MATEI TOMA

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A CLASS OF NON-ALGEBRAIC THREEFOLDS

by Matei TOMA

Introduction.

Let X be a non-algebraic compact complex surface. A holomorphic vector bundle E on X is called irreducible if it does not admit coherent subsheaves F with $0 < \operatorname{rank} F < \operatorname{rank} E$. In contrast with the algebraic case there exist such bundles on some non-algebraic surfaces. This phenomenon was brought forward by G. Elencwajg and O. Forster in [3] and further studied by C. Bănică and J. Le Potier in [1].

One may expect that the projective bundle $\mathbf{P}(E)$ also has strongly non-algebraic features. Assume X has no curves and $\operatorname{rank} E = 2$. Then $\mathbf{P}(E)$ is a threefold whose only curves are the vertical lines of the fibering $\mathbf{P}(E) \to X$. But when does $\mathbf{P}(E)$ have no surface? It turns out that this happens if and only if the bundle E remains irreducible after any base change $X' \xrightarrow{f} X$, where X' is again a compact complex surface and f a surjective map. We call these bundles strongly irreducible.

Combining the methods of [1] and [3], we prove in this paper the existence of strongly irreducible bundles on any 2–dimensional torus without curves and on any $K3$–surface without curves (see the theorem for the exact statement). Using these bundles one obtains as above families of analytic threefolds without divisors. Their Chern numbers depend on the invariants of the surface X and on the Chern numbers of the bundles E .

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In this way we specify a region in the "geography" of analytic threefolds. The only other compact complex surfaces without curves besides tori and $K3$–surfaces are of class VII (Inoue surfaces, for example) but we have not been able to find examples of strongly irreducible bundles in this case.

I express my gratitude to C. Bănică for suggesting the problem to me and for the useful discussions about it.

1. Strongly irreducible vector bundles.

A holomorphic vector bundle E of rank r on a complex manifold X is called irreducible if it does not admit coherent subsheaves of rank r' with $0 < r' < r$.

If E has rank 2 , then this is equivalent to $h^0(E \otimes L) = 0$, for every L in $\mathrm{Pic}(X)$, [3].

If $\operatorname{rank} E = 3$, irreducibility amounts to $h^0(E \otimes L) = h^0(E^* \otimes L) = 0$, for every L in $\mathrm{Pic}(X)$.

DEFINITION. — *We call E strongly irreducible if for every "base change" $X' \xrightarrow{f} X$, meaning by this a proper holomorphic surjective map between complex manifolds of the same dimension, f^*E is irreducible.*

From now on X , X' will always denote connected, non-singular, compact, complex surfaces while E will be a holomorphic vector bundle of rank 2 on X .

LEMMA 1. — *Let $X' \xrightarrow{f} X$, be a bimeromorphic mapping. Then E is irreducible (resp. strongly irreducible) on X if and only if f^*E is irreducible (resp. strongly irreducible) on X' .*

Proof. — If $L \hookrightarrow E$ for L in $\mathrm{Pic}(X)$, then $f^*L \to f^*E$ is injective on a Zariski open set hence the image of this morphism is a coherent subsheaf of rank 1 in f^*E .

Conversely, let $L \hookrightarrow f^*E$, with L in $\mathrm{Pic}(X')$. Applying f_* we get an injection

$$f_*L \hookrightarrow f_*f^*E ,$$

where $\operatorname{rank} f_*L = 1$. The natural morphism

$$E \to f_*f^*E$$

is an isomorphism on a Zariski open set so the inverse image of f_*L through it is a coherent subsheaf of rank 1 in E .

Coming now to the strong irreducibility, let $X'' \xrightarrow{g} X$ be a base change with g^*E reducible and $Y \to X'' \times_X X'$ a resolution of singularities. In the commutative diagram

$$\begin{tikzcd}
Y \arrow[dr] \arrow[drr, bend left=10, "\sigma"] \arrow[ddr, bend right, "\tau"'] & & \\
& X'' \times_X X' \arrow[r] \arrow[d] & X' \arrow[d, "f"] \\
& X'' \arrow[r, "g"] & X
\end{tikzcd}$$

τ is bimeromorphic, hence $\tau^* g^* E = \sigma^* f^* E$ is reducible, and so f^*E is not strongly irreducible. The converse is obvious.

Consequently, the bimeromorphic mappings do not change the irreducibility of bundles. The following example shows that not every base change has this property.

Example. — Let X' be a 2-dimensional complex torus with $a(X') = 0$ and $NS(X') \neq 0$ (NS denotes the Néron-Severi group), τ a translation on X' with $\tau^2 = \mathrm{id}_{X'}$, $G = \{\mathrm{id}, \tau\}$, $X = X'/G$, $H : \mathbf{C}^2 \times \mathbf{C}^2 \to \mathbf{C}^2$ a hermitian form with $\mathrm{Im}H(\Gamma' \times \Gamma') \subset \mathbf{Z}$ and $\mathrm{Im}H(\Gamma \times \Gamma) \not\subset \mathbf{Z}$, where Γ and Γ' are the lattices in $\mathbf{C}^2$ which give X' and X , respectively. (For example, one can choose Γ' generated by the vectors $(1,0), (0,1), i(1,\sqrt{2}), i(-\sqrt{2},1)$, Γ by $(1,0), (0,1), i(1/2, \sqrt{2}/2), i(-\sqrt{2},1)$ and

$$H = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} , \quad \text{see [3; Appendix]).}$$

By the theorem of Appel-Humbert (*cf.* [6]) there is a line bundle $L' = L'(H, \alpha)$ on X' such that $c_1(L')$ corresponds to H (we use the notations of loc. cit.). L' is not in $f^*(\mathrm{Pic}(X))$ by the choice of H .

If there existed an isomorphism

$$\varepsilon : \tau^* L' \to L' ,$$

then multiplying it with a suitable constant we should have $\tau^*(\varepsilon) \circ \varepsilon = 1$, hence L' would be invariant to the action of G , which contradicts the fact that L' is not in $f^*(\mathrm{Pic}(X))$. Consequently $\tau^* L' \not\cong L'$.

Let's consider $E' = L' \oplus \tau^* L'$ and the natural isomorphism $\tau^* E' \to E'$. It follows that E is invariant to the action of G and so there is a holomorphic vector bundle E on X of rank 2 such that $E' = f^* E$. E'

being reducible, by construction, E cannot be strongly irreducible. But E is irreducible. Indeed, if $L \hookrightarrow E$ were an injective morphism of coherent sheaves we would get

$$f^*L \hookrightarrow f^*E = L' \oplus \tau^*L' .$$

Composing with the projections it would follow that one of the morphisms $f^*L \to L'$ or $f^*L \to \tau^*L'$ would be nonzero. This would be an isomorphism ([3], § 2.1) because $a(X') = 0$ and so X' has no curves (being a torus). This would contradict the choice of L' .

The compact complex threefolds we study are projective bundles $\mathbf{P}(E)$ associated to holomorphic vector bundles E of rank 2 on X . We denote by $\pi : \mathbf{P}(E) \to X$ the natural projection and by $\mathcal{O}_{\mathbf{P}(E)}(-1)$ the tautological line subbundle in π^*E . In the sequel we use the standard notation $\mathcal{O}_{\mathbf{P}(E)}(n)$, $n \in \mathbf{Z}$, for its tensor powers. One has the following exact sequence on $\mathbf{P}(E)$:

$$(1) \qquad 0 \to \mathcal{O}_{\mathbf{P}(E)} \to \pi^*E \otimes \mathcal{O}_{\mathbf{P}(E)}(1) \to T_{\mathbf{P}(E)} \to \pi^*T_X \to 0$$

where the first morphism is induced by the inclusion $\mathcal{O}_{\mathbf{P}(E)}(-1) \hookrightarrow \pi^*E$. One also has $\mathrm{Pic}(\mathbf{P}(E)) \cong \mathrm{Pic}(X) \oplus \mathbf{Z}$, any invertible sheaf on $\mathbf{P}(E)$ being of the form $\pi^*L \otimes \mathcal{O}_{\mathbf{P}(E)}(n)$ for some L in $\mathrm{Pic}(X)$ and n in $\mathbf{Z}$. For $n > 0$ and $\mathcal{F} \in \mathrm{Coh}(X)$ the following isomorphisms are well known

$$\pi_*\mathcal{O}_{\mathbf{P}(E)}(n) \cong S^nE^*$$

$$\pi_*(\pi^*\mathcal{F} \otimes \mathcal{O}_{\mathbf{P}(E)}(n)) \cong \mathcal{F} \otimes S^nE^*$$

where S^nE are the symmetric powers of E .

DEFINITION. — *A horizontal divisor of $\mathbf{P}(E)$ is an effective divisor in $\mathbf{P}(E)$ such that the restriction of π to its support covers X .*

PROPOSITION. — *For a nonsingular compact complex surface X and a holomorphic vector bundle E of rank 2 on X the following statements are equivalent :*

1) E is strongly irreducible.

2) $\mathbf{P}(E)$ does not admit horizontal divisors.

3) $h^0(L \otimes S^nE) = 0$ for all L in $\mathrm{Pic}(X)$ and all positive integers n .

In the proof we shall use the

LEMMA 2. — *E is reducible if and only if $\mathbf{P}(E)$ admits a divisor whose projection on X is bimeromorphic.*

Proof. — Let D be an irreducible divisor as above and $\nu : \overline{D} \to D$ a resolution of singularities. One has the diagram

$$\begin{array}{ccccc} \overline{D} & \xrightarrow{\nu} & D & \overset{i}{\hookrightarrow} & \mathbf{P}(E) \\ & & {\scriptstyle p=\pi_{|D}} \searrow & & \downarrow \pi \\ & & & & X \end{array}$$

Having $\mathcal{O}_{\mathbf{P}(E)}(-1) \hookrightarrow \pi^*E$ and applying $(i \circ \nu)^*$ we get an injective bundle morphism on $\overline{D}$

$$(i \circ \nu)^* \mathcal{O}_{\mathbf{P}(E)}(-1) \hookrightarrow (i \circ \nu)^* \circ \pi^* E$$

hence $(p \circ \nu)^* E = (\pi \circ i \circ \nu)^* E$ is reducible and so E must be reducible according to lemma 1.

Conversely, if E is reducible there is an element L of $\operatorname{Pic} X$ and a nonzero section $\mathcal{O}_X \hookrightarrow E \otimes L^{-1}$, having Z as zero divisor. Then

$$\mathcal{O} \overset{s}{\hookrightarrow} E \otimes L^{-1} \otimes \mathcal{O}(-Z)$$

is a section which vanishes on a finite or empty set $A \subset X$. On $X \setminus A$, $L \otimes \mathcal{O}(Z)$ is a line subbundle of E , hence it induces a section

$$X \setminus A \longrightarrow \mathbf{P}(E) \setminus \pi^{-1}(A) \ .$$

The closure in $\mathbf{P}(E)$ of the image of this section is an analytic set (*cf.* [5], Prop. 10.6.3) which defines a horizontal divisor whose projection on X , is bimeromorphic.

Proof of the Proposition. — "1 $\Rightarrow$ 2" Suppose D is a horizontal divisor of $\mathbf{P}(E)$. Then as in the proof of lemma 2 one gets that $(p \circ \nu)^* E$ is reducible hence E cannot be strongly irreducible.

"2 $\Rightarrow$ 1" Suppose now there is a base change $X' \xrightarrow{f} X$ such that f^*E is reducible. One has the commutative diagram

$$\begin{array}{cccc} \mathbf{P}(E) \times_X X' \cong & \mathbf{P}(f^*E) & \xrightarrow{\bar{f}} & \mathbf{P}(E) \\ & {\scriptstyle \pi'} \downarrow & & \downarrow {\scriptstyle \pi} \\ & X' & \xrightarrow{f} & X \end{array}$$

where $\bar{f}$ is induced by the projection. By lemma 2 there is a horizontal divisor D' in $\mathbf{P}(f^*E)$. Since $f \circ \pi'(D') = X$ it follows by commutativity, that $\pi_{|\bar{f}(D')} : \bar{f}(D') \to X$ is surjective hence $\bar{f}(D')$ is a horizontal divisor of $\mathbf{P}(E)$.

"$1 \Leftrightarrow 3$" E is not strongly irreducible $\Leftrightarrow$ E^* is not strongly irreducible $\Leftrightarrow$ $\mathbf{P}(E^*)$ admits horizontal divisors.

For a horizontal divisor D in $\mathbf{P}(E^*)$

$$\mathcal{O}(D) \cong \pi^* L \otimes \mathcal{O}_{\mathbf{P}(E^*)}(n)$$

with L in $\mathrm{Pic}(X)$ and $n > 0$. On the other side

$H^0(\mathbf{P}(E^*), \pi^* L \otimes \mathcal{O}_{\mathbf{P}(E^*)}(n)) \cong$

$H^0(X, \pi_*(\pi^* L \otimes \mathcal{O}_{\mathbf{P}(E^*)}(n))) \cong H^0(X, L \otimes S^n E^{**}) \cong H^0(X, L \otimes S^n E)$

and the wanted equivalence follows.

Remark. — E strongly irreducible $\Rightarrow$ S^2E strongly irreducible.

Proof. — Since $f^*(S^2E) \cong S^2(f^*E)$ it will be enough to prove that the strongly irreducibility of E implies the irreducibility of S^2E . For this we have to show that $h^0(S^2E \otimes L) = h^0((S^2E)^* \otimes L) = 0$ for all L in $\mathrm{Pic}(X)$. But $(S^2E)^* \cong S^2(E^*)$ and the conclusion follows using the Proposition for both E and E^* .

2. On the existence of strongly irreducible bundles.

According to the proposition the existence of compact analytic threefolds of type $\mathbf{P}(E)$ without divisors is ensured if and only if the base X has no curves and E is strongly irreducible. The following theorem gives an answer to the problem of existence of such bundles. We first recall some notations from [1].

Let X be a non-algebraic compact complex surface. For every pair of cohomology classes (c_1, c_2) , $c_1 \in H^2(X, \mathbf{Z})$, $c_2 \in H^4(X, \mathbf{Z}) \cong \mathbf{Z}$ one defines the rational number

$$\Delta(c_1, c_2) = \frac{1}{2}\left(c_2 - \frac{1}{4}c_1^2\right) .$$

If E is a holomorphic vector bundle of rank 2 with Chern classes $c_i(E) = c_i$, then Δ is the discriminant of the bundle. For $\mu \in NS(X) \otimes \mathbf{Q}$ one defines ([1], p.21)

$$s(\mu) = -\frac{1}{2}\sup(\mu - \xi) , \quad \xi \in NS(X) .$$

If μ is of the form $\eta/2$ with η in $NS(X)$, $s(\mu)$ will coincide with the number $m(2, \eta)$ defined in loc. cit. p. 7. Note that if $\eta = 0$ (in fact if $\eta \in 2NS(X)$) then $s(\eta/2) = 0$.

THEOREM. — *Let X be a $K3$–surface without divisors or a 2–dimensional torus without divisors. Then there exist strongly irreducible holomorphic vector bundles of rank 2 on X . More precisely :*

– on $K3$–surfaces without curves every irreducible 2–bundle is strongly irreducible and for every pair $(c_1, c_2) \in NS(X) \times \mathbf{Z}$ verifying

$$\Delta(c_1, c_2) \geq \max(s(c_1/2), 2 - s(c_1/2))$$

there exist such bundles E with $c_i(E) = c_i$,

– on tori without curves there exist strongly irreducible bundles E with Chern classes $(c_1, c_2) \in NS(X) \times \mathbf{Z}$ as soon as

$$\Delta(c_1, c_2) \geq \sup(s(c_1/2), 1 - s(c_1/2)) .$$

Proof. — Any base change $X' \to X$ has a factorization $X' \xrightarrow{g} \widetilde{X} \xrightarrow{h} X$ where $\widetilde{X}$ is normal, g has connected fibers and h is finite. Since the branch locus of h on $\widetilde{X}$ is purely 1–codimensional ([4], p. 170), if X has no curves it follows that h is a finite unramified covering, $\widetilde{X}$ nonsingular and g bimeromorphic. By lemma 1 we can restrict ourselves to the study of base changes which are finite unramified coverings. When X is $K3$, hence simply connected, these are trivial and the statement of the theorem follows (for the existence see [1], § 5.10).

Let now X be a 2–torus without curves with $X = \mathbf{C}^2/\Gamma$, Γ a lattice in $\mathbf{C}^2$, $\mathbf{C}^2 \to X$ the universal covering. Every finite unramified covering $f : X' \to X$ is obtained from the universal covering factorizing through a sublattice $\Gamma' \subset \Gamma$, where $X' \cong \mathbf{C}^2/\Gamma'$. Hence X' is a complex torus without curves. The condition $\Delta \geq s(c_1/2)$ ensures the existence of an extension on X

$$0 \longrightarrow L_1 \longrightarrow E \longrightarrow L_2 \otimes I_Z \longrightarrow 0$$

where $L_1, L_2 \in \mathrm{Pic}(X)$, Z is a 2–codimensional subspace in X and E locally free sheaf of rank 2 having Chern classes c_1, c_2 (see [1], th. 2.3). (The extension is also called a "devissage" of E).

We want f^*E to be simple (i.e. $\mathrm{End}(f^*E) \cong \mathbf{C}$) for any base change $f : X' \to X$ as above. Since X' has no curves this happens if and only if in the extension

$$(2) \qquad 0 \longrightarrow f^*L_1 \longrightarrow f^*E \longrightarrow f^*L_2 \otimes I_{f^*Z} \longrightarrow 0$$

one has $f^*L_1 \not\cong f^*L_2$ (see [3], th. 2.2). It is necessary, therefore, to have for every sublattice $\Gamma' \subset \Gamma$:

$$f^*(L_2^{-1} \otimes L_1) \not\cong \mathcal{O}_{X'}$$

where $f : X' \to X$ is the associated covering. If this doesn't happen we modify firstly L_1 by tensorizing it with a suitable bundle L_0 in $\operatorname{Pic} X$. There exists such an L_0 because we can choose, for example $L_2^{-1} \otimes L_1 \otimes L_0 \in \operatorname{Pic}^0 X \cong \operatorname{Hom}(\Gamma, \{z \in \mathbf{C} \mid |z| = 1\})$ to correspond to an injective morphism $\alpha : \Gamma \to \{z \in \mathbf{C} \mid |z| = 1\}$ (it will remain injective hence nonzero on any sublattice), *cf.* [6], th. Appell-Humbert. Then we remark that after modifying L_1 as shown, $H^2(X, L_2^* \otimes L_1) \cong H^0(X, L_2 \otimes L_1^*) = 0$ ($L_1 \not\cong L_2$ and we have no curves), and this ensures the existence of a new extension with the required property.

The base $(S, 0)$ of the versal deformation $\mathcal{E} \to S \times X$, of $E = E_0$ (simple) will be smooth (see [3], § 3.6). Moreover, shrinking if necessary S around 0 , we can assume that all the bundles E_s , $s \in S$, are simple. It follows by Serre duality

$$\dim \operatorname{Ext}^2(E_s, E_s) = \dim \operatorname{Ext}^0(E_s, E_s) = 1$$

for $s \in S$, and by Riemann-Roch one gets $\dim \operatorname{Ext}^1(E_s, E_s)$ constant on S , hence equal to $\dim \operatorname{Ext}^1(E_0, E_0)$. This entails that the deformation $\mathcal{E} \to S \times X$ is versal in each $s \in S$. Therefore the conditions for S required in the proof of theorem 5.1. of [1] are fulfilled (without having to leave the centre $0 \in S$).

Let $D(\mathcal{E})$ be the relative Douady space of $\mathcal{E}$, $D \subset D(\mathcal{E})$ the open subset corresponding to the torsion-free quotients of rank 1 of E_s , $s \in S$, and $\pi : D \to S$ the projection. Let $s \in S$, and E'' in D quotient of E_S through a coherent subsheaf E' (E' will be a line bundle). One has the following exact sequence

$$0 \longrightarrow \operatorname{Hom}(E', E'') \longrightarrow T_{E''} D \xrightarrow{T_{E''}\pi} T_s S \xrightarrow{\omega_+} \operatorname{Ext}^1(E', E'')$$

where ω_+ is the composition

$$\begin{array}{ccc} T_s S & \xrightarrow{\omega} & \operatorname{Ext}^1(E_s, E_s) \\ & {\scriptstyle \omega_+} \searrow & \downarrow \\ & & \operatorname{Ext}^1(E', E'') \end{array}$$

and ω the Kodaira-Spencer morphism (see [1], § 5.5). Moreover, in the chosen situation for S and Δ one shows that $T_{E''}\pi$ isn't surjective for any E'' , fact which entails the existence of irreducible bundles on X (see [1], § 5.1).

We fix a covering $f : X' \to X$ and consider the deformation

$$f^* \mathcal{E} \longrightarrow X' \times S \ .$$

Since $f^* E_0$ is simple, we can choose a neighbourhood S' of 0 in S , such that $f^* E_s$ are simple for $s \in S'$, S' is Stein and $H^2(S', \mathbf{Z}) = 0$. These conditions will be necessary later in order to apply a result of [3]. Let D_f be the open set corresponding to the torsion-free quotients of rank 1 in the relative Douady space associated to the restriction of $f^*\mathcal{E}$ to S' and $\pi^f : D_f \to S'$ the projection. We denote $E' = L_1$, $E'' = L_2 \otimes I_Z$ and we derive from (2), as above, an exact sequence

$$T_{f^*E''}D_f \xrightarrow{T_{f^*E''}\pi^f} T_0 S \xrightarrow{\omega_+^f} \mathrm{Ext}^1(f^*E', f^*E'') .$$

We shall show that $T_{f^*E''}\pi^f$ isn't surjective or, equivalently, that $\omega_+^f \neq 0$.

Using the natural commutative diagram

$$\begin{array}{ccccc} T_0 S & \xrightarrow{\omega} & H^1(X, E^* \otimes E) & \xrightarrow{f^*} & H^1(X', f^*(E^* \otimes E)) \\ & \omega_+ \searrow & \downarrow & & \downarrow \\ & & H^1(X, E'^* \otimes E'') & \xrightarrow{f^*} & H^1(X', f^*(E'^* \otimes E'')) \end{array}$$

and the definitions through the double point $(0, \mathbf{C}[\varepsilon])$, one easily gets

$$\omega^f = f^* \circ \omega \quad \text{and} \quad \omega_+^f = f^* \circ \omega_+ .$$

Since $\omega_+ \neq 0$ it is enough to prove that

$$f^* : H^1(X, E'^* \otimes E'') \longrightarrow H^1(X', f^*(E'^* \otimes E''))$$

is injective. Let $\mathcal{F} = E'^* \otimes E''$. f^* is obtained by composition in the diagram

$$\begin{array}{ccc} H^1(X, \mathcal{F}) & \overset{f^*}{\searrow} & \\ \downarrow & & \\ H^1(X, f_* f^* \mathcal{F}) & \xrightarrow{\simeq} & H^1(X', f^* \mathcal{F}) \end{array}$$

hence we must only show that the vertical arrow is injective. Since the natural mapping $\mathcal{F} \to f_* f^* \mathcal{F}$ has a section $\mathrm{tr} : f_* f^* \mathcal{F} \to \mathcal{F}$ there exists a section at H^1–level too, hence the wanted injectivity.

$T_{f^*E''}\pi^f$ not being surjective we deduce now that the morphism $D_f \xrightarrow{\pi^f} S'$ is not surjective in the following way. Assuming its surjectivity we would have f^*E_s reducible and indecomposable for all s in S' . Then there would exist L , M in $\mathrm{Pic}(X' \times S')$, Y a 2–codimensional subspace in $X' \times S'$, flat over S' and an extension

$$0 \longrightarrow L \longrightarrow f^*\mathcal{E} \longrightarrow M \otimes I_Y \longrightarrow 0 \tag{3}$$

whose restriction to each fiber $X' \times \{s\}$ is the uniquely determined devissage of E_s . This follows from [3], th. 2.3 (it seems to us that in order to have the morphism q biholomorphic in loc. cit. one needs $\mathrm{Pic}^0 X$ to be compact, which is the case in our situation). The sheaf $M \otimes I_Y$ is S'–flat hence there exists an S'–morphism $\lambda : S' \to D_f$ such that (3) is the pull-back of the universal extension. In particular

$$\pi^f \circ \lambda = \mathrm{id}_{S'} \ . \tag{4}$$

f^*E_s being indecomposable, they have at most one devissage (see [3], th. 2.2), hence π^f is injective (even bijective in the hypothesis we made) and passing in (4) to the tangent morphism in 0 we get a contradiction.

$D_f \xrightarrow{\pi^f} S'$ not being surjective, there exist elements s in S such that f^*E_s is irreducible. We want to show that the set N_f of elements of S which do not have this property is a countable union of proper analytic subsets of S . Let

$$R_f = \{(\xi, s) \in \mathrm{Pic}(X') \times S \mid H^0(X', \mathbf{P}_\xi \otimes f^*E_s) \neq 0\}$$

where $\mathbf{P}_\xi$ is the fiber in ξ of the Poincaré bundle $\mathbf{P}$ of X' . By Grauert's semi-continuity theorem, it follows that R_f is an analytic subset in $\mathrm{Pic}(X') \times S$. Let $p : R_f \to S$ be the morphism induced by projection. We have

$$N_f = p(R_f) \ .$$

Thus p isn't surjective, by the above facts.

$\mathrm{Pic}(X')$ is a countable union of connected components each isomorphic to $\mathrm{Pic}^0(X')$ which in its turn is a 2–dimensional complex torus and therefore compact. The restriction of p to each such compact is proper, hence its image is a closed analytic set. It follows that N_f is a countable union of proper closed analytic subsets of S .

This closes the proof of the theorem because making the union of all N_f after all finite coverings $f : X' \to X$ (which form a countable set, up to isomorphisms) we find that the complementary set consisting of those s in S for which E_s is strongly irreducible is dense in S .

3. Some remarks.

1. The Chern numbers c_1^3, c_1c_2, c_3 of $\mathbf{P}(E)$ can be computed using (1) and one finds :

$$\begin{aligned} c_1^3 &= 2[c_1(E)^2 - 4c_2(E) + 3c_1(X)^2] \\ c_1c_2 &= 2[c_1(X)^2 + c_2(X)] \\ c_3 &= 2c_2(X) \ . \end{aligned}$$

We present the particular case $c_1(E) = 0$. The theorem provides then strongly irreducible 2–bundles on tori without divisors if $c_2(E) \geq 2$ and on $K3$–surfaces without divisors if $c_2(E) \geq 4$. For the corresponding threefolds one has

	c_1^3	c_1c_2	c_3
X a torus	$-8k$, k integer ≥ 2	0	0
X $K3$	$-8k$, k integer ≥ 4	48	48

2. If E is as in the theorem then $h^0(S^nE) = 0$ for all $n > 0$. In particular, for X a $K3$–surface with $NS(X) = 0$, since T_X is irreducible, hence strongly irreducible, we have

$$h^0(S^nT_X) = 0\ , \quad \text{for all } n > 0\ .$$

3. We couldn't obtain examples of strongly irreducible bundles on any compact complex surface without curves. Indeed, the only case left, that of the surfaces of class VII (*cf.* [2], p. 188), doesn't admit an analogous proof, because here $\mathrm{Pic}^0 X \cong \mathbf{C}^*$ isn't compact.

4. It is easy to get examples of strongly irreducible bundles on some surfaces having divisors (for all surfaces whose minimal model is as in the theorem, $K3$ or torus without curves, by lemma 1).

A torus X has no curves if and only if $a(X) = 0$, but there exist $K3$–surfaces X having curves and $a(X) = 0$. We didn't succeed in finding examples of strongly irreducible bundles for this class of minimal models too.

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Matei TOMA,
Department of Mathematics
INCREST
Bd. Pacii
79622 Bucharest (Roumania).