly $\nabla_0$ can be made to act on the de Rham complex of $\mathbb{P}(E^*)$. Then by [B1] eq. (1.30) and [BGS1] eq. (1.26), we find that

$$d = \nabla_0 + d^\nu + i_{L^E_y}.$$  \hfill (19)

Since $L^E_y$ is of type $(1, 1)$, we deduce from (19) that we have the identity of operators acting on $\Omega^{(0, \cdot)}(\xi_0)$

$$\delta^\bullet_0 (E^*) = \nabla_0^\nu + \partial^\nu.$$  \hfill (20)

Extending (20) to $\Omega^{(0, \cdot)}(\xi_p)$ is easy and is left to the reader. \hfill \Box

Remark 4. — The fibration $\pi : \mathbb{P}(E^*) \to X$ is locally Kähler in the sense of Bismut-Gillet-Soule [BGS1], [BGS2]. Part of the identities in (17) follows from [BGS1], Theorem 2.6.
Let $\tilde{\nabla''}^*$, $\tilde{\partial''}^*$, $\tilde{\partial''}^{P(E^*)}$, be the formal adjoints of $\tilde{\nabla''}$, $\tilde{\partial''}$, $\tilde{\partial''}^{P(E^*)}$, with respect to the obvious Hermitian product on $\Omega^{(0,\cdot)}(\xi_p)$ associated to the various metrics.

Observe that $\tilde{\partial''}^*$ restricts on each fibre of $\pi: \mathbb{P}(E^*) \to E^*$ to the fibrewise adjoint of $\tilde{\partial''}$.

**Theorem 5.** — The following identities of operators acting on $\Omega^{(0,\cdot)}(\xi_p)$ hold

\[
(21) \quad (\tilde{\nabla''}^* + \tilde{\partial''}^*)^2 = (\tilde{\nabla''} + \tilde{\partial''})^2 + (\tilde{\partial''}^{P(E^*)} + \tilde{\partial''}^{P(E^*)})^2.
\]

**Proof.** — (21) follows from Proposition 3. □

For $0 \leq q \leq \dim X$, $0 \leq q' \leq \dim E - 1$, let $\Omega^{(0,q,q')} (\xi_p)$ be the set of smooth sections of $\pi^*(\Lambda^q(T^{\ast\ast}(0,1)X)) \otimes \Lambda^{q'}(T^{\ast\ast}(0,1)\mathbb{P}(E^*)) \otimes \xi_p$. By (12), we know that $\Omega^{(0,q,q')} (\xi_p)$ is a vector subspace of $\Omega^{(0,q+q')} (\xi_p)$. More precisely, for any $q$, $0 \leq q \leq \dim X + \dim E - 1$

\[
(22) \quad \Omega^{(0,q)} (\xi_p) = \bigoplus_{q' + q'' = q} \Omega^{(0,q',q'')} (\xi_p).
\]

Set

\[
(23) \quad \Box_p^{P(E^*)} = (\tilde{\partial''}^{P(E^*)} + \tilde{\partial''}^{P(E^*)})^2.
\]

By Theorem 5, we find that

\[
(24) \quad \Box_p^{P(E^*)} = \tilde{\nabla''}^* \tilde{\nabla''}^* + \tilde{\nabla''} \tilde{\nabla''} + \tilde{\partial''}^* \tilde{\partial''}^* + \tilde{\partial''} \tilde{\partial''}.
\]

From (24), it is clear that the operator $\Box_p^{P(E^*)}$ acts on each $\Omega^{(0,q,q')} (\xi_p)$. Let $\Box_p^{P(E^*,q,q')}$ be the restriction of $\Box_p^{P(E^*)}$ to $\Omega^{(0,q,q')} (\xi_p)$.

We now have the following result directly inspired by Getzler [Ge].

**Theorem 6.** — For any $p \in N$, $0 \leq q \leq \dim X$, $t > 0$, the following identity holds

\[
(25) \quad \text{Tr} \left[ \exp \left( -t \Box_p^{X,q} \right) \right] = \sum_{q' = 0}^{\dim E - 1} (-1)^{q'} \text{Tr} \left[ \exp \left( -t \Box_p^{(P(E^*)^*,q,q')} \right) \right].
\]

**Proof.** — Let $F_p$ be the vector space of smooth sections of $\Lambda(T^{\ast\ast}(0,1)\mathbb{P}(E^*)) \otimes \xi_p$ over $\mathbb{P}(E^*)$. As in the proof of Proposition 3, we regard $F_p$ as an infinite dimensional vector bundle over $X$. If $x \in X$,
the fibre $F_{p,x}$ is the set of smooth sections of $\Lambda(T^*(\mathbb{P}(E^*)) \otimes \xi_p$ over $\mathbb{P}(E^*)_x$. $F_p$ is a $U(k)$-equivariant Hermitian vector bundle on $X$, and $\nabla_p$ is the corresponding holomorphic Hermitian connection. Also $\Omega^{(0,\cdot)}(F_p)$ is canonically isomorphic to $\Omega^{(0,\cdot)}(\xi_p)$.

The operator $(\overline{\partial}^\nu_p + \partial^\nu_p)^2$ is $U(k)$-equivariant. Therefore the spectrum of $(\overline{\partial}^\nu_p + \partial^\nu_p)^2$ acting on a fibre $F_{p,x}$ does not depend on $x \in X$. In the sequel $\lambda \geq 0$ varies in the spectrum of $(\overline{\partial}^\nu_p + \partial^\nu_p)^2$.

The vector bundle $F_p$ over $X$ then splits into a direct orthonormal sum of finite dimensional vector spaces $F^\lambda_p$ which are eigenspaces of $(\overline{\partial}^\nu_p + \partial^\nu_p)^2$ associated with the eigenvalues $\lambda$, i.e.

\begin{equation}
F_p = \bigoplus_{\lambda > 0} F^\lambda_p.
\end{equation}

For $0 \leq q' < \dim E - 1$, let $F^q_p$ be the set of smooth sections of $\Lambda^q(T^*(\mathbb{P}(E^*)) \otimes \xi_p$ over $\mathbb{P}(E^*)$. Clearly $F_p = \bigoplus_{q' = 0}^{\dim E - 1} F^q_p$. Also the operator $(\overline{\partial}^\nu_p + \partial^\nu_p)^2$ preserves each $F^q_p$. To the splitting (26) of $F_p$ corresponds the splitting

\begin{equation}
F^q_p = \bigoplus_{\lambda > 0} F^q_p^\lambda.
\end{equation}

of each $F^q_p$. Using (10) and Hodge theory, we know that

\begin{equation}
F^q_p^{(0)} = S^p(E) \otimes \xi \quad \text{if} \quad q' = 0
= 0 \quad \text{if} \quad q' > 0.
\end{equation}

Moreover since $U(k)$ acts irreducibly on $S^p(E)$, the metric on $S^p(E)$ induced from the $L_2$ metric on the fibers of $\mathbb{P}(E^*)$ coincides (up to an irrelevant constant) with the metric $\| \|_{S^p(E)}$.

Let $\square^q_{p,\lambda}$ be the restriction of the operator $(\nabla^p + \nabla^p)^2$ to the set of smooth sections of $\Lambda^q(T^*(\mathbb{P}(E^*)) \otimes F^q_p^\lambda$ over $X$. From Theorem 5, we get

\begin{equation}
\sum_{q' = 0}^{\dim E - 1} (-1)^{q'} \Tr [\exp (-t \square^p_{p}(E^*)_{q,q'})] \\
= \sum_{\lambda > 0} \exp (-t \lambda) \sum_{q' = 0}^{\dim E - 1} (-1)^{q'} \Tr [\exp (-t \square^p_{q,\lambda})].
\end{equation}
From (28) and from the considerations which follow, we find that
\[
\dim E-1 \sum_{q'=0} (-1)^{q'} \operatorname{Tr} \left[ \exp \left( -t \Box_p q',0 \right) \right] = \operatorname{Tr} \left[ \exp \left( -t \Box_p X,0 \right) \right].
\]

On the other hand, for $\lambda > 0$, we have a $U(k)$-equivariant exact sequence of vector bundles on $X$
\[
0 \to F_p^{0,\lambda} \to F_p^{1,\lambda} \to \cdots \to F_p^{\dim E-1,\lambda} \to 0.
\]

From (31), we easily deduce that for $\lambda > 0$
\[
\sum_{q'=0} (-1)^{q'} \operatorname{Tr} \left[ \exp \left( -t \Box_p q',\lambda \right) \right] = 0.
\]

Using (30), (32), we get (25).

\[\Box\]

Remark 7. - As $t \to 0$, the left-hand side of (25) has a singularity $t^{-\dim X}$. A priori, the right-hand side has a singularity $t^{-(\dim X+\dim E-1)}$. Therefore a cancellation process occurs in the right-hand side of (25) as $t \to 0$.

Proof of Theorem 1. - Let $r_d$ be the analogue of $r_d'$ on $\mathbb{P}(E^*)$. Namely if $w'_1, \ldots, w'_{r+k-1}$ is an orthonormal base of $T\mathbb{P}(E^*)$, if $w^1, \ldots, w^{r+k-1}$ is the corresponding base of $T^*\mathbb{P}(E^*)$, set
\[
r_d = - \sum r(w'_i, w'_j) \tilde{w}^{ij} \wedge i_{\bar{w}_i}.
\]

Then $r_d$ acts as a derivation of
\[
\Lambda(T^*^{(0,1)}\mathbb{P}(E^*)) = \pi^*(\Lambda(T^*^{(0,1)}X)) \otimes \Lambda(T^{r'}^{(0,1)}\mathbb{P}(E^*)�.
\]

We identify $r$ with the self-adjoint matrix $r \in \operatorname{End}(T\mathbb{P}(E))$ such that $U, V \in T\mathbb{P}(E)$
\[
r(U, V) = \langle U, rV \rangle.
\]

By (1), (2), it is clear that $r_d$ preserves $\pi^*(\Lambda^q(T^*^{(0,1)}X)) \otimes \Lambda^{r'}(T^{r'}^{(0,1)}\mathbb{P}(E^*)�$. Let $r_{d'}^{q'}$ be the corresponding restriction of $r_d$.

Let $dz$ be the volume form on $\mathbb{P}(E^*)$ with respect to the metric $\| \|_{TP(E^*)}$. 
Clearly as $t \to 0$, we have the asymptotic expansion

$$\begin{align*}
(35) \quad (2\pi)^{-(\dim X + \dim E - 1)} & \int_{\mathcal{P}(E^*)} \det (\mathcal{R}) \frac{\text{Tr} [e^{q,q'}]}{\det (1 - e^{-T})} \, dz \\
& = \sum_{j = -\ell - k + 1}^{m} b_{j}^{q,q'} t^{j} + o(t^m).
\end{align*}$$

For any $p \in \mathbb{N}$, $0 \leq q, q' \leq \dim E - 1$, as $t \to 0$, we have the asymptotic expansion

$$\begin{align*}
(36) \quad p^{-(\dim X + \dim E - 1)} & \text{Tr} \left[ \exp \left( -\frac{t}{p} \mathcal{P}(E^*, q, q') \right) \right] = \sum_{j = -\ell - k + 1}^{m} b_{j}^{q,q'} t^{j} + o(t^m).
\end{align*}$$

By a straightforward adaptation of [B2]. Theorem 1.5, and [BV] Theorem 2, we know that for any $t > 0$

$$\begin{align*}
(37) \quad \lim_{p \to +\infty} p^{-(\dim X + \dim E - 1)} & \text{Tr} \left[ \exp \left( -\frac{t}{p} \mathcal{P}(E^*, q, q') \right) \right] \\
& = (2\pi)^{-(\dim X + \dim E - 1)} \frac{\text{det} (\mathcal{R}) \text{Tr} [e^{q,q'}]}{\det (1 - e^{-T})} \\
& = rk(\xi) \int_{\mathcal{P}(E^*)} \frac{\text{det} (\mathcal{R}) \text{Tr} [e^{q,q'}]}{\det (1 - e^{-T})} \, dz.
\end{align*}$$

and the convergence is uniform as $t$ varies in compact subsets of $R^*$. Also as $p \to +\infty$

$$\begin{align*}
(38) \quad b_{p,j}^{q,q'} &= rk(\xi) b_{j}^{q,q'} + O\left( \frac{1}{\sqrt{p}} \right).
\end{align*}$$

Moreover in (36), $o(t^m)$ is uniform with respect to $p \in \mathbb{N}$.

By (2) $\mathcal{R}$ map $T^Y \mathcal{P}(E^*)$ into itself. Let $\mathcal{R}^Y$ be the restriction of $\mathcal{R}$ to $T^Y \mathcal{P}(E^*)$. We then find that

$$\begin{align*}
(39) \quad \sum_{0}^{\dim E - 1} (-1)^{q'} \text{Tr} [e^{tr_{q,q'}}] &= \text{Tr} [e^{tr_{H,q}}] \text{det} (1 - e^{-tr_Y}) \\
& = \text{det} (1 - e^{-tr_Y}) = \text{det} (1 - e^{-tr_H}) \text{det} (1 - e^{-tr_Y}).
\end{align*}$$

By (25), (37), (39), we get

$$\begin{align*}
(40) \quad \lim_{p \to +\infty} p^{-(\dim X + \dim E - 1)} & \text{Tr} \left[ \exp \left( -\frac{t}{p} \mathcal{P}(E^*, q, q') \right) \right] \\
& = rk(\xi) \int_{\mathcal{P}(E^*)} \frac{\text{Tr} [e^{tr_{H,q}}]}{\det (1 - e^{-tr_H})} \frac{1}{2\pi} \, dz.
\end{align*}$$
Clearly

\[(41) \quad \det \left( \frac{t}{2\pi} \right) dz = \left[ \exp \left( \frac{-r}{2i\pi} \right) \right]^{\text{max}}.\]

Using (40), (41), we get (8). From the previous considerations, we also obtain the full proof of Theorem 1. \(\square\)

3. The asymptotics of the Ray-Singer analytic torsion as \(p \to \infty\).

From now on, we assume that the holomorphic Hermitian vector bundle \((E, \| \cdot \|_E)\) is positive, i.e. that if \(U \in TX\setminus\{0\}, e \in E\setminus\{0\}\)

\[(42) \quad \langle L^E(U, \bar{U})e, \bar{e} \rangle > 0.\]

From (2), we find that if \(y \in E^*\setminus\{0\}\) represent \(z \in \mathbb{P}(E^*)\), then

\[(43) \quad r_z = r^y + \pi^* \frac{\langle L^E y, y \rangle}{|y|^2}.\]

Classically [GrH] p. 30, the restriction of the line \((\mu, \| \cdot \|_\mu)\) to the fibres \(\mathbb{P}(E^*)\) is positive, i.e. if \(U \in T^V\mathbb{P}(E^*)\setminus\{0\}, r^V(U, \bar{U}) > 0\). From (43), we deduce that the Hermitian line bundle \((\mu, \| \cdot \|_\mu)\) is positive on \(\mathbb{P}(E^*)\). This is of course a well-known result [K] Theorem III 6.19.

**Theorem 8.** - There exists \(C > 0, c > 0, c' > 0\) such that for any \(p \in \mathbb{N}, 1 \leq q \leq \ell, t \geq 1, \) then

\[(44) \quad p^{-(\dim X + \dim \mathcal{E} - 1)} \text{Tr} \left[ \exp \left( -\frac{t}{p} \Box_p^{X,q,q} \right) \right] \leq C \exp \left( -\left( c - \frac{c'}{p} \right) t \right).\]

**Proof.** - By [BV] Theorems 1 and 2, there exist \(C > 0, c > 0, c' > 0\) such that for \(p \in \mathbb{N}, 0 \leq q \leq \ell, 0 \leq q' < k - 1, q + q' \geq 1, t \geq 1\)

\[(45) \quad \text{Tr} \left[ \exp \left( -\frac{t}{p} \Box_p^{\mathbb{P}(E^*), q,q'} \right) \right] \leq C \exp \left( -\left( c - \frac{c'}{p} \right) t \right).\]

Using (25) and (45), (44) follows. \(\square\)
Remark 9. — Let $\lambda_p^q$ be the lowest eigenvalue of $\Box_p X^q$. From (44), we deduce that if $q \geq 1$

\begin{equation}
\lambda_p^q \geq cp - c'.
\end{equation}

(46) is also an easy consequence of Theorem 5, of the considerations in the proof of Theorem 6 and of [BV], Theorem 1. [BV] Theorem 1 is itself a consequence of the Bochner-Kodaira-Nakano formula of Demailly [De] for the operator $\Box_p^p(E)$. Strangely enough, (46) does not seem to be a straightforward consequence of a similar formula for $\Box_p X^q$.

By Theorem 8 or by (46), there exists $p_0 \in \mathbb{N}$ such that if $p \geq p_0$, $1 \leq q \leq \ell$, the operator $\Box_p^p(E) \otimes \xi$ is invertible.

**Definition 10.** — For $p \geq p_0$, $s \in \mathbb{C}$, $\text{Re} (s) \geq \ell$, set

\begin{equation}
\zeta_p(s) = -\frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \left( \sum_{i=1}^{\ell} (-1)^q \text{Tr} \left[ \exp \left( -t \Box_p^p X^q \right) \right] \right) dt.
\end{equation}

By a well-known result of Seeley [Se], $\zeta_p(s)$ extends into a meromorphic function of $s \in \mathbb{C}$ which is holomorphic at $s = 0$. By definition $\exp \left( -\zeta_p'(0) \right)$ is the Ray-Singer analytic torsion [RS] of the Hermitian vector bundle $S^p(\xi) \otimes \xi$.

We now state the main result of this paper.

**Theorem 11.** — As $p \to +\infty$

\begin{equation}
\zeta_p'(0) = \text{rk}(\xi) \frac{1}{2} \int_{p(E^*)} \text{Log} \left[ \left| \det \left( \frac{p_p R}{2\pi} \right) \right| \exp \left( \frac{-p_p}{2i\pi} \right) \right. \left. + o(p^{\text{dim} X + \text{dim} E - 1}) \right].
\end{equation}

In particular as $p \to +\infty$

\begin{equation}
\zeta_p'(0) = O(p^{\text{dim} X + \text{dim} E - 1} \text{Log} p).
\end{equation}

**Proof.** — In view of Theorems 1 and 8, which are the obvious extensions of [BV] Theorem 2, the proof of Theorem 11 proceeds formally as the proof of [BV] Theorems 4 and 8. Details are left to the reader.
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