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Two problems of Calderón-Zygmund theory on product-spaces


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Robert Fefferman introduced in [1] the notion of a rectangle atom on $\mathbb{R}^n \times \mathbb{R}^m$ and proved the following theorem.

**Theorem A.** - Let $T$ be a bounded linear operator on $L^2(\mathbb{R}^n \times \mathbb{R}^m)$. Suppose that for any $H^p(\mathbb{R}^n \times \mathbb{R}^m)(0 < p \leq 1)$ rectangle atom $a$ supported on the rectangle $R$ we have

$$\int_{(\gamma R)^c} |Ta|^p dx_1 dx_2 \leq c\gamma^{-\delta}$$

for some fixed $\delta > 0$ and all $\gamma \geq 2$. Then $T$ is a bounded operator from $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ to $L^p(\mathbb{R}^n \times \mathbb{R}^m)$.

The definitions and tools involved in this theorem and its proof have been generalized to product spaces with an arbitrary number of factors [2], [3], but the question of whether Theorem A extends for three or more factors or not, raised implicitly in [4] and explicitly in [2] was open. Our purpose is to show that in the case $p = 1$, and of the space $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$, Theorem A does not extend without any further assumptions on the nature of $T$. If, however, one supposes that $T$ is supported by N.S.F.

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a convolution operator and if $\delta > (1/8)$, then Theorem A extends. As will be apparent from the proof $1/8$ is probably not sharp and it is reasonable to conjecture that $\delta > 0$ should suffice.

The second question which we shall answer has been raised by Raphy Coifman and concerns the $L^2$-boundedness of the operator $c_a$ defined for $a \in L^\infty_0(\mathbb{R}^2)$ and $||a||_{\infty} < 1$, by the kernel

$$c_a(x, y) = \frac{1}{(x_1 - y_1)(x_2 - y_2) + \int_{x_1}^{y_1} \int_{x_2}^{y_2} a(u_1, u_2) du_1 du_2}.$$  

The case $||a||_{\infty} < \varepsilon$ was handled in [3] and was a consequence of the estimate

$$||L_{k,a}||_{2,2} \leq c_k ||a||_{\infty}^k$$

where $L_{k,a}$ is the operator defined by the kernel

$$\frac{1}{(x_1 - u_1)(x_2 - y_2)} \left[ \int_{x_1}^{y_1} \int_{x_2}^{y_2} a(u_1, u_2) du_1 du_2 \right]^k \left( x_1 - y_1 \right) \left( x_2 - y_2 \right).$$

Here we improve this estimate and obtain $||L_{k,a}||_{2,2} \leq c_\delta (1 + k)^{2+\delta}$ for all $\delta > 0$, which yields the general case $||a||_{\infty} < 1$.

In Section 1 we recall some facts about bounded mean oscillation over rectangles, and state Theorem A, restricted to $p = 1$, in this dual setting. In Section 2 we present the counterexample to the extension of Theorem A for $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and $p = 1$. In Section 3 we show how the positive result for convolution operators can be reduced to a problem on finite families of convolution operators, which is handled in Section 4. In Section 5 we treat the operators $L_{k,a}$. This section essentially combines ideas already contained elsewhere, and for this reason, is rather sketchy.

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1. Bounded mean oscillation over rectangles.

The space $B(R \times R)$, introduced in [5], is the dual space of the atomic $H^1$-space, constructed from rectangle atoms. In other words, let $b \in L^2_{loc}(R^2)$. For every rectangle $R = I \times J$ in $R^2$ let $\text{Osc}_R b = \inf_{b_1, b_2} \left( \frac{1}{|R|} \int_R |b(x_1, x_2) - b_1(x_1) - b_2(x_2)|^2 dx_1 dx_2 \right)^{1/2}$, where the inf is taken over all $b_1, b_2$ respectively in $L^2(I)$ and $L^2(J)$. Then $b \in B$ if and only if

$$1.1 \quad \sup_R \text{Osc}_R b < +\infty.$$ 

The left hand side of 1.1 is denoted $||b||_B$.

An equivalent definition can be given in terms of Carleson measures over rectangles. Let $\psi$ be a real even $C^\infty(R)$ function such that $\int \psi \, dx = 0$. For $t > 0$ and $i \in \{1, 2\}$, let $Q_{t_i}$ be the convolution operator on $R^2$ of symbol $\widehat{\psi}(t \xi)$. We normalize $\psi$ so that $\int_0^{+\infty} Q_{t_i}^2 \frac{dt_i}{t_i} = I$. For each rectangle $R$, the set $S(R)$ denotes the subset of $R_+^2 \times R^2_+$ of those $(x_1, t_1, x_2, t_2) = (x, t)$ such that $|x_1 - t_1, x_1 + t_1[\times]x_2 - t_2, x_2 + t_2[\subset] R$.

**Lemma 1.** - A function $b \in L^2_{loc}(R^2)$ is in $B$ if and only if for some constant $c_b$

$$1.2 \quad \int_{S(R)} |Q_{t_1} Q_{t_2} b|^2 dx_1 \frac{dt_1}{t_1} dx_2 \frac{dt_2}{t_2} \leq c_b |R|.$$ 

Moreover if $c_b$ is optimal, $c_b \approx ||b||^2_B$.

Notice that $1.1 \implies 1.2$ is clear since $\psi$ has compact support. We shall prove the converse in the non-product setting but the proof we give extends easily.

It is enough to show that if $b \in B(R) = \text{BMO}(R)$, and for all interval $I \subseteq R$

$$1.3 \quad \int_{(x,t), [x-t,x+t] \subseteq I} |Q_{t} b|^2 dx \frac{dt}{t} \leq |I|.$$
then \(||b||_B = ||b||_{\text{BMO}} \leq c\) where \(c\) depends only on \(\psi\). For all \(t > 0\), let \(P_t\) be the operator \(I - \int_0^t Q_s^2 \frac{ds}{s}\). Then for all \(t > 0\), \(P_t b\) is \(C^\infty\) and \(||P_t b'||_{\infty} \leq c ||b||_B t^{-1}\). It follows that if \(I\) is an interval of center \(x_0\) and \(t_I = K|I|\) for \(K\) fixed large enough, 
\[
\left(\frac{1}{|I|} \int_I |P_t b(x) - P_t b(x_0)|^2 dx\right)^{1/2} \leq \frac{1}{2} ||b||_B.
\]
Therefore
\[
\left(\frac{1}{|I|} \int_I |b(x) - P_t b(x_0)|^2 dx\right)^{1/2} \leq \left(\frac{1}{|I|} \int_I |b(x) - P_t b(x)|^2 dx\right)^{1/2} + \frac{1}{2} ||b||_B.
\]
By taking the sup over \(I\), we see that \(||b||_B \leq 2 \sup_t \left(\frac{1}{|I|} \int_I |b(x) - P_t b(x)|^2 dx\right)^{1/2}\). To estimate the right hand side we let \(g\) be in \(L^2(I)\), with \(||g||_2 = 1\) and dominate \(<g, b - P_t b>\) by 
\[
\int_{s \leq t_I} |Q_s g(x)| |Q_s b(x)| \frac{ds}{s} dx.
\]
The conditions \(Q_s g \neq 0, s \leq t_I,\) and \(g \in L^2(I)\), imply \(x K' I\) for some \(K'\) fixed. Using Cauchy-Schwarz, 1.3 and \(||g||_2 = 1\), we can dominate 1.5 by an absolute constant, which proves the lemma.

In the following lemma the notations and definitions are those of [3].

**Lemma 2.** – Let \(T\) be a translation invariant \(\delta - \text{CZO}\) on \(\mathbb{R} \times \mathbb{R}\). Then \(T\) is bounded on \(B\).

This lemma is an easy consequence of lemma 1. For simplicity we shall consider the non-product situation, but give a proof which extends trivially.
Let $T$ be a translation invariant $\delta -$ CZO on $\mathbb{R}$. The kernel $(Q_t TQ_{t'}) (x - y)$ of $T$ is easily seen to satisfy

$$ |(Q_t TQ_{t'}) (x - y)| \leq c w_{\delta', t\vee t'} (x - y) \left( \frac{t \wedge t'}{t \vee t'} \right)^{\delta'} $$

for some $\delta' < \delta$, where $w_{\delta', t}(z) = \frac{t^{\delta'}}{t^{1+\delta'} + |z|^{1+\delta'}}$. For $(x, t) \in \mathbb{R}^2_+$ and $b \in B |Q_t T b(x)| = \left| \int_{\mathbb{R}^2_+} (Q_t TQ_{t'}) (x, y) (Q_{t'} b)(y) \, dy \, \frac{dt'}{t'} \right|$. By Cauchy-Schwarz and because of 1.6 this is less than

$$ c \left[ \int_{\mathbb{R}^2_+} w_{\delta', t\vee t'} (x - y) \left( \frac{t \wedge t'}{t \vee t'} \right)^{\delta'} |Q_{t'} b(y)|^2 \, dy \, \frac{dt'}{t'} \right]^{1/2}. $$

It follows that if $|Q_{t'} b(y)|^2 \, dy \, \frac{dt'}{t'}$ is a Carleson measure, then $|Q_t T b(x)|^2 \, dx \, \frac{dt}{t}$ is a Carleson measure. The same proof using Carleson measures over rectangles yields the result in the product case. Lemma 2 is proved, by Lemma 1.

We conclude this section by stating Theorem A, restricted to $p = 1$, in dual form [4].

**Theorem A'.** Let $T$ be a linear operator bounded on $L^2(\mathbb{R} \times \mathbb{R})$. Suppose that for any rectangle $R$, and any $L^\infty$-function $a$ supported out of $\gamma R$,

$$ \text{Osc}_R T a \leq c \gamma^{-\delta} $$

for some $\delta > 0$ and all $\gamma > 2$. Then $T$ maps $L^\infty$ to $\text{BMO}(\mathbb{R} \times \mathbb{R})$.


Any counterexample in this kind of question has to be related to the counterexample of Carleson [6] showing that rectangular Carleson measures are not, on the bidisc, a good substitute for classical Carleson-measures. As shown by R. Fefferman in [5], this
counterexample implies that there can be no a priori estimate $\|b\|_{BMO} \leq c\|b\|_B$ on the bidisc. We shall denote for each $k \geq 0$ by $b_k$ a function on $\mathbb{R} \times \mathbb{R}$ such that $\|b_k\|_{BMO} = 1$ and $\|b\|_B \leq 2^{-k}$.

Using $b_k$ we form a $\delta - \text{CZO}$ on $T_k$ on $\mathbb{R} \times \mathbb{R}$ by letting $T_k f = \int \int Q_{t_1} Q_{t_2} \{(Q_{t_1} Q_{t_2} b_k) (P_{t_1} P_{t_2} f)\} \frac{dt_1}{t_1} \frac{dt_2}{t_2}$ for $f \in C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})$. The fact that $T_k$ is a $\delta - \text{CZO}$ is clear, and proved in [3]. Let also $S_k$ be defined on $\mathbb{R}$ as the operator of convolution by $\frac{1}{x} \{-\phi(x) + \phi(2^{-k} x)\}$, where $\phi$ is a $C_c^\infty(\mathbb{R})$ function equal to 1 near 0, followed by the multiplication by $e^{ix}$. Finally let $U_k = T_k \otimes S_k$.

We claim that $U_k$ satisfies uniformly the assumptions of Theorem A, adapted to the case of three factors. Clearly $\|U_k\|_{2,2} \leq c$.

To check 1.7, only the oscillation of $b_k$ over rectangles is used, introducing a gain of $2^{-k}$. On the other hand the kernel $s_k(x, y)$ of $S_k$ satisfies $|\nabla_x s_k(x, y)| \leq \frac{c}{|x - y|}$, but since on the support of $s_k(\cdot, \cdot), |x - y| \leq 2^k$, one also has $|\nabla_x s_k(x, y)| \leq \frac{c 2^k}{|x - y|^2}$, and $|\nabla_y s_k(x, y)| \leq \frac{c}{|x - y|^2}$. Therefore, writing $U_k = (2^k T_k) \otimes (2^{-k} S_k)$, we see that $U_k$ satisfies 1.7 as any tensor product of a $\delta - \text{CZO}$ on $\mathbb{R} \times \mathbb{R}$ with a $1 - \text{CZO}$ on $\mathbb{R}$. The contradiction comes from the fact that the functions $u_k = U_k (1 \otimes \text{sgn } x) = b_k \otimes S_k \text{sgn } x$, are not a bounded sequence in $\text{BMO}(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$. Indeed $\|u_k\|_{\text{BMO}} \approx \|S_k \text{sgn } (x)\|_{\text{BMO}} \approx k$.

3. Extension of Theorem A in the convolution case.

We wish to prove the following.

**THEOREM 1.** Let $T$ be a bounded convolution operator on $L^2(\mathbb{R}^3)$. Suppose that for any rectangle $R$, and any $L^\infty$-function supported out of $\gamma R$,

$$\text{Osc}_R Ta \leq c \gamma^{-\delta}$$

for some $\delta > 1/8$ and all $\gamma > 2$. Then $T$ maps $L^\infty(\mathbb{R}^3)$ to $\text{BMO}(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$. 

In [3] the $L^\infty - \text{BMO}$ boundedness of operators bounded on $L^2$, whose kernels satisfy some assumptions of Calderón-Zygmund type, is proved using the characterization of BMO in terms of Carleson measures [7] and some geometrical lemmas. Assumption 3.1 can be thought of as a very weak Calderón-Zygmund type of assumption and is reminiscent of the weak vector-valued standard estimates satisfied by the kernel of the square function operator of J.-L. Rubio de Francia [8]. It turns out that the technique used in [3] to extend Rubio de Francia’s theorem in several parameters also applies here. Indeed this technique can be summarized in the following lemma.

**Lemma 3.** Let $T$ be a bounded operator on $L^2(\mathbb{R}^n)$. Let $i \in \{1,2,\ldots,n\}$ and let $(\ell_1,\ldots,\ell_i) \in \mathbb{Z}^i$. Let $(x_1,\ldots,x_i) \in \mathbb{R}^i$. Let $b \in L^2_{\text{loc}}(\mathbb{R}^n)$ be supported in \{(z_1,\ldots,z_i) \in \mathbb{R}^n, 2^{\ell_k} \leq |z_k - x_k| \leq 2^{\ell_k+1} \text{ for } 1 \leq k \leq i\}, such that for all $(z_1,\ldots,z_i)$,

$$3.2 \quad \int_{z_{i+1},\ldots,z_n} |b(z_1,\ldots,z_n)|^2 dz_{i+1},\ldots,dz_n \leq 1.$$ 

Suppose that for $(t_1,\ldots,t_i)$ such that $t_k \leq 2^{\ell_k-1}$ for $1 \leq k \leq i$,

$$3.3 \quad \int_{z_{i+1},\ldots,z_n} |Q_{t_1} \ldots Q_{t_i} Tb(x_1,\ldots,x_n)|^2 dx_{i+1},\ldots,dx_n \leq c \prod_{1 \leq k \leq i} \left( \frac{t_k}{2^{\ell_k}} \right)^\varepsilon$$

for some $\varepsilon > 0$ and that all of this remains true if the set \{1,\ldots,i\} is replaced by any other non-empty subset of \{1,\ldots,n\}.

Then $T$ maps $L^\infty(\mathbb{R}^n)$ to $\text{BMO}(\mathbb{R} \times \cdots \times \mathbb{R})$.

Of course when $i = n$, then 3.2 and 3.3 simply mean that when

$$||b||_\infty \leq 1, |Q_{t_1} \ldots Q_{t_n} Tb(x_1,\ldots,x_n)| \leq c \prod_{1 \leq i \leq n} \left( \frac{t_k}{2^{\ell_k}} \right)^\varepsilon,$$

which follows trivially from 3.1.

Now we are going to see how to reduce the proof of Theorem 1 to a problem on finite families of convolution operators. In this reduction we shall suppose that $n = 2$ and $i = 1$. We want to prove
that if $T$ satisfies the assumptions of Theorem 1, then it satisfies the assumptions of Lemma 3.

We shall assume that the function $\psi$ defining $Q_{t_1}$, $i \in \{1, 2\}$, is of the form $\tilde{\psi} \ast \psi$, where $\psi$ is real even $C^\infty$, supported in $[-\frac{1}{2}, \frac{1}{2}]$ and with mean-value 0. Then if $|x_1 - z_1| \geq 2t_1$ we can define an operator $(Q_{t_1}T)_{x_1-z_1}$ acting on functions of the second variable by letting, for

$$f, g \in C^\infty_0(\mathbb{R}) < g, (Q_{t_1}T)_{x_1-z_1}f > = < \tilde{\psi}^{z_1}_{t_1} \otimes g, T\tilde{\psi}^{z_1}_{t_1} \otimes f >$$

where $\tilde{\psi}^{z_1}_{t_1}(u)$ is defined as $\frac{1}{t_1} \tilde{\psi} \left( \frac{u - z_1}{t_1} \right)$ and similarly for $\tilde{\psi}^{z_1}_{t_1}(u)$.

Let $(x_1,t_1) \in \mathbb{R}^2$; $\ell_1 \in \mathbb{Z}$ such that $2^{\ell_1} \geq 2t_1$ and $b \in L^2_{\text{loc}}(\mathbb{R}^2)$ such that supp $b \subseteq \{(z_1, z_2) \in \mathbb{R}^2, 2^{\ell_1} \leq |x_1 - z_1| \leq 2^{\ell_1+1}\}$ and for all $z_1 \in \mathbb{R}$, $\int |b(z_1, z_2)|^2 dz_2 \leq 1$. Then $Q_{t_1}Tb(x_1, \cdot) = \int (Q_{t_1}T)_{x_1-z_1} b(z_1, \cdot) \, dz_1$. In order to prove that $||Q_{t_1}Tb(x_1, \cdot)||_2 \leq c \left( \frac{t_1}{2^{\ell_1}} \right)^{\epsilon}$ for some $\epsilon > 0$, it suffices to prove that for all finite sequence $(z_{1,k})_{1 \leq k \leq N}$ such that $|z_{1,k} - z_{1,k+1}| = 2t_1$ and $2^{\ell_1} \leq |x_1 - z_{1,k}| \leq 2^{\ell_1+1}$ for all $1 \leq k \leq N$,

$$3.4 \quad \left\| \sum_{k=1}^N (Q_{t_1}T)_{x_1-z_{1,k}} b(z_{1,k}, \cdot) \right\|_2 \leq c \frac{t_1}{2^{\ell_1}} \left( \frac{t_1}{2^{\ell_1}} \right)^{\epsilon}.$$

On the other hand we are going to see that if $||b||_\infty \leq 1$, 3.1 implies

$$3.5 \quad \left\| \sum_{k=1}^N (Q_{t_1}T)_{x_1-z_{1,k}} b(z_{1,k}, \cdot) \right\|_{B(\mathbb{R})} \leq c \frac{t_1}{2^{\ell_1}} \left( \frac{t_1}{2^{\ell_1}} \right)^{\delta}.$$

Indeed, using the factorization of $Q_{t_1}$ as $\tilde{Q}^2_{t_1}$, we can rewrite the sum

$$\sum_{k=1}^N (Q_{t_1}T)_{x_1-z_{1,k}} b(z_{1,k}, \cdot)$$

as

$$\int (\tilde{Q}_{t_1}T)_{x_1-y_1} \left[ \sum_k \widetilde{\psi}_{t_1} (y_1 - z_{1,k}) b(z_{1,k}, \cdot) \right] dy_1.$$
As a function of \((y_1, \cdot)\), \(\sum_k \tilde{\psi}_k (y_1 - z_{1,k}) b(z_{1,k}, \cdot)\) is bounded of norm \(\frac{\xi}{t_1}\) and supported in a strip \(\{(y_1, \cdot) \in \mathbb{R}^2; |x_1 - y_1| \approx 2^t_1\}\). It follows that 3.1 implies 3.5.

Observe that now we just have to show that 3.5 for \(\delta > 1/8\) implies 3.4 for some \(\varepsilon > 0\). Since \(N \leq \frac{2t_1}{t_1}\), it will be a consequence of the following.

**Proposition 1.** - Let \(N\) be an integer and \((T_j)_{1 \leq j \leq N}\) be a family of convolution operators on \(L^2(\mathbb{R})\). Suppose that \((b_j)_{1 \leq j \leq N}\) is a sequence of bounded functions

\[
3.6 \quad \left\| \sum T_j b_j \right\|_B \leq \sup_j \|b_j\|_\infty.
\]

Then if \((f_j)_{1 \leq j \leq N}\) is a sequence of \(L^2\)-functions

\[
3.7 \quad \left\| \sum T_j f_j \right\|_2 \leq CN^{1+\eta}_8 \sup_j \|f_j\|_2
\]

for all \(\eta > 0\).

Of course to prove Theorem 1 we need an analogue of Proposition 1 in a higher number of parameters. The extension of the proof of Proposition 1 which we shall give in the next section relies only on Lemma 2, and on the characterization of \(BMO\) in terms of \(L^\infty\) and partial Hilbert transforms \([7]\). Therefore we shall leave it to the reader. We shall however use in our proof the symbols \(\| \cdot \|_B\) and \(\| \cdot \|_{BMO}\), even though the norms they denote coincide in the one-parameter case, in order to indicate which one should be used in several parameters, at which place.


**Lemma 4.** - Let \((\xi_m)_{1 \leq m \leq M}\) be a finite collection of distinct real numbers and let \((c_m)_{1 \leq m \leq M}\) be a finite collection of complex numbers. The function \(b = \sum c_m e^{i \xi_m \cdot} \) is in \(B\) with a norm at
To see this, it suffices to test the oscillation of \( b \) on intervals whose length tends to \( \infty \). We omit the details.

From this lemma it follows that, under the assumption 3.6 applied when the \( b_j \)'s are characters,

\[
\sum_{i \leq j \leq N} ||T_j||_{2,2}^2 \leq 1.
\]

Let us prove that 4.1 allows us to make the assumption that \( ||T_j||_{2,2} \leq \frac{1}{\sqrt{N}} \) for all \( j \in [1,N] \), without loss of generality. Indeed let \( c(N) \) be the best constant for which 3.7 holds with \( c(N) \) instead of \( cN^{1+\eta} \), and similarly for \( c'(N) \) but assuming \( ||T_j||_{2,2} \leq \frac{1}{\sqrt{N}} \) for \( j \in [1,N] \). Let \( \alpha > 0 \). Let \((T_j)_{1 \leq j \leq N}\) be a collection of operators satisfying to 3.6. By 4.1, the number of \( j \)'s for which \( ||T_j||_{2,2} \geq N^{\alpha - \frac{1}{2}} \) is less than \( \lfloor N^{1-2\alpha} \rfloor \), where \( \lfloor \cdot \rfloor \) denotes the integer part. It follows, by considering the set of \( j \)'s for which \( ||T_j||_{2,2} \leq N^{\alpha - \frac{1}{2}} \), and those for which \( ||T_j||_{2,2} \geq N^{\alpha - \frac{1}{2}} \), that

\[
4.2 \quad c(N) \leq c'(N)N^\alpha + c(\lfloor N^{1-2\alpha} \rfloor).
\]

Hence if \( c'(N) \) grows at most like \( N^{\frac{1}{2}+\eta} \) for all \( \eta > 0 \), so does \( c(N) \).

We now suppose that for all \( 1 \leq j \leq N \), \( ||T_j||_{2,2} \leq \frac{1}{\sqrt{N}} \). Since the Hilbert transform is bounded on \( B \) and any BMO function \( b \) can be written as \( a_1 + H a_2 \), where \( a_1 \) and \( a_2 \) are in \( L^\infty \) and satisfy \( ||a_1||_\infty + ||a_2||_\infty \leq c||b||_{\text{BMO}} \), we see that 3.6 implies

\[
4.3 \quad \left| \sum_j T_j b_j \right|_B \leq c \sup_j ||b_j||_{\text{BMO}}.
\]

We can also assume that the symbol of each \( T_j \) vanishes on \( \bigcup_{k \in \mathbb{Z}} [2^k, \frac{5}{4}, 2^k] \cup \mathbb{R}_- \). We let \( \Delta_k \) be the multiplier of symbol \( X_{\left[ \frac{a}{4}, \frac{a+1}{2} \right]} \) and let \( T_{j,k} \) be \( T_j \Delta_k \).
LEMMA 5. — For all $k \in \mathbb{Z}$, $\sum \|T_{j,k}\|_{2,2} \leq c$.

In order to prove this lemma it suffices to show that if $(S_j)_{1 \leq j \leq N}$ is a family of convolution operators whose symbols $\sigma_j$ are supported in $\{1 \leq \xi \leq 1 + \beta\}$ for some $\beta > 0$ and if for all sequences $(\xi_j)_{1 \leq j \leq N}$ of real numbers and all sequence $(c_j)_{1 \leq j \leq N}$ of complex numbers

$$4.4 \quad \left\| \sum_j S_j c_j e^{i<\xi_j, \cdot>} \right\|_B \leq \sup_j |c_j|$$

then

$$4.5 \quad \sum_j \|S_j\|_{2,2} = \sum_j \|\sigma_j\|_{\infty} \leq c.$$

To prove 4.5 it suffices to show that for any sequence $(\xi_j)_{1 \leq j \leq N},$

$$4.6 \quad \sum_j |\sigma_j(\xi_j)| < c.$$

We may assume that the $\xi_j$'s take their values in $[1, 1 + \beta]$ and that the $\sigma_j(\xi_j)$'s are valued in $[0,1]$. Then we just have to show that

$$4.7 \quad \frac{1}{\beta} \left| \sum_j \sigma_j(\xi_j)e^{i<\xi_j, \cdot>} - m_{[0,1]} \left( \sum_j \sigma_j(\xi_j)e^{i<\xi_j, \cdot>} \right) \right|^2 \geq c \left( \sum_j \sigma_j(\xi_j) \right)^2.$$

Equivalently it suffices to prove for all $j$

$$4.8 \quad \int_0^1 |e^{i<\xi_j, \cdot>} - m_{[0,1]} e^{i<\xi_j, \cdot>}|^2 \, dx \geq c$$

and for $j, \ell, j \neq \ell,$

$$4.9 \quad \text{Re} \int_0^1 (e^{i<\xi_j, \cdot>} - m_{[0,1]} e^{i<\xi_j, \cdot>}) (e^{i<\xi_\ell, \cdot>} - m_{[0,1]} e^{i<\xi_\ell, \cdot>}) \, dx \geq c.$$

This is clear if $\beta$ is small enough, which proves Lemma 5. We may therefore assume that $\sum_j \|T_{j,k}\|_{2,2} \leq 1$ for all $k \in \mathbb{Z}$. 

The last lemma we shall need is classical.

**LEMMA 6.** - Let \((c_k)_{k\in\mathbb{Z}} \in \ell^2_c(\mathbb{Z})\) and \((\xi_k)_{k\in\mathbb{Z}}\) be a sequence of real numbers such that \(\xi_k \in \left[\frac{5}{4}2^k, 2^{k+1}\right]\) for all \(k \in \mathbb{Z}\). Then \(\sum_k c_k e^{i\xi_k} \) is in BMO and \(\left\| \sum c_k e^{i\xi_k} \right\|_{\text{BMO}} \leq c \left( \sum |c_k|^2 \right)^{1/2} \).

Let us now indicate the strategy to go from 3.6 to 3.7, assuming that \(\|T_j\|_{2,2} \leq \frac{1}{\sqrt{N}}\) and \(T_j = T_j \sum_{k \in \mathbb{Z}} \Delta_k\), for all \(j\), and that \(\sum_j \|T_{j,k}\|_{2,2} \leq 1\) for all \(k \in \mathbb{Z}\).

Suppose there exists for each \(k\) a number \(\xi_k\) in \(\left[\frac{5}{4}2^k, 2^{k+1}\right]\), such that for all \(j\),

4.10 \(\|T_{j,k}\|_{2,2} \leq c |m_j(\xi_k)|.\)

For all \(j \in [1, N]\), let \((c_{j,k})_k\) be such that \(\sum_k |c_{j,k}|^2 \leq 1\). Then by Lemmas 4 and 6 and by 4.3 we obtain \(\sum_k \left| \sum_{j} m_j(\xi_k) c_{j,k} \right|^2 \leq c\) and even

4.11 \(\sum_k \left( \sum_j |m_j(\xi_k)| |c_{j,k}| \right)^2 \leq c.\)

Now if \((f_j)_{1 \leq j \leq N}\) are \(L^2\)-functions with norm 1, by 4.10 and 4.11,

\[
\left\| \sum_j T_j f_j \right\|_2^2 = \sum_k \left\| \sum_j T_{j,k} \Delta_k f_j \right\|_2^2 \\
\leq \sum_k \left( \sum_j \|T_{j,k}\|_{2,2} \|\Delta_k f_j\|_2 \right)^2 \\
\leq c \sum_k \left( \sum_j |m_j(\xi_k)| \|\Delta_k f_j\|_2 \right)^2 \leq c,
\]

since for all \(j\), \(\sum_k \|\Delta_k f_j\|_2^2 \leq 1\).
Unfortunately the existence of these miraculous $\xi_k$'s is not
guaranteed in general and matters are slightly more complicated.
The point will be to select a small number of $\xi_{k,\ell}$'s for each $k$, in such
a way that $\sup_{j,k} \left( \frac{||T_{j,k}||_{2,2}}{\sup_{\ell} m_j(\xi_{k,\ell})} \right)$ be not too large, and then apply
essentially the previous argument. We are now going to describe how
to select these $\xi_{k,\ell}$'s.

Let $k$ be fixed, $p$ be a large fixed integer, and $\mu = \frac{3}{4p}$. Let $r$
and $s$ be two integers such that $0 \leq s \leq r \leq p - 1$. We pick up a $\xi$, denoted $\xi_{k,1}^{r,s}$ such that

$$\sum_j |m_j(\xi)|^2 \geq \frac{1}{p^2} n^{-\frac{3}{4}}$$

where the sum runs over those $j$'s such that

$$N^{-\frac{3}{4} + \mu r} \leq ||T_{j,k}||_{2,2} \leq N^{-\frac{3}{4} + \mu (r+1)}$$

and

$$N^{-\frac{3}{4} + \mu s} \leq |m_j(\xi)| \leq N^{-\frac{3}{4} + \mu (s+1)}.$$

We take off all the $j$'s satisfying 4.13 and 4.14 and select another $\xi$, denoted $\xi_{k,2}^{r,s}$, in the same fashion. When we cannot go on for a
fixed $r, s$, we choose another couple $r', s'$, and obtain a collection of
$(\xi_{k,\ell}^{r',s'})_{\ell}$. Finally when the process stops we can conclude that for all $\xi$'s in $\left[\frac{5}{4} 2^k, 2^{k+1}\right]$,

$$\sum_j |m_j(\xi)|^2 \leq N^{-\frac{3}{4}},$$

where the sum is restricted to those $j$'s which have not been taken
off during the selection process. We call this set $E_k$. So we have a
decomposition of $[1,N]$ as $E_k \cup \bigcup_{r,s,\ell} E_{k,\ell}^{r,s}$ where $E_{k,\ell}^{r,s}$ is the set of $j$'s
which have been taken off after selecting $\xi_{k,\ell}^{r,s}$. Notice that all these
sets are pairwise disjoints. We define $T_{j}^{r,s} = T_j \left( \sum_k \Delta_k \right)$, where the
sum is extended to those $k$'s such that $j$ belongs to $\bigcup_\ell E_{k,\ell}^{r,s}$. Notice that for each $(r, s)$, the collection $(T_j^{r,s})_{1 \leq j \leq N}$ satisfy 4.3 uniformly.

**Lemma 7.** Let $(f_j)_{1 \leq j \leq N}$ be a sequence of $L^2$-functions of norm less than 1. Then $\left\| \sum_j T_j^{r,s} f_j \right\|_2 \leq c \, p \, N^\frac{\mu}{2} + 2$.

To prove this lemma we first observe that for each $\ell$, the set $E_{k,\ell}^{r,s}$ has at least $p^{-2} N^{\frac{3}{4} - 2\mu(s+1)}$ elements because of 4.12 and 4.14. On the other hand, since $\sum_j \left\| T_j f \right\|_{2,2} < 1$, the set $\bigcup_\ell E_{k,\ell}^{r,s}$ has at most $N^{\frac{3}{4} - \mu r}$ elements by 4.13. Hence there is at most $p^2 N^{\mu(2s+2-r)}$ distinct values of $\xi^{r,s}_k$, for $r,s$ and $k$ fixed. For each $j$ we consider a sequence $(c_{j,k})_{k \in \mathbb{Z}}$ in $\ell^2(\mathbb{Z})$ such that $\sum_k |c_{j,k}|^2 \leq 1$, and $c_{j,k} = 0$ if $T_j^{r,s} \Delta_k = 0$. Let $b_j$ be the BMO function $\sum_k c_{j,k} e^{i < \xi^{r,s}_k, \cdot >}$, where $\xi^{r,s}_k$ is the element of $\{ \xi^{r,s}_k, 1 \leq \ell \leq p^2 N^{\mu(2s+2-r)} \}$ for which $j \in E_{k,\ell}^{r,s}$.

By Lemmas 4 and 6 and by 4.3 and 4.14

$$\sum_{k,\ell} \left( \sum_j |c_{j,k}| \right)^2 \leq c \, N^{\frac{3}{4} - 2\mu s},$$

where the sum in $j$ runs over $E_{k,\ell}^{r,s}$. If $(f_j)_{1 \leq j \leq N}$ are $L^2$-functions of norms less than 1, $\left\| \sum_j T_j^{r,s} f_j \right\|_2^2 \leq \sum_k \left( \sum_j \left\| T_j^{r,s} f_j \right\|_{2,2} \left\| \Delta_k f_j \right\|_2 \right)^2$, which is, by 4.13, dominated by $\sum_k \left( \sum_j \left\| \Delta_k f_j \right\|_2 \right)^2 N^{-\frac{3}{4} + 2\mu(r+1)}$, the sum in $j$ running over $\bigcup_\ell E_{k,\ell}^{r,s}$. Hence, for each $k$,

$$\left( \sum_{j \in \bigcup_\ell E_{k,\ell}^{r,s}} \left\| \Delta_k f_j \right\|_2 \right)^2 \leq p^2 N^{\mu(2s+2-r)} \sum_\ell \left( \sum_{j \in E_{k,\ell}^{r,s}} \left\| \Delta_k f_j \right\|_2 \right)^2.$$
By the assumption \( \|T_j\|_{2,2} \leq \frac{1}{\sqrt{N}} \) for all \( j \), we see that \( \mu(r+1) \leq \frac{1}{4} \). We then deduce from Lemma 7 that

\[
\left\| \sum_{r,s} \sum_{j} T_j^{r,s} f_j \right\|_2 \leq c_p^2 N^{\mu(r+4)}
\]

and Lemma 7 is proved. By the assumption \( \mu(r+1) \leq \frac{1}{4} \), we see that

\[
\left\| \left( \sum_{j} T_j^{r,s} f_j \right) \right\|_2 \leq c_p^3 N^{\frac{3\mu+1}{2}} \sup_j \|f_j\|_2.
\]

Since \( \mu \) can be made arbitrarily small, we just have to estimate the remainder term \( \left\| \sum_j T_j f_j - \sum_{r,s} \left( \sum_j T_j^{r,s} f_j \right) \right\|_2 \). Using 4.15, Plancherel and Cauchy-Schwarz we easily obtain a domination by \( c N^\frac{1}{8} \). This concludes the proof of Proposition 1.

5. Tensor-products of multilinear singular integral operators.

We wish to prove the following.

**Theorem 2.** - For all \( \delta > 0 \), there exists \( c_\delta > 0 \), such that for all \( a \in L^\infty_\delta (\mathbb{R}^2) \) and \( k \in \mathbb{N} \)

\[
\|L_k a\|_{2,2} \leq c_\delta (1 + k)^{2+\delta} \|a\|_\infty^k.
\]

This theorem will essentially follow from a general result on multilinear singular integral operators.

Let \( T_1 \) and \( T_2 \) be two bounded operators on \( L^2(\mathbb{R}) \). Then it is a simple consequence of Fubini's Theorem that \( T_1 \otimes T_2 \), defined on \( L^2(\mathbb{R} \times \mathbb{R}) \), extends boundedly to all of \( L^2(\mathbb{R} \times \mathbb{R}) \). If however one considers two bilinear operators bounded from \( L^\infty(\mathbb{R}) \times L^2(\mathbb{R}) \) to \( L^2(\mathbb{R}) \), then their tensor-product, defined on \( [L^\infty(\mathbb{R}) \otimes L^\infty(\mathbb{R})] \times [L^2(\mathbb{R}) \otimes L^2(\mathbb{R})] \), is not in general bounded from \( L^\infty(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \) to \( L^2(\mathbb{R}^2) \) [9]. It is a surprising fact that it is bounded when the bilinear operators are of Coifman-Meyer type.

We shall see that this is also true for multilinear singular operators.
As in [10] we shall deal with multilinear singular integral forms instead of multilinear operators. We refer to [10] for the precise definition of a multilinear singular integral form, $\delta - n\text{SIF}$, and for a boundedness criterion concerning them. We shall denote by $U$ a $\delta - n\text{SIF}$ on $\mathbb{R}$ and refer to [10] for the notations $U_i1$, $i \in [1, n]$, $|U|_\delta$, $|U|_W$, $||U||_i$, $j$ and $|U_{ij}|_\delta$. We recall that $U$ is bounded if for $1 \leq i \leq j \leq n$,

$$
5.1 \quad |U(h_1, h_2, \ldots, h_n)| \leq c_{ij} \left( \prod_{k \neq i, j} ||h_k||_\infty \right) ||h_i||_2 ||h_j||_2
$$

for all $h_\ell$, $1 \leq \ell \leq n$, in $C_0^\infty(\mathbb{R})$.

If $U$ and $U'$ are two bounded $\delta - n\text{SIF}$'s, then their tensor-product $U \otimes U'$ is well defined on $[C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})]^n$. We then say that $U$ is bounded if for $1 \leq i \leq j \leq n$,

$$
5.2 \quad |U \otimes U'(h_1, h_2, \ldots, h_n)| \leq c_{ij} \left( \prod_{k \neq i, j} ||h_k||_\infty \right) ||h_i||_2 ||h_j||_2
$$

for all $h_\ell$, $1 \leq \ell \leq n$, $C_0^\infty(\mathbb{R})$, and we denote by $|||U \otimes U'||||$, sup$_{i,j} \tilde{c}_{ij}$, where $\tilde{c}_{ij}$ is the best constant in 5.2.

**Theorem 3.** If $U$ and $U'$ are two bounded $\delta - n\text{SIF}$'s, then $U \otimes U'$ is bounded and

$$
5.3 \quad |||U \otimes U'|||| \leq c \left\{ \sum_i ||U_i1||_{\text{BMO}} + n^2(|U|_\delta + |U|_W) \right\}
$$

Notice that the constant $c$ appearing in 5.3 is independent of $n$.

An application of Theorem 3 in the case where $U = U'$ is the form determined by the $(n - 2) - nd$ Calderón-commutator (see [10] section 4) yields $||L_{k,a}||_{2,2} \leq c_\delta (1 + n)^{4+\delta}$ for all $n \in \mathbb{N}$ and $\delta > 0$. As in [10] the antisymmetry of the kernel $\frac{1}{x - y}$ permits to improve this estimate and to obtain Theorem 2. Since this will be clear from the proof of Theorem 3 which we shall now outline, we omit the details.
The proof of Theorem 3 is along the same lines as the proof of Theorem 2 in [10], which we shall assume familiar to the reader. Let us recall however that the main ingredients of this proof are Carleson measures, quadratic estimates that have been developed in [5], [7], and [11] in the context of product-spaces. Another important element is the equivalence between \( H^1 \to L^1 \) and \( L^\infty \to \text{BMO} \) boundedness for a singular integral operator. The proof of this equivalence given in [12, p. 49] relies on the atomic decomposition of \( H^1 \) in \( H^{1,\infty} \) -atoms. Such a decomposition does not seem to exist on product-spaces where the atoms are only in \( L^2 \). However this equivalence is still true on product-spaces.

**Proposition 2.** Let \( T \) be a \( \delta - \text{SIO} \) on \( \mathbb{R} \times \mathbb{R} \). Then \( T \) is bounded on \( L^2 \) if and only if \( T \) maps \( L^\infty \) to \( \text{BMO} \).

We refer to [3] for the definition of a \( \delta - \text{SIO} \) on \( \mathbb{R} \times \mathbb{R} \). Notice that by applying Proposition 2 simultaneously to \( T \) and \( T^* \) we see that a \( \delta - \text{SIO} \) maps \( L^\infty \) to \( \text{BMO} \) if and only if it maps \( H^1 \) to \( L^1 \).

The fact that the \( L^2 \)-boundedness of \( T \) implies its \( L^\infty - \text{BMO} \) boundedness is already known [3]. The converse is then an easy consequence. Suppose that \( T \) is a \( \delta - \text{SIO} \) bounded from \( L^\infty \) to \( \text{BMO} \), and let us also assume that \( ||T||_{L^2} < +\infty \). Then, by the direct part of Proposition 2 applied to \( T^* \) we obtain \( ||T||_{H^{1,L^1}} \leq c||T||_{L^2} + c(T) \), where \( c(T) \) depends only on the constants for the standard estimates of the kernel of \( T \). By interpolation [13] \( ||T||_{L^2} \leq c(||T||_{L^\infty, \text{BMO}}||T||_{H^{1,L^1}})^{1/2} \). It follows that, \( ||T||_{L^2} \leq c(||T||_{L^\infty, \text{BMO}} + c(T)) \), which easily implies Proposition 2.

The connection between \( \delta - \text{SIO}'s \) on \( \mathbb{R} \times \mathbb{R} \) and tensor products of \( \delta - \text{nSIF}'s \) on \( \mathbb{R} \) is provided by the following lemma.

**Lemma 8.** Let \( U \) and \( U' \) be two bounded \( \delta - \text{nSIF}'s \) on \( \mathbb{R} \). For all \( 1 \leq i \leq j \leq n \) and all \( h_k \in C_0^\infty(\mathbb{R}^d) \otimes C_0^\infty(\mathbb{R}^d) \), \( k \neq i, j \), the operator \( T = (U \otimes U')_{i,j}(h_1, \ldots, h_k, \ldots, h_n) \) defined by \( < U_{i,j}, Th_j > = (U \otimes U')(h_1, \ldots, h_n) \), is a \( \delta - \text{SIO} \) on \( \mathbb{R} \times \mathbb{R} \), of norm less than

\[
c(||U||_{i,j} + |U_{i,j}| \delta)|U'_{i,j}| \delta + (||U'||_{i,j} + |U'_{i,j}| \delta)|U_{i,j}| \delta \prod_{k \neq i,j} ||h_k||_{\infty}.
\]
The proof is routine and we omit it.

We turn to the proof of Theorem 3. Let $\phi$ be a non-negative function in $C^\infty_0(\mathbb{R})$ such that $\int \phi \, dx = 1$. For all $t > 0$, we denote by $P_t$ the convolution operator on $\mathbb{R}^2$ of symbol $\sigma(\xi, \xi') = \hat{\phi}(t\xi)$. Similarly $P_{t'}$ is the operator of symbol $\hat{\phi}(t'\xi')$. Finally $Q_t = -t \frac{\partial}{\partial t} P_t$ and $Q_{t'} = -t' \frac{\partial}{\partial t'} P_{t'}$.

As in [10] we choose $h_1, \ldots, h_n$ in $C^\infty_0(\mathbb{R}^d) \otimes C^\infty_0(\mathbb{R}^d)$ and express $U \otimes U'(h_1, \ldots, h_n)$ as the sum of $n^2$ double integrals of two different types:

$$I = \int \int U \otimes U'(Q_t Q_{t'}', h_1, P_t P_{t}', h_2, \ldots, P_t P_{t}'', h_n) \frac{dt}{t} \frac{dt'}{t'}$$

$$II = \int \int U \otimes U'(Q_t P_{t}', h_1, P_t Q_{t}', h_2, P_t P_{t}', h_3, \ldots, P_t P_{t}', h_n) \frac{dt}{t} \frac{dt'}{t'}.$$

It is clear that the estimates of the $n$ integrals of type I can be reduced to Carleson measure estimates. To see that this is also true for the $n^2 - n$ integrals of type II we need the following.

**Lemma 9.** Let $T$ be a $\delta - $SIO on $\mathbb{R} \times \mathbb{R}$. If $T_1$ and $T^{*1}$ vanish (or are in BMO), and the partial adjoints of $T$ are bounded on $L^2$, then $T$ is a $\delta - $CZO.

This lemma follows immediately from the T1-Theorem and Theorem 3 of [3].

Let $V'$ be the $\delta - $nSIF obtained from $U'$ by letting $V'(f_1, f_2, \ldots, f_n) = U'(f_2, f_1, f_3, \ldots, f_n)$. By lemmas 8 and 9 we see that

$$\left| \int \int U \otimes V'(Q_t Q_{t}'', h_1, P_t P_{t}', h_2, \ldots) \frac{dt}{t} \frac{dt'}{t'} \right| \leq c_1 \|h_1\|_2 \|h_2\|_2 \prod_{k \geq 3} \|h_k\|_\infty$$

implies

$$\left| \int \int U \otimes U'(Q_t P_{t}', h_1, P_t Q_{t}', h_2, \ldots) \frac{dt}{t} \frac{dt'}{t'} \right|$$
\[ \leq c_2 ||h_1||_2 ||h_2||_2 \prod_{k \geq 3} ||h_k||_\infty. \]

Since \( V' \) has the same properties as \( U' \), we are reduced to estimating integrals of type I.

We are going to prove the following estimate:

\[
\left| \int \int U \otimes U'(Q_i Q'_i h_1, P_i P'_i h_2, \ldots) \frac{dt}{t} \frac{dt'}{t'} \right| \leq c ||U_1||_{BMO} + n(||U|_0 + ||U||_W)) \prod_{k \geq 3} ||h_k||_\infty ||h_1||_2 ||h_2||_2.
\]

It is easy to see from the previous remarks that 5.4 implies Theorem 3. In turn 5.4 is itself an immediate consequence, after reduction to a Carleson-measure estimate, of an extension of Theorem 1 of [10] to the setting of product spaces, which we now describe. We first need to recall the notion of an \( \varepsilon \)-family introduced in [10].

**Definition 1.** - A family \( S = (s_t)_{t > 0} \) of operators given by kernels satisfying

\[ |s_t(x, y)| \leq c \frac{t^\varepsilon}{t^{1+\varepsilon} + |x - y|^{1+\varepsilon}} \]

\[ |s_t(x, y) - s_t(x, z)| \leq c \frac{t^\varepsilon}{t^{1+\varepsilon} + |x - y|^{1+\varepsilon}} \left( \frac{y - z}{t + |x - y|} \right)^\varepsilon,
\]

for all \( x, y, \) and \( z \) such that \( |y - z| \leq \frac{1}{2}(t + |x - y|) \), is an \( \varepsilon \)-family.

It is bounded if for all \( f \in L^2 \),

\[ \left[ \int_0^{+\infty} ||s_t f||_2^2 \frac{dt}{t} \right]^{1/2} \leq c ||f||_2. \]

Following the procedure of [3], to extend this notion to product spaces, we first put a norm on the space of \( \varepsilon \)-families by letting

\[ ||S||_\varepsilon = ||S||_2 + ||S||_\varepsilon, \text{ where } ||S||_2 \text{ is the best constant in 5.7 and} \]

\[ ||S||_\varepsilon = \sup_{n} ||S_n||_\varepsilon, \]

where \( S_n \) is the \( \varepsilon \)-family obtained by truncating \( S \) at \( n \).
|S|_e in 5.5 and 5.6. An \( \varepsilon \)-family on \( \mathbb{R} \times \mathbb{R} \) will then be a two-parameter family \( (T_{t,t'})_{t,t' > 0} \) of operators given by integrable kernels \( T_{t,t'}(x,x,y,y) \). For \( t,x,y \) fixed, we shall denote by \( (T_{t,t'}[x,y])_{t' > 0} \), the one-parameter family of operators acting on the second variable, and of kernels \( (T_{t,t'}[x,y])(x',y') = T_{t,t'}(x,x',y,y') \), and similarly for \( (T_{t,t'}[x',y'])_{t > 0} \). Then \( (T_{t,t'})_{t,t' > 0} \) is an \( \varepsilon \)-family if

\[
\|(T_{t,t'}[x,y])_{t' > 0}\|_e \leq c \frac{t^\varepsilon}{|x-y|^{1+\varepsilon} + t^{1+\varepsilon}}
\]

\[
\|(T_{t,t'}[x,y] - T_{t,t'}[x,z])_{t' > 0}\|_e \leq c \left( \frac{|y-z|}{t + |x-y|} \right) t^\varepsilon \frac{t^\varepsilon}{t^{1+\varepsilon} + |x-y|^{1+\varepsilon}}
\]

when \( |y-z| \leq \frac{1}{2} (t + |x-y|) \) and similarly for \( (T_{t,t'}[x',y'])_{t' > 0} \).

We denote by \( |T_{t,t'}|_e \) the best constant in 5.8 and 5.9. The family \( (T_{t,t'})_{t,t' > 0} \) is bounded if for all \( f L^2 \)

\[
\left( \int \int \|T_{t,t'}f\|^2 \frac{dt}{t} \frac{dt'}{t'} \right)^{1/2} \leq c\|f\|_2.
\]

We also introduce a "Carleson norm" on functions \( w \) from \( \mathbb{R}^2_+ \times \mathbb{R}^2_+ \) into \( \mathbb{C} \), by letting

\[
|w|_c = \sup_{\Omega \subseteq \mathbb{R}^2} \left( \frac{1}{|\Omega|} \right) \int \int_{S(\Omega)} |w(x,x',t,t')|^2 dx dx' \left( \frac{dt}{t} \frac{dt'}{t'} \right)^{1/2},
\]

where \( \Omega \) is an arbitrary bounded open subset of \( \mathbb{R}^2 \), and \( S(\Omega) \) consists of these \( (x,x',t,t') \) such that \( |x-t, x+t[x] x' - t', x' + t'| \subseteq \Omega \).

**Theorem 4.** - Let \( (T_{t,t'})_{t,t' > 0} \) be an \( \varepsilon \)-family. It is bounded if and only if \( |(T_{.,1})(.,.)|_e < +\infty \). In this case for all \( a \in L^\infty(\mathbb{R}^2) \)

\[
|(T_{.,a})(.,.)|_c \leq \|a\|_\infty \|(T_{.,1})(.,.)\| + c_e |a|_e |T_{t,t'}|_e.
\]

By the same argument as for Theorem 1 for [10], we need to consider only the case where \( T_{t,t'}1 = 0 \) for all \( t,t' > 0 \). We then decompose \( T_{t,t'} \) as \( X_{t,t'} + Y_{t,t'} \) where \( X_{t,t'}f(x,x') = \)
\[ \int \int T_{t,t'}(x, x', y, y')(P'_{t'}f)(y, x') \, dy \, dy' \] notice that \((X_{t,t'})_{t,t'>0}\) is itself an \(\varepsilon\)-family, as well as \((Y_{t,t'})_{t,t'>0}\). Furthermore if \(f\) does not depend on the first variable \(X_{t,t'}f = 0\), while if it depends only on the first variable \(Y_{t,t'}f = 0\). Therefore we are reduced to the case where not only \(T_{t,t'}1 = 0\) but also \(T_{t,t'}f = 0\) for all functions \(f\) of the first variable. To show that \((T_{t,t'})_{t,t'>0}\) is bounded in this case it suffices to show that

\[ Z = \int \int T^{*}_{t,t'} T_{t,t'} \frac{dt}{t} \frac{dt'}{t'} \]

is bounded on \(L^2\). But \(Z\) is an SIO on \(\mathbb{R} \times \mathbb{R}\), to which it is easy to see that the T1-Theorem of [3] applies. To deduce 5.11 from the boundedness of \((T_{t,t'})_{t,t'>0}\) one proceeds exactly as in the proof of Theorem 3 on [3]. This proves Theorem 4. Routine arguments, which we shall omit, now yield 5.4 and then Theorem 3.

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