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On functions with bounded remainder


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ON FUNCTIONS WITH BOUNDED REMAINDER

by P. HELLEKALEK & G. LARCHER

0. Introduction.

Let $\lambda$ denote normalized Haar measure on the one-dimensional torus $\mathbb{R}/\mathbb{Z}$. The following two classes of $\lambda$-preserving measurable transformations on $\mathbb{R}/\mathbb{Z}$ are important in ergodic theory as well as in the theory of uniform distribution modulo one.

Let $\alpha$ be an irrational number and $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, $Tx := \{x + \alpha\}$, $\{\cdot\}$ the fractional part. $T$ is called an "irrational rotation" on $\mathbb{R}/\mathbb{Z}$.

Let $q \geq 2$ be an integer and $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, $Tx := x - (1 - q^{-k}) + q^{-(k+1)}$, whenever $x \in [1 - q^{-k}, 1 - q^{-(k+1)}]$, $k = 0, 1, \ldots$. $T$ is called a "$q$-adic von Neumann-Kakutani adding machine transformation" on $\mathbb{R}/\mathbb{Z}$. In the following, $T$ will be called a "$q$-adic transformation".

Let $\varphi : [0, 1] \to \mathbb{R}$ be a Riemann-integrable function with $\int_0^1 \varphi(t) \, dt = 0$ and let $T$ be either an irrational rotation or a $q$-adic transformation on $\mathbb{R}/\mathbb{Z}$. Define

$$\varphi_n(x) := \sum_{k=0}^{n-1} \varphi(T^k x),$$

where $x \in \mathbb{R}/\mathbb{Z}$ and $n \in \mathbb{N}$ (we shall always identify $\mathbb{R}/\mathbb{Z}$ with $[0, 1]$).

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The following two questions are of importance in ergodic theory – for the study of skew products – as well as for the study of irregularities in the distribution of sequences in \(\mathbb{R}/\mathbb{Z}\):

1. Under which conditions (on \(\varphi\) and \(x\)) one has \(\sup_n |\varphi_n(x)| < +\infty\)?

2. What can be said about limit points of \((\varphi_n(x))_{n\geq 1}\)?

The classical example. — Let \(\varphi(x) = 1_{[0,\beta]}(x) - \beta\), \(0 < \beta \leq 1\).

In this now “classical” example, the first question leads to the study of irregularities in the distribution of the sequence \((T^k x)_{k\geq 0}\), \(\varphi_n(x)\) being the so-called discrepancy function. For \(x = 0\) one gets well-known sequences: in the first case \(\{k\alpha\}_{k\geq 0}\), in the second case the Van-der-Corput-sequence to the base \(q\).

For this example, the first question has been solved completely by elementary and by ergodic methods (for the first type of \(T\) see Kesten [8] and Petersen [11], for the second type Faure [2] and Hellekalek [4]). The numbers \(\beta\) with \(\sup_n |\varphi_n(0)| < +\infty\), respectively \(\sup_n |\varphi_n(x)| < +\infty\), are all known.

The second question is closely related to ergodicity of the skew product (cylinder flow) \(T_\varphi : T_\varphi(x,y) = (Tx, y + \varphi(x))\) on the cylinder \(\mathbb{R}/\mathbb{Z} \times \mathbb{R}\) (see Oren [10] and Hellekalek [5]). In exactly this context Oren has solved the problem.

In this paper we shall be interested in question 1,2 and ergodicity of the cylinder flow \(T_\varphi\) on \(\mathbb{R}/\mathbb{Z} \times \mathbb{R}\) in the case of a \(q\)-adic transformation \(T\) and \(\varphi \in C^1([0,1])\).

1. Results.

Throughout this paper we shall assume \(q \geq 2\) to be an integer and \(T\) to be a \(q\)-adic transformation on \(\mathbb{R}/\mathbb{Z}\).

**Theorem 1.** — Let \(\varphi \in C^1([0,1])\), let \(\int_0^1 \varphi(t)\, dt = 0\) and \(\varphi(1) \neq \varphi(0)\). Then every number \(c\) such that \(|c| \leq |\varphi(1) - \varphi(0)|/2\) is a limit point of the sequence \((\varphi_{n^k}(x))_{k\geq 0}\) for almost all \(x \in \mathbb{R}/\mathbb{Z}\), in particular for any \(x\) normal to base \(q\).

**Theorem 2.** — Let \(\varphi \in C^1([0,1])\), let \(\int_0^1 \varphi(t)\, dt = 0\) and let \(\varphi'\) be Lipschitz continuous on \([0,1]\). Then
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(1) \( \varphi(0) = \varphi(1) \Rightarrow \sup_n |\varphi_n(x)| < \infty \) for all \( x \in \mathbb{R}/\mathbb{Z} \);

(2) \( \sup_n |\varphi_n(x)| < \infty \) for some \( x \in \mathbb{R}/\mathbb{Z} \) \( \Rightarrow \varphi(0) = \varphi(1) \);

(3) \( \varphi(1) < \varphi(0) \Rightarrow -\infty < \liminf_{n \to \infty} \varphi_n(0) \) and \( \limsup_{n \to \infty} \varphi_n(0) = +\infty \);

(4) \( \varphi(1) > \varphi(0) \Rightarrow -\infty = \liminf_{n \to \infty} \varphi_n(0) \) and \( \limsup_{n \to \infty} \varphi_n(0) < +\infty \);

\( (\text{if } \omega(\delta) := \sup \{|\varphi'(x) - \varphi'(y)| : |x - y| < \delta, 0 \leq x, y \leq 1\}, \delta > 0, \text{denotes the modulus of continuity of } \varphi' \), then \( \varphi' \) called Lipschitz-continuous if \( \omega(\delta) \leq L \cdot \delta \), \( \forall \delta > 0 \), \( L \) a positive constant).

The reader might want to compare theorem 2 (1) with theorem 7.8 in [7], and theorem 2 (3) and (4) with results on the one-sided boundedness of the discrepancy function (see [1]).

**Theorem 3.** — Let \( \varphi \in C^1([0,1]) \) and let \( f \varphi(t) dt = 0 \). Then \( \varphi(1) \neq \varphi(0) \Rightarrow \forall \bar{z} \in \mathbb{R}/\mathbb{Z} \) normal to base \( q \) : \( (\varphi_n(x))_{n \geq 1} \) is dense in \( \mathbb{R} \).

In particular, if \( \varphi(1) \neq \varphi(0) \) and if \( x \) is normal to base \( q \), then \( \liminf_{n \to \infty} \varphi_n(x) = -\infty \) and \( \limsup_{n \to \infty} \varphi_n(x) = +\infty \).

The reader might want to compare theorem 3 with corollary \( C \) in [10].

**Theorem 4.** — Let \( \varphi \) be as in theorem 3 and let \( T_\varphi : \mathbb{R}/\mathbb{Z} \times \mathbb{R} \to \mathbb{R}/\mathbb{Z} \times \mathbb{R} \), \( T_\varphi(x,y) = (Tx, y + \varphi(x)) \). Then

(1) \( \varphi(1) \neq \varphi(0) \Rightarrow T_\varphi \) ergodic;

(2) let \( \varphi' \) be Lipschitz-continuous on \([0,1]\). Then \( T_\varphi \) is ergodic if and only if \( \varphi(1) \neq \varphi(0) \).

2. The proofs.

Let \( A(g) = \left\{ \sum_{i=0}^{\infty} z_i q^i : z_i \in \{0,1,\ldots,q-1\} \right\} \) denote the compact Abelian group of \( q \)-adic integers with the metric

\[ \rho(z,z') := q^{-\min\{i:z_i \neq z'_i\}} \]

for \( z = \sum_{i=0}^{\infty} z_i q^i \neq z' = \sum_{i=0}^{\infty} z'_i q^i \) and \( \rho(z,z) := 0 \).
The homeomorphism $S : A(q) \to A(q)$, $Sz = z + 1$ ($z \in A(q)$, $1 := 1 \cdot q^0 + 0 \cdot q^1 + 0 \cdot q^2 + \cdots$) has a unique invariant Borel probability measure on $A(q)$: the normalized Haar measure. The dynamical system $(A(q), S)$ is minimal (see [4]).

The map $\Phi : A(q) \to \mathbb{R}/\mathbb{Z}$, $\Phi\left(\sum_{i=0}^{\infty} z_i q^i\right) := \sum_{i=0}^{\infty} z_i q^{-(i+1)} \mod 1$, is measure preserving, continuous and surjective.

The $q$-adic representation of an element $x$ of $\mathbb{R}/\mathbb{Z}$, $x = \sum_{i=0}^{\infty} x_i q^{-(i+1)}$ with digits $x_i \in \{0, 1, \ldots, q-1\}$, is unique under the condition $x_i \neq q-1$ for infinitely many $i$. From now on we shall assume this uniqueness condition to hold for all $x$. Numbers $x$ with $x_i \neq 0$ for infinitely many $i$ will be called non-$q$-adic. In the following $z = z(x)$ will denote the element $z = z(x) := \sum_{i=0}^{\infty} x_i q^i$ of $A(q)$ associated with $x$. One has

$$Tx = \Phi(z + 1)$$

and it is elementary to see:

- $T \circ \Phi(z) = \Phi \circ S(z)$, $\forall z \in A(q)$
- $x \in [aq^{-k}, (a + 1)q^{-k}]$, $0 \leq a < q^k$, $k = 1, 2, \ldots \Rightarrow T^k x \in [aq^{-k}, (a + 1)q^{-k}]$ and therefore $|T^k x - x| < q^{-k}$.
- $T$ permutes the open elementary $q$-adic intervals $[aq^{-k}, (a + 1)q^{-k}]$, $0 \leq a < q^k$, of length $q^{-k}$, $k = 1, 2, \ldots$.

**Proposition 1.** Let $\varphi$ be continuously differentiable on the closed interval $[0, 1]$ and let $\int_0^1 \varphi(t) dt = 0$. If $\omega$ denotes the modulus of continuity of $\varphi'$, then for all $k \in \mathbb{N}$ and for all $x \in \mathbb{R}/\mathbb{Z}$

$$\varphi(q^k x) = (\varphi(1) - \varphi(0))\left((\rho_k + \sigma_k - 1/2) + O(\omega(q^{-k})) + O(\rho_k \cdot \omega(c(q) \cdot (q^k - z(k)))^{-1}\log(q^k - z(k))))
+ O(\rho_k \cdot \omega(c(q) \cdot z(k))^{-1}\log z(k))\right),$$

where

$$x = \sum_{i=0}^{\infty} x_i q^{-(i+1)}$$

$$z = z(x) := \sum_{i=0}^{\infty} x_i q^i$$
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\[ z(k) := \sum_{i=0}^{k-1} x_i q^i \quad k = 1, 2, \ldots \]
\[ \rho_k := (q^k - z(k)) \cdot \Phi(z - z(k)) \]
\[ \sigma_k := z(k) \cdot \Phi(z - z(k) + q^k) \]

and \( c(q) \) is a constant that depends only on \( q \). The \( O \)-constants that appear in identity (1) are all bounded from above by a constant that depends only on \( q \) and \( \varphi \).

Proof. — It is easy to prove

\[ \varphi q_k(x) = \sum_{i=0}^{q^k-1} \varphi(a_i q^{-k}) + \sum_{i=0}^{q^k-1} \varphi'(a_i q^{-k})(T^i x - a_i q^{-k}) + O(\omega(q^{-k})) , \]

where \( a_i \) is the uniquely determined integer with \( 0 \leq a_i < q^k \) and \( T^i x \in [a_i q^{-k}, (a_i + 1) q^{-k} ] \). From proposition 1 in [6] it follows that

\[ \sum_{i=0}^{q^k-1} \varphi(a_i q^{-k}) = - (\varphi(1) - \varphi(0))/2 + O(\omega(q^{-k})) . \]

Further

\[ T^i x - a_i q^{-k} = \begin{cases} \Phi(z - z(k)) & 0 \leq i < q^k - z(k) \\ \Phi(z - z(k) + q^k) & q^k - z(k) \leq i < q^k . \end{cases} \]

By theorem 5.4, chapter 2 of [9]

\[ (q^k - z(k))^{-1} \sum_{i=0}^{q^k-z(k)-1} \varphi'(a_i q^{-k}) = \varphi(1) - \varphi(0) + O(\omega(D_{q^k-z(k)})) , \]

where \( D_{q^k-z(k)} \) denotes the discrepancy of \((a_i q^{-k})_{i=0}^{q^k-z(k)-1}\). As \( a_i q^{-k} = \Phi(z(k) + i) \), this is a string in the Van-der-Corput-sequence to base \( q \), and therefore the following discrepancy estimate holds (see [9] chapter 2, theorem 3.5 for the idea of the proof):

\[ D_{q^k-z(k)} \leq c(q)(q^k - z(k))^{-1} \log(q^k - z(k)) , \quad k = 1, 2, \ldots, \]

\( c(q) \) a constant that depends only on \( q \).

With the same arguments one proves

\[ z(k)^{-1} \sum_{i=q^k-z(k)}^{q^k-1} \varphi'(a_i q^{-k}) = \varphi(1) - \varphi(0) + O(\omega(c(q)z(k)^{-1} \log z(k))) . \]
COROLLARY 1. — Let \( n \in \mathbb{N} \), \( n = \sum_{i=0}^{s} n_i q^i \), with \( n_i \in \{0,1,\ldots, q-1\} \), \( 0 \leq i \leq s \), \( n_s \neq 0 \), and let \( n(k) := \sum_{i=0}^{k-1} n_i q^i \) if \( k = 1, \ldots, s + 1 \), 
\( n(0) := 0 \).

If \( \sum_{k=0}^{s} \) denotes \( \sum_{k=0 \atop k \neq k_0}^{s} \) then
\[
\varphi_n(x) = \sum_{k=0}^{s} \sum_{\ell=0}^{n_k-1} \varphi(T^{n(k)}+\ell q^k+jx).
\]

Let
\[
T^{n(k)+\ell q^k} x := x^{k, \ell} = \sum_{i=0}^{\infty} x_i^{k, \ell} q^{-i(i+1)}
\]
\[
z^{k, \ell} := \sum_{i=0}^{\infty} x_i^{k, \ell} q^i
\]
\[
z^{k, \ell}(m) := \sum_{i=0}^{m-1} x_i^{k, \ell} q^i \quad (m = 1, 2, \ldots)
\]
\[
\rho_{k, \ell} := (q^k - z^{k, \ell}(k)) \cdot \Phi(z^{k, \ell} - z^{k, \ell}(k))
\]
\[
\sigma_{k, \ell} := z^{k, \ell}(k) \cdot \Phi(z^{k, \ell} - z^{k, \ell}(k) + q^k).
\]

Then proposition 1 implies:
\[
\varphi_n(x) = (\varphi(1) - \varphi(0)) \sum_{k=0}^{s} \sum_{\ell=0}^{n_k-1} (\rho_{k, \ell} + \sigma_{k, \ell} - 1/2)
\]
\[
(2) \quad + O \left( \sum_{k=0}^{s} n_k \omega(q^{-k}) \right)
\]
\[
+ O \left( \sum_{k=0}^{s} \sum_{\ell=0}^{n_k-1} (\rho_{k, \ell} \omega(c(q)(q^k - z^{k, \ell}(k))^{-1} \log(q^k - z^{k, \ell}(k)))
\]
\[
+ \sigma_{k, \ell} \omega(c(q)z^{k, \ell}(k))^{-1} \log z^{k, \ell}(k)) \right).
\]

The \( O \)-constants in identity (2) are bounded from above by a constant that depends only on \( q \) and \( \varphi \).

Proof of theorem 1. — Let \( x \) be normal to base \( q \) and let \( d = 0, d_0 d_1 d_2 \ldots \) be an arbitrary number in \([0,1[\). For any index \( k \) such that
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\[ x_k < q - 1 \] we have

\[ \rho_k + \sigma_k = (q^k - z(k)) \sum_{i \geq k} x_{i}q^{-i-1} + z(k) \left( \sum_{i \geq k} x_{i}q^{-i-1} + q^{-k-1} \right) \]

\[ = \sum_{i \geq 0} x_{i}q^{-|i-k|-1}. \]

Let \( \varepsilon > 0 \) be arbitrary. Choose \( m \) such that \( q^{-m} < \varepsilon \). As \( x \) is normal there are infinitely many \( k \) such that \( x_k < q - 1 \)

\[ |\rho_k + \sigma_k - d| = |0, x_k x_{k+1} x_{k+2} \cdots + 0, 0x_{k-1} x_{k-2} \cdots x_0 - d| < q^{-m} \]

(this imposes a condition on the digits \( x_k, x_{k\pm 1}, \ldots, x_{k\pm m-1} \))

\[ x_{k-m} = q - 1, \quad x_{k-m-1} = 0. \]

Then

\[ z(k) \geq q^{k-m}, \quad q^k - z(k) \geq q^{k-m-1} \]

and, if we choose \( k \) sufficiently large,

\[ \omega(q^{-k}) < \varepsilon \quad \text{and} \quad \omega(c(q)q^{-k+m+1} \log q^k) < \varepsilon. \]

If we put \( c := (\varphi(1) - \varphi(0))(d - 1/2) \), then it follows directly that

\[ |\varphi_{q^k}(x) - c| = O(\varepsilon). \]

\[ \Box \]

Proof of theorem 2. — (1): Let \( \varphi(1) = \varphi(0) \). It is \( \Phi(z - z(k)) < q^{-k} \) and \( \Phi(z - z(k) + q^k) < q^{-k}, \quad k = 1, 2, \ldots. \) Hence for the third term in identity (2) we get the estimate

\[ \sum_{k=0}^{\infty} q^{-k} \log q^k < +\infty. \]

Thus the first part of the theorem is proved.

(2): Let \( \sup |\varphi_n(x)| < +\infty \) for some \( x \in \mathbb{R}/\mathbb{Z} \) and let \( z := z(x) \). The map \( \varphi \circ \Phi : \mathbb{A}(q) \to \mathbb{R} \) is continuous and \( (\mathbb{A}(q), S) \) is a minimal (topological) dynamical system. We have

\[ \sup_n |\varphi_n(x)| = \sup_n \sum_{k=0}^{n-1} \varphi \circ \Phi(S^kz) < +\infty. \]

By theorem 14.11 of [3] there is a continuous function \( g : \mathbb{A}(q) \to \mathbb{R} \) such that \( \varphi \circ \Phi(z) = g(z) - g(Sz) \), \( \forall z \in \mathbb{A}(q) \). Hence

\[ -(\varphi(1) - \varphi(0))/2 = \lim_{k \to \infty} \varphi_{q^k}(0) = \lim_{k \to \infty} \sum_{i=0}^{q^k-1} \varphi \circ \Phi(S^i0) \]

\[ = \lim_{k \to \infty} (g(0) - g(q^k)) = 0; \]
(here we use proposition 1 in [6] to prove the first equality).

(3) : We shall prove $-\infty < \liminf_{n \to \infty} \varphi_n(0)$, then part (2) will imply the remaining statement. Because of identity (2) and inequality (3) it is enough to show, for $x = 0$,

$$
\Sigma_n := \sum_{k=0}^{s} \sum_{\ell=0}^{n_k-1} (\rho_{k,\ell} + \sigma_{k,\ell} - 1/2) \leq K, \quad \forall n \in \mathbb{N}
$$

with some constant $K$. If $x = 0$ then $z^{k,\ell} = n(k) + \ell q^k$ and $z^{k,\ell}(k) = n(k)$. Hence $\rho_{k,\ell} = (q^k - n(k))\ell q^{-(k+1)}$ and $\sigma_{k,\ell} = n(k)(\ell + 1)q^{-(k+1)}$. Thus

$$
\Sigma_n = \sum_{k=0}^{s} n_k((n_k - 1)/(2q) + n(k)q^{-(k+1)} - 1/2).
$$

The statement then follows because $(n_k - 1)/(2q) + n(k)q^{-(k+1)} - 1/2 < 0$.

(4) : The idea of the proof is the same as in (3).

**Remark.** — In theorem 2 (1), (3) and (4) one can weaken the condition on the modulus of continuity of $\varphi'$ to $\omega(\delta) = O(|\log \delta|^{-1-\varepsilon})$ with some $\varepsilon > 0$.

**Proof of theorem 3.** — The idea of the proof is as follows. Let $(k_m)_{m \geq 1}$ be a strictly increasing sequence of positive integers. If $n = q^{k_1} + \cdots + q^{k_s}$ then

$$
\varphi_n(x) = (\varphi(1) - \varphi(0)) \sum_{m=1}^{s} (\rho_{k_m} + \sigma_{k_m} - 1/2) + O\left(\sum_{m=1}^{s} \omega(q^{-k_m})\right)
$$

$$
+ O\left(\sum_{m=1}^{s} \rho_{k_m} \omega(c(q)(q^{k_m} - z^{k_m}(k_m)))^{-1} \log(q^{k_m} - z^{k_m}(k_m))\right)
$$

$$
+ \sigma_{k_m} \omega(c(q)(z^{k_m}(k_m))^{-1} \log z^{k_m}(k_m))
$$

with $x = 0, x_0 x_1 x_2 \cdots$, $z = z(x) = \sum_{i=0}^{\infty} x_i q^i, z^{k_m} = z + q^{k_1} + \cdots + q^{k_{m-1}}$ and, if $x_{k_m} \leq q - 2$,

$$
\rho_{k_m} + \sigma_{k_m} = 0, \quad x_{k_m} x_{k_m+1} \cdots + 0, \quad 0 x_{k_m-1} x_{k_m-2} \cdots x_0.
$$

Now, let $d \in \mathbb{R}, \varepsilon > 0$ and $x \in [0,1]$ normal to base $q$ be given. We shall prove that there is a positive integer $m_0$ and a strictly increasing sequence $(k_m)_{m \geq m_0}$ such that

$$
|\varphi_n(x) - d| < \varepsilon \quad \text{for all} \quad n = q^{k_{m_0}} + \cdots + q^{k_s} \text{ sufficiently large.}
$$
Let $m_0$ be such that $\sum_{m \geq m_0} q^{-m} < \varepsilon$. Let $(a_m)_{m \geq m_0}$ be a sequence in $[0,1]$ such that  

$$d = (\varphi(1) - \varphi(0)) \sum_{m \geq m_0} (a_m - 1/2).$$

The number $x$ is normal to base $q$. Hence there are infinitely many $k = k(m)$ such that

1. $x_k \leq q - 2$
2. $x_{k-2m} = 1$
   \[x_{k-2m-1} = x_{k+2m} = x_{k+2m+1} = 0\]
3. $|\rho_k + \sigma_k - a_m| < q^{-m}(\varphi(1) - \varphi(0))^{-1}$, $\forall m \geq m_0$; (this condition defines a string of digits $x_{k-2m+1}, \ldots, x_{k+2m-1}$). Hence we may choose a strictly increasing sequence $(k_m)_{m \geq m_0}$ such that these three conditions hold for every $k_m$ and such that

4. $k_m + 2m + 1 < k_{m+1}$
5. $\sum_{m \geq m_0} \omega(q^{-k_m}) < \varepsilon$
6. $\sum_{m \geq m_0} \omega(c(q)q^{-k_m+2m+1} \log q^{k_m}) < \varepsilon$.

Then if $n = q^{k_{m_0}} + \cdots + q^{k_s}$ ($s \geq m_0$),

$$|\varphi_n(x) - d| = o(\varepsilon),$$

and therefore the sequence $(\varphi_n(x))_{n \geq 1}$ is dense in $\mathbb{R}$.

Remark. — Theorem 3 gives an alternative to the proof of theorem 2 (2), this time without a condition on the modulus of continuity of $\varphi'$:

If $\sup |\varphi_n(x)| < \infty$ for some $x \in [0,1]$, then this holds for all $x$ by the theorem of Gottschalk and Hedlund. Hence $\varphi(1) = \varphi(0)$, otherwise a contradiction to theorem 3 would arise for any $x$ normal to base $q$.

Proof of theorem 4.

(1) is proved in the very same way as the theorem of [6].

(2) : Let $L_2$ stand for $L_2(\mathbb{R}/\mathbb{Z}, \lambda)$. Then $\varphi(1) = \varphi(0)$ implies $\sup \|\varphi_n\|_{L_2} < +\infty$. By Lemma 2.2 in [4] there exists an element $g$ of $L_2$ such that $\varphi = g - g \circ T$ (mod $\lambda$). This implies that $(x,y) \mapsto$
(Tx, y + \varphi(x) \mod 1) is not ergodic on \( \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \) and therefore \( T_\varphi \) cannot be ergodic on \( \mathbb{R}/\mathbb{Z} \times \mathbb{R} \) (see [5], part. I : remarks).

\[ \square \]

BIBLIOGRAPHIE


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