GERTRUDE K. IMMINK

On meromorphic equivalence of linear difference operators


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ON MEROMORPHIC EQUIVALENCE OF LINEAR DIFFERENCE OPERATORS

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0. Introduction.

We consider linear difference operators of the type

\[ \Delta_A y(z) = y(z+1) - A(z) y(z), \]

where \( A \in \text{Gl}(n; \mathbb{C}[z^{-1}]) \), \( n \in \mathbb{N} \).

Two difference operators \( \Delta_A \) and \( \Delta_B \) will be called formally equivalent, if there exists a matrix \( F \in \text{Gl}(n; \mathbb{C}[z^{-1}]) \) such that \( F(z+1)^{-1} A(z) F(z) = B(z) \). The difference operators \( \Delta_A \) and \( \Delta_B \) will be called meromorphically equivalent if there exists a matrix \( F \in \text{Gl}(n; \mathbb{C}[z^{-1}]) \) with this property.

Meromorphically equivalent difference operators are also formally equivalent. This paper is concerned with the meromorphic classification of difference operators belonging to the same formal equivalence class. The meromorphic equivalence classes will be characterized by a system of "meromorphic invariants". It is shown that this system is both complete and free. In other words, if two difference operators have the same set of invariants they must be equivalent. Moreover, every possible set of invariants corresponds to some equivalence class.

Neither the results nor the methods of this paper are entirely new. We have merely rearranged and extended some results already established by Birkhoff (cf. [1], [2]).

In §1 we define a system of meromorphic invariants. Its completeness is proved by an argument familiar from the theory of differential equations (cf. [7], [8], [11]). The main difficulty is to solve the inverse problem, i.e. to establish the existence of a difference operator having a given set of invariants. This is done in §2. The problem is reduced to a Riemann-Hilbert boundary value problem (also called Riemann problem by some and Hilbert problem by others) on two intersecting contours (cf. also [4]). Here we have resorted to well-known existence theorems (cf. [9], [13]), rather than adopting Birkhoff’s constructive but elaborate method. In both sections we pay special attention to the important subclass of difference operators with rational coefficients.

We do not go into the difficult problem of the actual evaluation of the meromorphic invariants. An attempt in that direction was undertaken in [6]. For a very profound study of the analytic invariants of various local objects we refer the reader to the work of J. Ecalle (cf. [3]).

1. A complete system of meromorphic invariants.

We use the following notations
\[ K = \mathbb{C}\{z^{-1}\}[z], \quad \hat{K} = \mathbb{C}\{z^{-1}\}[z], \quad K_p = \mathbb{C}\{z^{-1/p}\}[z^{1/p}], \quad \hat{K}_p = \mathbb{C}\{z^{-1/p}\}[z^{1/p}], \quad p \in \mathbb{N}. \]

Let \( A \in \text{Gl}(n; K) \). It is known that there exists a positive integer \( p \) and a matrix \( F \in \text{Gl}(n; \hat{K}_p) \) such that the transformation
\[ A \mapsto A^F \equiv F(z + 1)^{-1}A(z)F(z) \]
changes \( A \) into a matrix function \( \hat{A} \) of the form
\[ \hat{A}(z) = \exp\{Q(z + 1) - Q(z)\}(1 + \frac{1}{z})^G \]
where \( G = \text{diag}\{G_1, \ldots, G_m\}, Q = \text{diag}\{q_1 I_{n_1}, \ldots, q_m I_{n_m}\} \) with \( m \in \mathbb{N} \) and, for all \( i \in \{1, \ldots, m\}, G_i = \gamma_i I_{n_i} + N_i, \gamma_i \in \mathbb{C}, 0 \leq \Re \gamma_i < \frac{1}{p}, N_i \)
is a nilpotent \( n_i \times n_i \) matrix, \( q_i(z) = d_i z \log z + \sum_{h=1}^{p} \mu_{i,h} z^{h/p}, d_i \in \frac{1}{p} \mathbb{Z}, \mu_{i,h} \in \mathbb{C}, 0 \leq \Im \mu_{i,p} < 2\pi. \)
We shall call \( \hat{A} \) a canonical form of \( A \). It is uniquely determined by \( A \) up to permutations of the diagonal blocks (cf. [10]).

We shall write
\[
d_i - d_j = d_{ij}, \quad q_i - q_j = q_{ij}, \quad \gamma_i - \gamma_j = \gamma_{ij}, \quad \mu_{i,h} - \mu_{j,h} = \mu_{ij,h}
\]
for all \( i, j \in \{1, \ldots, m\} \), \( h \in \{1, \ldots, p\} \).

**Lemma 1.1.** Let \( A \in \text{Gl}(n; K) \), \( F_1, F_2 \in \text{Gl}(n; \hat{K}_p) \) with \( p \in \mathbb{N} \), and suppose that
\[
A^F_1 = A^F_2 = \hat{A}.
\]
Then there exists a constant invertible \( n \times n \) matrix \( C \) such that
\[
[H^C, F] = 0 \quad \text{and} \quad F_2 = F_1 C.
\]

**Proof.** Let \( F = F_1^{-1} F_2 \). Then we have
\[
F(z + 1) = A(z) F(z) A(z)^{z-1}.
\]
Hence the block \( F_{ij} \) in the partition of \( F \) induced by \( \hat{A} \), must satisfy the equation
\[
Y(z + 1) = \exp\{q_{ij}(z + 1) - q_{ij}(z)\}(1 + 1/z)^{r_{ij}} (1 + 1/z)^{N_{ij}} Y(z)(1 + 1/z)^{-N_{ij}},
\]
for all \( i, j \in \{1, \ldots, m\} \).

This equation has no nonvanishing solutions \( \in \text{Hom}(\hat{K}_p^{n_j}, \hat{K}_p^{n_i}) \) unless \( q_{ij} \equiv 0 \), \( \gamma_{ij} = 0 \). In the latter case the only solutions \( \in \text{Hom}(\hat{K}_p^{n_j}, \hat{K}_p^{n_i}) \) are the constant \( n_i \times n_j \) matrices \( C_{ij} \) with the property that
\[
N_{ij} C_{ij} = C_{ij} N_{ij}.
\]
Thus \( F \) is a constant matrix with the property that
\[
\hat{A}_{ii} F_{ij} = F_{ij} \hat{A}_{jj} \quad \text{for all} \quad i, j \in \{j, \ldots, m\}
\]
and hence \([\hat{A}, F] = 0\). Obviously, \( \det F \neq 0 \).

**Definition.** (i) A quadrant is a region \( \Gamma \) of \( \mathbb{C} \) of the following form:
\[
\Gamma = \{ z \in \mathbb{C} : k \frac{\pi}{2} < \arg(z - z_0) < (k + 1) \frac{\pi}{2}, \ |z| > R_0 \}
\]
where \( z_0 \in \mathbb{C}, \ k \in \mathbb{Z} \) and \( R_0 \) is a positive number.
(ii) Let $\Gamma$ be a region of $\mathbb{C}$ and $R$ a positive number. By $\Gamma^*$ and $\Gamma(R)$ we shall denote the regions

$$\Gamma^* = \{z \in \mathbb{C} : \bar{z} \in \Gamma\}$$

and

$$\Gamma(R) = \{z \in \Gamma : |z| > R\}.$$

**Definition.** — Let $\Gamma$ be an unbounded region of $\mathbb{C}$, $\varphi$ an analytic function on $\Gamma$ and $f$ a formal series of the form $f = \sum_{n \geq n_0} f_n z^{-n/p}$, $n_0 \in \mathbb{Z}$.

We shall say that $\varphi$ is represented asymptotically by $f$ (or admits the asymptotic expansion $f$) as $z \to \infty$ in $\Gamma$, if

$$\sup_{z \in \Gamma} \left| (\varphi(z) - \sum_{n=n_0}^{N-1} f_n z^{-n/p}) z^N/p \right| < \infty \text{ for all } N > n_0.$$

In that case we write

$$\varphi(z) \sim f, \quad z \to \infty \text{ in } \Gamma,$$

and put $f = \widehat{\varphi}$.

**Definition.** — A matrix function $\Phi$ will be called non singular on a set $S$ if $\det \Phi(z) \neq 0$ for all $z \in S$.

The following theorem is a slightly improved version of theorem 3.2 in [5].

**Theorem 1.2.** — Let $A \in \text{Gl}(n; K)$ and let $\hat{A}$ be a canonical form of $A$. Let $\Gamma$ be a quadrant. There exists a positive number $R$ and a matrix function $\Phi$ with the following properties:

(i) $\Phi$ is non singular and analytic in $\Gamma(R) \cup \Gamma(R)^*$

(ii) $\Phi$ is represented asymptotically by a matrix $F \in \text{Gl}(n; \hat{K}_p)$ ($p \in \mathbb{N}$), as $z \to \infty$ in $\Gamma(R)$

(iii) $A^\Phi = \hat{A}$.

**Proof.** — We shall consider the case that $\Gamma$ (and hence $\Gamma^*$) is contained in some right half plane $\text{Re} \ z > a$, $a \in \mathbb{R}$. The proof for a left half plane is analogous. Without loss of generality we may assume that $\Gamma$ contains a strip $S_0$ of the form $S_0 = \{z \in \mathbb{C} : \text{Re} \ z > b, \ |\text{Im} \ z| < c\}$ where
Let $b > 0$, $c > 0$. Let $R$ be a large positive number and let $\Gamma_+ = \Gamma(R)$ if $\text{Im } z$ is bounded from below on $\Gamma$, and $\Gamma_+ = \Gamma(R)^*$ otherwise. Let $\Gamma_- = \Gamma^*$.

According to theorem 3.2 in [5], if $R$ is sufficiently large, there exist matrix functions $\Phi^+$ and $\Phi^-$ with the following properties:

(i) $\Phi^\pm$ is analytic in $\Gamma_\pm$

(ii) $\Phi^\pm$ admits an asymptotic expansion $F^\pm \in \text{Gl}(n; \mathbb{R})$ as $z \to \infty$ in $\Gamma_\pm$

(iii) $A^\Phi = \hat{A}$.

The second property implies that

\[ \det \Phi^\pm(z) \neq 0 \text{ for all } z \in \Gamma_\pm \]

provided $R$ has been chosen sufficiently large. In view of lemma 1.1 we may assume that $F^+ = F^- = F$ (if this is not the case, $\Phi^-$ can be replaced by $\Phi^- C$, where $C = (F^-)^{-1} F^+$). Let

\[ Y^\pm(z) = \Phi^\pm(z) \exp\{Q(z)\} z^G \]

(we take $\text{arg } z \in (-\pi, \pi)$) and

\[ P = (Y^+)^{-1} Y^- ; \]

$P$ is periodic matrix function of period 1, analytic in $S \equiv \Gamma_+ \cap \Gamma_-$. Furthermore, in the partition induced by $\hat{A}$

\[ P_{ij}(z) = \exp\{-q_{ij}(z)\} (z^{-G(\Phi^+)^{-1}} \Phi^- z^G)_{ij} \]

for all $i, j \in \{1, \ldots, m\}$. Due to the fact that $(\Phi^+)^{-1} \Phi^- \sim I$ as $z \to \infty$ in $S$, \[ \lim_{z \to \infty} P_{ij}(z) = \delta_{ij} \]

and hence $P_{ij} \equiv \delta_{ij}$, unless $\exp q_{ij}(z) \to 0$ as $z \to \infty$ in $S$. For convenience we shall assume that the blocks of $\hat{A}$ are ordered in such a way that for all sufficiently large positive numbers $z$

\[ \text{Re } q_{ij}(z) \geq 0 \text{ if } i \geq j . \]

This implies that, for $i \geq j$, $\text{Re}(q_{ij})$ is bounded from below on the positive real axis and hence $P_{ij} \equiv \delta_{ij}$. Following Birkhoff and Trjitzinsky (cf. [2]) we are going to look for periodic matrix functions $P^+$ and $P^-$, upper-block-triangular in the partition induced by $\hat{A}$, analytic in $\Gamma_+$ and $\Gamma_-$ respectively, such that

\[ P = P^+ P^- . \]

We choose $P^+_{ii} = P^-_{ii} = I_n$, for all $i \in \{1, \ldots, m\}$. The remaining blocks can be determined recursively from the relations

\[ P_{ij} = P^+_{ij} + P^-_{ij} + \sum_{i < h < j} P^+_{ih} P^-_{hj}, \ i < j \]
by means of induction on $j - i$. Obviously, the factorization (1.7) is not unique. We may impose the additional condition that, for $i < j$, $P^+_ij$ and $P^-_{ij}$ have the form

\begin{equation}
(1.9) \quad P^+_ij(z) = \sum_{n \geq n_{ij}} (P^+_n)_{ij} e^{2n\pi i z}, \quad P^-_{ij}(z) = \sum_{n < n_{ij}} (P^-_n)_{ij} e^{2n\pi i z}
\end{equation}

where the numbers $n_{ij}$ are arbitrary integers. Now let

\begin{equation}
(1.10) \quad \Phi(z) = Y^+(z)P^+(z) \exp\{-Q(z)\}z^{-G}, \quad z \in \Gamma_+.
\end{equation}

One easily verifies that $A^+ = \hat{A}$. Moreover, due to (1.5) and (1.7), $\Phi$ may be continued analytically to $\Gamma_-$ and we have

\begin{equation}
(1.11) \quad \Phi(z) = Y^-(z)P^-(z)^{-1} \exp\{-Q(z)\}z^{-G}, \quad z \in \Gamma_-.
\end{equation}

Noting that $\det P^\pm = 1$ and using (1.3), (1.4), (1.10) and (1.11), we conclude that $\det \Phi(z) \neq 0$ for all $z \in \Gamma_+ \cup \Gamma_-.

Now let us consider the asymptotic behaviour of $\Phi$ as $z \to \infty$ in $\Gamma_+ \cup \Gamma_-$. For all, $i,j \in \{1, \ldots, m\}$ we have

\begin{equation}
(1.12) \quad \Phi_{ij}(z) - \Phi^\pm_{ij}(z) = \sum_{h<j} \Phi^\mp_{ih}(z) \exp\{q_{hj}(z)\}z^G P^\pm_{hj}(z)z^{-G}, \quad z \in \Gamma_\pm.
\end{equation}

First, suppose that, for some $h < j$, $\Re q^j_{hj}(z) = 0$ for all sufficiently large positive values of $z$. Due to (1.6), the same is true of $\Re q^k_{hk}$ for all $k \in \{h + 1, \ldots, j\}$. Consequently, $P^+_{hk} \equiv 0$ for all $k \in \{h + 1, \ldots, j\}$. With (1.8) and (1.9) it follows that $P^+_{hj} \equiv P^-_{hj} \equiv 0$ for all $k \in \{h + 1, \ldots, j\}$. In particular, $P^\pm_{hj} \equiv 0$. Thus the only non-vanishing terms in the right-hand side of (1.12) are the ones for which $\Re q^j_{hj}(z) < 0$ for sufficiently large positive $z$. This implies that either $d_{hj} < 0$, or else $d_{hj} = 0$ and there is a number $h_0 \in \{1, \ldots, p\}$ such that $\Re \mu_{hj,k} = 0$ for all $k < h_0$ whereas $\Re \mu_{hj,h_0} < 0$. In both cases there exists a number $\delta \in (0,1)$ such that

\begin{equation}
\exp(q_{hj}(z)) = O(\exp\{-z^\delta\}), \quad z \to \infty \text{ in } S.
\end{equation}

As $\Phi^\pm(z) \sim F \in G\ell(n; \hat{K})$ as $z \to \infty$ in $S$, it follows that there exists a $\delta \in (0,1)$ such that

\begin{equation}
(1.13) \quad \Phi_{ij}(z) - \Phi^\pm_{ij}(z) = O(\exp\{-z^\delta\}), \quad z \to \infty \text{ in } S.
\end{equation}

If $\Gamma(R) = \Gamma_+$ we choose the integers $n_{ij}$ in (1.9) in such a way that

\[ 2n_{ij}\pi + d_{ij}\pi + \Im \mu_{ij,p} > 0, \quad \text{for } i < j. \]

Then it is easily seen that all terms in the right-hand side of (1.12) decrease exponentially as $z \to \infty$ in $\Gamma_+$, provided $\arg z > \varepsilon$ for some
positive number $\varepsilon$. Combining this with (1.13) and applying a well-known theorem of Phragmén-Lindelöf (cf. [12], p. 177) we conclude that, for all $i, j \in \{1, \ldots, m\}$,

$$\Phi_{ij}(z) - \Phi_{ij}^+(z) = O(\exp\{-z^\delta\}), \quad z \to \infty \text{ in } \Gamma_+$$

and, consequently, $\Phi^+(z) \sim F$ as $z \to \infty$ in $\Gamma_+$. If, on the other hand, $\Gamma(R) = \Gamma_-$ we choose the integers $n_{ij}$ in such a way that

$$2(n_{ij} - 1)\pi - d_{ij} \pi + \text{Im } \mu_{ij,p} < 0 \text{ for } i < j.$$

By means of an argument similar to the one used above we find that

$$\Phi^-(z) \sim F \text{ as } z \to \infty \text{ in } \Gamma_- \text{ in that case.}$$

The next theorem follows immediately from theorem 1.2 and lemma 1.1.

**Theorem 1.14.** — Let $A, B \in \text{Gl}(n; K)$ and suppose there exists a matrix $F \in \text{Gl}(n; \hat{K}_p)$ such that $A^F = B$. Let $\Gamma$ be a quadrant. There exists a positive number $R$ and a matrix function $\Phi$ with the following properties:

(i) $\Phi$ is non singular and analytic in $\Gamma(R) \cup \Gamma(R)^*$

(ii) $\Phi(z) \sim F(z)$ as $z \to \infty$ in $\Gamma(R)$

(iii) $A^\Phi = B$.

**Proof.** — Let $\hat{A}$ be a canonical form of $A$. Obviously, $\hat{A}$ is a canonical form of $B$ as well. According to theorem 1.2 there exists a positive number $R$ and matrix functions $\Phi_1$ and $\Phi_2$ with the following properties:

(i) $\Phi_1$ and $\Phi_2$ are non singular and analytic in $\Gamma(R) \cup \Gamma(R)^*$

(ii) $\Phi_j(z) \sim \hat{\Phi}_j \in \text{Gl}(n; \hat{K}_p)$ as $z \to \infty$ in $\Gamma(R), j = 1, 2$

(iii) $A^{\Phi_1} = B^{\Phi_2} = \hat{A}$.

Consequently, $A^{\hat{\Phi}_1} = B^{\hat{\Phi}_2} = A^{F^{\hat{\Phi}_2}} = \hat{A}$. By lemma 1.1 this implies that $F^{\hat{\Phi}_2} = \hat{\Phi}_1 C$, where $C$ is a constant invertible matrix which commutes with $\hat{A}$. Hence it follows that

$$A^{\Phi_1 C^{\Phi_2^{-1}}} = A^{C^{\Phi_2^{-1}}} = B.$$

One easily verifies that the matrix function $\Phi \equiv \Phi_1 C^{\Phi_2^{-1}}$ has the required properties.
DEFINITION. — Let $\Phi$ be a meromorphic matrix function in $\mathbb{C}$. By $P_\Phi$ we shall denote the set
$$P_\Phi = \{z \in \mathbb{C} : \Phi \text{ has a pole in } z\}.$$

COROLLARY of theorem 1.14. — If, in addition to the assumptions made in theorem 1.14, $A$ and $B$ are matrices of rational functions, then any matrix $\Phi$ with the properties (i)–(iii) can be continued analytically to a meromorphic matrix function. Moreover

$$(1.15) \quad P_\Phi \subseteq P_A \cup P_{B^{-1}} + \mathbb{N}, \quad P_{\Phi^{-1}} \subseteq P_{A^{-1}} \cup P_B + \mathbb{N}$$

if $\Gamma$ is contained in a left half plane, whereas

$$(1.16) \quad P_\Phi \subseteq P_{A^{-1}} \cup P_B - \mathbb{N}_0, \quad P_{\Phi^{-1}} \subseteq P_A \cup P_{B^{-1}} - \mathbb{N}_0$$

if $\Gamma$ is contained in a right half plane.

Proof. — If $\Gamma$ is contained in a left half plane the matrix function $\Phi$ with the properties (i)–(iii) mentioned in theorem 1.14 may be continued analytically to the right by means of the relation

$$(1.17) \quad \Phi(z + 1) = A(z)\Phi(z)B(z)^{-1}.$$

If, on the other hand, $\Gamma$ is contained in a right half plane, $\Phi$ may be continued to the left by means of

$$(1.18) \quad \Phi(z) = A(z)^{-1}\Phi(z + 1)B(z).$$

(1.15) and (1.16) follow immediately from (1.17) and (1.18), respectively.

THEOREM 1.19. — Let $A, B, M \in \text{Gl}(n; K)$ and suppose that $\Delta_A$ and $\Delta_B$ are formally equivalent to $\Delta_M$. Let $\Gamma_j, j = 1, \ldots, 4$, be quadrants such that $\Gamma_{j+1} = e^{i\frac{\pi}{2}}\Gamma_j$ for $j = 1, 2, 3$, and $\Gamma_1, \ldots, \Gamma_4$ cover a neighbourhood of $\infty$. $\Delta_A$ and $\Delta_B$ are meromorphically equivalent if and only if there exists a positive number $R$ and matrix functions $\Phi_j$ and $\Psi_j$, $j = 1, \ldots, 4$, with the following properties:

(i) $\Phi_j$ and $\Psi_j$ are non singular and analytic in $\Gamma_j(R)$

(ii) $\Phi_j$ and $\Psi_j$ admit asymptotic expansions $\hat{\Phi}_j$ and $\hat{\Psi}_j \in \text{Gl}(n; \hat{K})$ as $z \to \infty$ in $\Gamma_j(R)$ and both $\hat{\Phi}_j$ and $\hat{\Psi}_j$ are independent of $j \in \{1, \ldots, 4\}$

(iii) $A^{\Phi_j} = B^{\Psi_j} = M$ for all $j \in \{1, \ldots, 4\}$

(iv) $\Phi_j^{-1}\Phi_{j+1} = \Psi_j^{-1}\Psi_{j+1}$ for $j \in \{1, 2, 3\}$ and $\Phi_4^{-1}\Phi_1 = \Psi_4^{-1}\Psi_1$. 

Proof. — Suppose that $\Delta_A$ and $\Delta_B$ are meromorphically equivalent. Then there exists a matrix function $F \in \text{Gl}(n; K)$ such that $A^F = B$. According to theorem 1.14 there exists a positive number $R$ and matrix functions $\Phi_j$, $j = 1, \ldots, 4$, with the properties (i) – (iii) mentioned above. Let $\Psi_j = F^{-1}\Phi_j$, $j \in \{1, \ldots, 4\}$. One easily verifies that conditions (i) – (iv) of theorem 1.19 are satisfied.

Conversely, suppose that (i) – (iv) hold. (iv) implies that
\[ \Phi_j(z)\Psi_j^{-1}(z) = \Phi_{j+1}(z)\Psi_{j+1}^{-1}(z) , \quad z \in \Gamma_j(R) \cap \Gamma_{j+1}(R), \quad j = 1, 2, 3 \]
and
\[ \Phi_4(z)\Psi_4^{-1}(z) = \Phi_1(z)\Psi_1^{-1}(z) , \quad z \in \Gamma_4(R) \cap \Gamma_1(R) . \]
Hence the matrix function $F \equiv \Phi_1\Psi_1^{-1}$ may be continued analytically to $\bigcup_{j=1}^4 \Gamma_j(R)$, i.e. to a reduced neighbourhood of $\infty$. Moreover, $F$ admits an asymptotic expansion $\hat{F} = \Phi_j\Psi_j^{-1} \in \text{Gl}(n; \hat{K})$ as $z \to \infty$ in $\bigcup_{j=1}^4 \Gamma_j(R)$. Therefore, $F \in \text{Gl}(n; K)$. Obviously,
\[ A^F = A^{\Phi_1}\Psi_1^{-1} = M^{\Psi_1^{-1}} = B . \]

The connection matrices $T_j \equiv \Phi_j^{-1}\Phi_{j+1}$, $j = 1, 2, 3$, and $T_4 = \Phi_4^{-1}\Phi_1$ are uniquely determined by $M$ and $A$ up to transformations of the following type
\[ T_j \rightarrow S_j^{-1}T_jS_{j+1} , \quad j = 1, 2, 3 \]
\[ T_4 \rightarrow S_4^{-1}T_4S_1 \]
where $S_j$ is a non singular and analytic matrix function in $\Gamma_j(R)$, admitting an asymptotic expansion $\hat{S}_j \in \text{Gl}(n; \hat{K})$ as $z \to \infty$ in $\Gamma_j(R)$, independent of $j$, with the additional property that $M^{S_j} = M$ for all $j \in \{1, \ldots, 4\}$. The set of connection matrices $\{T_1, \ldots, T_4\}$ modulo transformations of the form (1.20) constitutes a complete system of meromorphic invariants of the difference operator $\Delta_A$.

It is easily seen that theorem 1.19 remains valid if (i) is replaced by

(i)' $\Phi_j$ and $\Psi_j$ are non singular and analytic in $\Gamma_j(R) \cup \Gamma_j(R)^*$.

This implies that, in the case that $A$, $B$ and $M$ have rational coefficients, the matrix functions $\Phi_j$ and $\Psi_j$ may be continued analytically to meromorphic functions in $\mathbb{C}$ (cf. the corollary of theorem 1.14). Thus the meromorphic equivalence classes of matrices of rational functions which are formally equivalent to a matrix of rational functions, can be characterized by a set
of meromorphic connection matrices \( \{T_1, \ldots, T_4\} \) with the property that 
\[ T_1T_2T_3T_4 = I, \]
modulo transformations of the type (1.20), where \( S_j \) is meromorphic in \( \mathbb{C} \) for all \( j \in \{1, \ldots, 4\} \).

2. The inverse problem.

**Definition.** — Let \( \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}, z_0 \in \hat{\mathbb{C}} \) and let \( C \) be a simple closed contour in \( \hat{\mathbb{C}} \setminus \{z_0\} \). \( C \) is the positively oriented boundary of a domain \( D^+ \subset \hat{\mathbb{C}} \setminus \{z_0\} \) and the negatively oriented boundary of a domain \( D^- \subset \hat{\mathbb{C}} \setminus \{z_0\} \). We shall call \( D^+ \) the interior and \( D^- \) the exterior of \( C \).

We shall call \( \Phi \) a sectionally holomorphic function in \( \hat{\mathbb{C}} \setminus \{z_0\} \), relative to \( C \), if

(i) \( \Phi \) is holomorphic in \( D^+ \cup D^- \), and

(ii) for any \( t \in C \), \( \Phi \) approaches a definite limiting value \( \Phi^+(t) \) or \( \Phi^-(t) \) as \( z \to t \) along any path in \( D^+ \) or \( D^- \), respectively.

A matrix function \( \Phi \) will be called non singular and sectionally holomorphic in \( \hat{\mathbb{C}} \setminus \{z_0\} \), relative to \( C \), if in addition to (i) and (ii) above, \( \Phi \) is non singular in \( D^+ \cup D^- \) and both \( \Phi^+ \) and \( \Phi^- \) are non singular on \( C \).

We begin by stating a well-known result (cf. [9], [13]).

**Theorem 2.1.** — Let \( z_0 \in \hat{\mathbb{C}} \) and let \( C \) be a simple, smooth, closed contour in \( \hat{\mathbb{C}} \setminus \{z_0\} \). Let \( A \) be a non singular, Hölder continuous matrix function on \( C \).

There exists a matrix function \( \Phi \), non singular and sectionally holomorphic in \( \hat{\mathbb{C}} \setminus \{z_0\} \), relative to \( C \), with the following properties :

(i) \( \Phi^+(z) = \Phi^-(z)A(z) \) for all \( z \in C \)

(ii) \( \Phi \) has at most a pole in \( z_0 \).

Moreover, \( \Phi^+ \) and \( \Phi^- \) are Hölder continuous on \( C \).

**Remark.** — The usual version of this theorem applies to the case that \( z_0 = \infty \). However, the general situation can be easily reduced to that case by means of a linear fractional transformation \( \varphi \) of the form

\[ \varphi(z) = a \frac{z - z_1}{z - z_0}, \]

with \( a \neq 0, z_1 \neq z_0 \).
Theorem 2.1 will enable us to solve the inverse problem mentioned in the introduction. We shall take \( z_0 = 0 \) and put \( \tilde{C} \setminus \{0\} = \tilde{C}^* \). Throughout this section \( \{\Gamma_j, j = 1, \ldots, 4\} \) will denote a set of quadrants such that \( \Gamma_{j+1} = e^{i\frac{\pi}{2}}\Gamma_j \) for \( j = 1, 2, 3 \), and \( \Gamma_1, \ldots, \Gamma_4 \) cover a neighbourhood of \( \infty \). We define: \( S_j = \Gamma_j \cap \Gamma_{j+1} \) for \( j = 1, 2, 3 \) and \( S_4 = \Gamma_4 \cap \Gamma_1 \). Furthermore, we shall assume that

\[
\sup_{z \in S_j} \Re(\zeta - z) > 1 \quad \text{for all } j \in \{1, \ldots, 4\}.
\]

Let \( j \in \{1, \ldots, 4\}, R > 0, \varepsilon > 0 \). By \( \Gamma_{j,\varepsilon}(R) \) we shall denote the quadrant

\[
\Gamma_{j,\varepsilon}(R) = \{z \in \Gamma_j(R) : |z - \zeta| > \varepsilon \text{ for all } \zeta \in \partial \Gamma_j(R)\}.
\]

**Theorem 2.2.** — Suppose we are given a matrix function \( M \in \mathcal{G}_\ell(n; K) \) and matrix functions \( T_j, j = 1, \ldots, 4 \), with the following properties:

1. \( T_j \) is analytic in \( S_j \)
2. \( T_j \sim I \) as \( z \to \infty \) in \( S_j \)
3. \( M^{T_j} = M \).

There exists a positive number \( R \), a matrix function \( A \in \mathcal{G}_\ell(n; K) \) and matrix functions \( \Phi_j, j = 1, \ldots, 4 \), with the following properties:

1. \( \Phi_j \) is analytic in \( \Gamma_j(R) \) and admits an asymptotic expansion \( F \in \mathcal{G}_\ell(n; \tilde{K}) \), independent of \( j \), as \( z \to \infty \) in \( \Gamma_{j,\varepsilon}(R) \), for every \( \varepsilon > 0 \)
2. \( A^{\Phi_j} = M \)
3. \( \Phi_j^{-1}\Phi_{j+1} = T_j \) for \( j = 1, 2, 3 \) and \( \Phi_4^{-1}\Phi_1 = T_4 \).

We shall prove theorem 2.2 in two steps. Let \( C_1 \) be a smooth contour in \( \Gamma_2 \cup \Gamma_3 \) consisting of a half line \( L_1 \) in \( S_1 \), a half line \( L_2 \) in \( S_3 \) and an arc connecting the starting points of \( L_1 \) and \( L_2 \). Since \( T_j \sim I \) as \( z \to \infty \) in \( S_j \), there exists a positive number \( R_0 \) such that

\[
\text{det} T_j(z) \neq 0 \quad \text{for all } z \in S_j(R_0), \quad j \in \{1, \ldots, 4\}.
\]

Let \( T \) be a matrix function on \( C_1 \cup S_1(R_0) \cup S_3(R_0) \) with the following properties:

1. \( T \) is Hölder continuous on \( C_1 \)
2. \( \text{det} T \neq 0 \) on \( C_1 \)
3. \( T(z) = T_1(z) \) for \( z \in S_1(R_0) \), \( T(z) = T_3(z)^{-1} \) for \( z \in S_3(R_0) \).
PROPOSITION 2.4. — There exists a positive number \( R \), a matrix function \( \Phi^+ \), non singular and holomorphic in \( \Gamma_2(R) \cup \Gamma_3(R) \) and a matrix function \( \Phi^- \), non singular and holomorphic in \( \Gamma_4(R) \cup \Gamma_1(R) \) such that

(i) \( \Phi^+(z) = \Phi^-(z)T(z) \) for all \( z \in S_1(R) \cup S_3(R) \)

(ii) \( \Phi^+ \) and \( \Phi^- \) admit the same asymptotic expansion \( \check{\Phi} \in \mathcal{O}(n;\Omega[z^{-1}]) \) as \( z \to \infty \) in \( \Gamma_{2,\varepsilon}(R) \cup \Gamma_{3,\varepsilon}(R) \) and \( \Gamma_{4,\varepsilon}(R) \cup \Gamma_{1,\varepsilon}(R) \), respectively, for any \( \varepsilon > 0 \).

Proof. — Let \( D^+ \) and \( D^- \) denote the interior and exterior of \( C_1 \), respectively. According to theorem 2.1 there exists a matrix function \( \Phi \), non singular and sectionally holomorphic in \( \hat{\mathbb{C}}^* \), relative to \( C_1 \), such that

\[
\Phi^+(z) = \Phi^-(z)T(z) \quad \text{for all} \quad z \in C_1
\]

and \( \Phi \) has at most a pole in 0. By Cauchy’s theorem,

\[
\Phi(z) = \frac{z}{2\pi i} \int_{C_1} \frac{\Phi^+(t)}{t(t-z)} dt, \quad z \in D^+
\]

and

\[
\int_{C_1} \frac{\Phi^-(t)}{t(t-z)} dt = \int_{C_0} \frac{\Phi(t)}{t(t-z)} dt, \quad z \in D^+,
\]

where \( C_0 \) is a simple, closed contour in \( D^- \) enclosing 0. With (2.5) it follows that

\[
\Phi(z) = \frac{z}{2\pi i} \int_{C_1} \frac{\Phi^-(t)(T(t) - I)}{t(t-z)} dt + \frac{z}{2\pi i} \int_{C_0} \frac{\Phi(t)}{t(t-z)} dt, \quad z \in D^+.
\]

Similarly, we have

\[
\Phi(z) = \frac{z}{2\pi i} \int_{C_1} \frac{\Phi^+(t)(I - T(t)^{-1})}{t(t-z)} dt + \frac{z}{2\pi i} \int_{C_0} \frac{\Phi(t)}{t(t-z)} dt, \quad z \in D^-
\]

where \( C_0(z) \) is a simple closed contour in \( D^- \), enclosing 0, but not \( z \). In view of (2.3) and the third property of \( T \), both \( T \) and \( T^{-1} \) are holomorphic in \( S_1(R_0) \cup S_3(R_0) \). Hence we deduce, by deforming the contour \( C_1 \) in (2.6) and (2.7), that \( \Phi^+ \) and \( \Phi^- \) may be continued analytically to \( D^+ \cup S_1(R_0) \cup S_3(R_0) \) and \( D^- \cup S_1(R_0) \cup S_3(R_0) \), respectively. Consequently, (2.5) holds for all \( z \in S_1(R_0) \cup S_3(R_0) \). As \( \Phi \) is non singular in \( \hat{\mathbb{C}}^* \) and \( T \)

Next we consider the asymptotic behaviour of \( \Phi^+ \) and \( \Phi^- \) as \( z \to \infty \). Note that the second integral in the right-hand side of (2.6) and (2.7) is
holomorphic at $\infty$. Furthermore, $T(z) \sim I$ as $z \to \infty$ in $S_1(R_0) \cup S_3(R_0)$, hence
\[ \sup_{t \in S_1(R_0) \cup S_3(R_0) \cup C_1} |(T(t) - I)t^n| < \infty \quad \text{for all } n \in \mathbb{N}. \]
Putting
\[ -\frac{1}{2\pi i} \int_{C_1} \Phi^-(t)(T(t) - I)t^{n-1} dt = F_n \]
we have, for all $z \in D^+$,
\[ \frac{z}{2\pi i} \int_{C_1} \frac{\Phi^-(t)(T(t) - I)}{t(t - z)} dt - \sum_{n=0}^{N} F_n z^{-n} = \frac{1}{2\pi i} z^{-N} \int_{C_1} \frac{\Phi^-(t)(T(t) - I)}{t - z} t^N dt. \]
The right-hand side of this identity is $O(z^{-N})$ as $z \to \infty$, uniformly on $D^+_\varepsilon \equiv \{ z \in D^+ : |z - t| > \varepsilon \text{ for all } t \in C_1 \}$ for any $\varepsilon > 0$. Hence, it follows that $\Phi$ admits an asymptotic expansion $\hat{\Phi}^+$ as $z \to \infty$ in $D^+_\varepsilon$ for any $\varepsilon > 0$. Moreover, $\lim_{z \to \infty} \Phi(z) = \Phi^+(\infty)$ and $\det \Phi^+(\infty) \neq 0$.

This implies that $\hat{\Phi}^+ \in \text{Gl}(n; \mathbb{C}[z^{-1}])$. By varying the contour $C_1$ we find that $\Phi^+(z) \sim \hat{\Phi}^+(z)$ as $z \to \infty$ in $\Gamma_{2,\varepsilon}(R) \cup \Gamma_{3,\varepsilon}(R)$ for any $\varepsilon > 0$ and a sufficiently large number $R$. In a similar manner one proves that $\Phi^-$ admits an asymptotic expansion $\hat{\Phi}^-$ as $z \to \infty$ in $\Gamma_{4,\varepsilon}(R) \cup \Gamma_{1,\varepsilon}(R)$ for any $\varepsilon > 0$ and a sufficiently large number $R$. By (2.5),
\[ \hat{\Phi}^+ = \hat{\Phi}^- T = \hat{\Phi}^- . \]

This completes the proof of proposition 2.4.

**Proof of theorem 2.2.** — Now let $R_1$ be some sufficiently large positive number and let $C_2$ be a smooth contour in $\Gamma_3(R_1) \cup \Gamma_4(R_1)$ consisting of a half line $L_3$ in $S_2(R_1)$, a half line $L_4$ in $S_3(R_1)$ and an arc connecting the starting points of $L_3$ and $L_4$. Let $S$ be a matrix function on $C_2 \cup S_2(R_1) \cup S_3(R_1)$ with the following properties:

(i) $S$ is Hölder continuous on $C_2$

(ii) $\det S \neq 0$ on $C_2$

(iii) $S(z) = \Phi^+(z)T_2(z)\Phi^+(z)^{-1}$ for $z \in S_2(R_1)$,

\[ S(z) = \Phi^-(z)T_4^{-1}(z)\Phi^-(z)^{-1} \]

for $z \in S_4(R_1)$,

where $\Phi^+$ and $\Phi^-$ are matrix functions with the properties mentioned in proposition 2.4.

Obviously, $S(z) \sim I$ as $z \to \infty$ in $S_2(R_1) \cup S_4(R_1)$. Thus we may apply proposition 2.4, with $T_2 \cup \Gamma_3$ replaced by $\Gamma_3(R_1) \cup \Gamma_4(R_1)$, $T$ by $S$, etc. Hence there exists a positive number $R$, a matrix function $\Psi^+$, non
singular and holomorphic in \( \Gamma_3(R) \cup \Gamma_4(R) \) and a matrix function \( \Psi^- \), non singular and holomorphic in \( \Gamma_1(R) \cup \Gamma_2(R) \), such that

1. \( \Psi^+(z) = \Psi^-(z)S(z) \) for all \( z \in S_2(R) \cup S_4(R) \)

2. \( \Psi^+ \) and \( \Psi^- \) admit the same asymptotic expansion \( \Psi \in \text{GL}(n; \mathbb{C}[z^{-1}]) \), as \( z \to \infty \) in \( \Gamma_{3,\varepsilon}(R) \cup \Gamma_{4,\varepsilon}(R) \) and \( \Gamma_{1,\varepsilon}(R) \cup \Gamma_{2,\varepsilon}(R) \), respectively, for any \( \varepsilon > 0 \).

We now define the matrix functions \( \Phi_j, j = 1, \ldots, 4 \) as follows:

\[
\Phi_1(z) = \Psi^-(z)\Phi^-,(z), \quad z \in \Gamma_1(R) \\
\Phi_2(z) = \Psi^-(-z)\Phi^+(z), \quad z \in \Gamma_2(R) \\
\Phi_3(z) = \Psi^+(z)\Phi^+(z), \quad z \in \Gamma_3(R) \\
\Phi_4(z) = \Psi^+(z)\Phi^-(z), \quad z \in \Gamma_4(R),
\]

\( \Phi_j \) is non singular and holomorphic in \( \Gamma_j(R) \) and represented asymptotically by \( F \equiv \Psi \Phi \) as \( z \to \infty \) in \( \Gamma_{j,\varepsilon} \), for every \( \varepsilon > 0 \), \( j \in \{j, \ldots, 4\} \).

Furthermore, we have

\[
\Phi_1(z)^{-1}\Phi_2(z) = \Phi^-(z)^{-1}\Phi^+(z) = T_1(z), \quad z \in S_1(R) \\
\Phi_2(z)^{-1}\Phi_3(z) = \Phi^+(z)^{-1}S(z)\Phi^+(z) = T_2(z), \quad z \in S_2(R) \\
\Phi_3(z)^{-1}\Phi_4(z) = \Phi^+(z)^{-1}\Phi^-(z) = T_3(z), \quad s \in S_3(R) \\
\Phi_4(z)^{-1}\Phi_1(z) = \Phi^-(z)^{-1}S(z)^{-1}\Phi^-(z) = T_4(z), \quad z \in S_4(R).
\]

Hence it follows that

\[
M^{\Phi_j^{-1}}(z) = M^{T_j\Phi_{j+1}^{-1}}(z) = M^{\Phi_{j+1}^{-1}}(z), \quad z \in S_j(R) \cap S_{j+1}(R) - 1, \; j \in \{1, 2, 3\}
\]

and

\[
M^{\Phi_4^{-1}}(z) = M^{T_4\Phi_1^{-1}}(z) = M^{\Phi_1^{-1}}(z), \quad z \in S_4(R) \cap S_4(R) - 1.
\]

Consequently, the matrix function \( A \) defined by

\[
A(z) = M^{\Phi_i^{-1}}(z), \quad z \in \Gamma_1(R) \cap \Gamma_1(R) - 1
\]

may be continued analytically to a reduced neighbourhood of \( \infty \). Moreover,

\[
A(z) \sim F(z+1)M(z)F(z)^{-1} \quad \text{as} \quad z \to \infty \quad \text{in} \quad \bigcup_{j=1}^{4} \Gamma_{j,\varepsilon}(R) \cap \Gamma_{j,\varepsilon}(R) - 1
\]

for every \( \varepsilon > 0 \), and this implies that \( A \in \text{GL}(n; K) \).

Remark. — An alternative proof of theorem 2.2 can be given by adapting an argument used by J. Martinet and J.P. Ramis in [14], which makes essential use of the theorem of Newlander-Nirenberg.
Finally, we consider the particular case that $M$ is a matrix of rational functions and the matrix functions $T_j$ are meromorphic in $\mathbb{C}$ for each $j \in \{1, \ldots, 4\}$.

**Definition.** If $\Phi$ is a meromorphic matrix function in $\mathbb{C}$ with the property that $\det \Phi \neq 0$, then $\sum(\Phi)$ will denote the set of all singularities of $\Phi$, i.e.,

$$\sum(\Phi) = P_\Phi \cup P_{\Phi^{-1}}.$$

**Theorem 2.8.** In addition to the assumptions made in theorem 2.2, suppose that $M$ is a matrix of rational functions, that $T_j$ is meromorphic in $\mathbb{C}$ for each $j \in \{1, \ldots, 4\}$ and that

$$T_1T_2T_3T_4 = I.$$

Then there exists a matrix $A$ of rational functions and meromorphic matrix functions $\Phi_j$, $j \in \{1, \ldots, 4\}$, with the properties mentioned in theorem 2.2. Moreover

$$\sum(\Phi_j) \subset \bigcup_{i=1}^{4} \sum(T_i), \quad j \in \{1, \ldots, 4\},$$

and

$$\sum(A) \subset \bigcup_{i=1}^{4} \sum(T_j) \cup \sum(T_j) - 1 \cup \sum(M).$$

**Proof.** From property (3) in theorem 2.2 we deduce that the matrix function $\Phi_1$ can be continued analytically to a meromorphic function in some reduced neighbourhood $U$ of $\infty$. Moreover, the singular points of $\Phi_1$ in $U$ will form a subset of $\bigcup_{j=1}^{4} \sum(T_j)$. Using an idea of Birkhoff (cf. [1]), we shall remove the singularities of $\Phi_1$ outside $U$ by means of a simple transformation.

Let $C$ be a simple closed contour in $\mathbb{C}$ with interior $D^+$ and exterior $D^-$, such that $\Phi_1$ is non singular and analytic on $C$ and $\mathbb{C} \setminus U \subset D^+$. According to theorem 2.1 (with $z_0 = \infty$) there exists a matrix function $X$, non singular and sectionally holomorphic in $\mathbb{C}$, relative to $C$, with the property that

$$X^+(z) = X^-(z)\Phi_1(z) \quad \text{for all } z \in C.$$

Moreover, $X$ has at most a pole at $\infty$. Consequently, at $\infty$ $X$ admits a Laurent series representation $\hat{X} \in \mathfrak{gl}(n; K)$. 


For each $j \in \{1, \ldots, 4\}$ let $\Phi_j$ be defined by
\[
\Phi_j(z) = X(z) \phi_j(z), \quad z \in D^- \cap \Gamma_j(R).
\]
Thus, for a sufficiently large number $R_1$, $\Phi_j$ is a non-singular and analytic function in $\Gamma_j(R_1)$, admitting the asymptotic expansion $\tilde{X}F \in \text{Gl}(n; K)$ as $z \to \infty$ in $\Gamma_j, \varepsilon (R_1)$, for every $\varepsilon > 0$, $j \in \{1, \ldots, 4\}$. Moreover, $\Phi_1$ is meromorphic in $D^-$. Due to (2.9), it may be continued analytically to $D^+$. Thus it becomes a meromorphic function in $\mathbb{C}$ and the same is true of $\Phi_1^{-1}$. Furthermore, we have
\[
\tilde{\Phi}_j^{-1} \Phi_{j+1} = \Phi_j^{-1} \Phi_{j+1} = T_j \quad \text{in} \quad S_j(R_1), \ j = 1, 2, 3,
\]
and
\[
\tilde{\Phi}_4^{-1} \Phi_1 = \Phi_4^{-1} \Phi_1 = T_4 \quad \text{in} \quad S_4(R_1).
\]
Hence it follows that all $\Phi_j$ may be continued to meromorphic functions in $\mathbb{C}$ with the property that
\[
\sum (\Phi_j) \subset \bigcup_{i=1}^4 \sum (T_i), \quad j \in \{1, \ldots, 4\}.
\]
Now let $A$ be defined by
\[
A(z) = M \Phi_1^{-1} (z), \quad z \in \mathbb{C}.
\]
One easily verifies that $A$ has the properties mentioned in theorem 2.2 (with respect to $\Phi_j$ instead of $\Phi_j$). In particular, $A \in \text{Gl}(n; K)$. At the same time, $A$ is a meromorphic matrix function in $\mathbb{C}$. Hence its entries must be rational functions. Moreover,
\[
\sum (A) \subset \sum (\Phi_1) \cup \sum (\Phi_1) - 1 \cup \sum (M).
\]
With (2.10) the last statement of the theorem follows.

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Gertrude K. IMMINK,
University of Groningen
Institute of Econometrics
P.O. Box 800
9700 AV Groningen (Pays Bas).