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CONNECTIONS WITH PRESCRIBED CURVATURE

by D. DETURCK [1] and J. TALVACCHIA

Introduction.

Geometry has served as a rich source of interesting (and difficult) problems in partial differential equations. Conversely, the study of such equations often leads to new geometrical insight. In this note, we consider some aspects of the problem of prescribing the curvature tensor of the connection on a principal bundle. This problem is easily cast as a nonlinear system of first-order differential equations (see (1.1)), and we are particularly interested in knowing when the system does (or does not) have local solutions. While this problem has much in common with other problems of prescribing curvature tensors (see e.g., [5], [6]), there are several new wrinkles here which have yet to be completely ironed out. They seem to stem from the interaction of the various groups which arise naturally in the problem : the structure group of the bundle and the diffeomorphism group of the base manifold. As one expects in such a problem, the equivariance of curvature under changes of gauge and coordinates yields the Bianchi identities, and these must be taken into account in any existence proof. However, some *nonlinear* identities also arise as obstructions to solvability when the structure group is semisimple — we give a simple derivation of these identities (of course, once one knows that an identity is there, it is always easy to derive it), and conjecture (perhaps naively) that we have found all of the obstructions to generic local solvability in the semisimple case.

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We give the proof of this conjecture for the group SU (2) (we also have the proof for other simple groups of low rank, see [14]). We note that the case of SU (2) had been done independently by S. Tsarev (without the new identity, or any indication of how to proceed for other groups) and R. Bryant (private communication). Also, A. Asada [15] has considered the problem for GL(n), and has expressed solvability conditions in terms of an infinite sequence of equalities which must be satisfied by the curvature candidate. Our results here show that generically there is local solvability over three-dimensional base manifolds, (perhaps modulo finitely many equations in the semisimple but not simple case), so that Asada's conditions cannot all be independent in dimension 3.

In this paper section 2 introduces the problem of prescribing the curvature of a connection on a principal bundle, and makes precise our notation. Section 3 is devoted to the case where the structure group of the bundle is nilpotent; in particular, a complete proof of local solvability is given for the Heisenberg group. We thank J. R. Vanstone for instigating the results of this section. In section 4, we turn to the case of (semi)simple structure groups, prove the new identities, prove generic local solvability for bundles with structure group SU (2), and indicate how to proceed for simple structure groups of higher rank. In the appendix, we review some basic techniques used to prove local solvability of systems of partial differential equations, and give a sketch of how the techniques work for the problem of prescribing Ricci curvature (a case where everything goes right).

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1. Connections on principal bundles.

We begin our discussion of the equation $\mathrm{Curv}(\Gamma) = F$ by making precise our notation and deriving a few general facts. To start, let G be a Lie group with Lie algebra $\mathfrak{g}$, and let P be a principal G-bundle over some manifold M. For any linear representation $\rho : G \to GL(V)$, there is a vector bundle E_ρ (whose fibers are isomorphic to the vector space V) associated to P, and a connection on P gives rise to a connection on E_ρ in a canonical way. Conversely, if ρ_* is a *faithful* representation of $\mathfrak{g}$, then a connection on E_ρ gives rise to a unique connection on P. We will freely pass among the various E_ρ associated to a given P; the ρ we consider most often are the fundamental representation of G and, in the case of semisimple G, the adjoint representation.

To be totally explicit, we work over a local coordinate chart on M with coordinates x^i, $i = 1, \ldots, n$, and consider the vector bundle E associated to the fundamental representation ρ of G. Let e_A, $A = 1, \ldots, r$ be a basis (over $C^\infty(M)$) of local sections of E, so that any local section v of E can be expressed as $v^A(x)e_A$ (we are using the summation convention). For any connection ∇ on E, ∇v should be a section of $T^* \otimes E$, and to achieve this we set

$$\nabla v = \left(\frac{\partial v^A}{\partial x^i} + (\Gamma_i(v))^A\right) dx^i \otimes e_A$$

for some choice of $\Gamma_i \in \rho_*(\mathfrak{g})$ (we abuse the notation and often write $\Gamma_i \in \mathfrak{g}$). We will not worry about how various objects transform under changes of coordinates on the manifold or basis in the bundle (gauge) since our point of view will be purely local.

The connection gives rise to exterior-differential-type operators d^∇ by tensoring with the deRham complex of M: If α is any p-form with coefficients in E (i.e., α is a section of $\Lambda^p T^* \otimes E$), then the $p + 1$-form $d^\nabla \alpha$ is given by

$$\begin{aligned} d^\nabla \alpha &= d\alpha + \Gamma \wedge \alpha \\ &= \left(\frac{\partial \alpha_\Lambda^A}{\partial x^i} + (\Gamma_i(\alpha_\Lambda))^A\right)(dx^i \wedge dx^\Lambda) \otimes e_A \end{aligned}$$

where Λ is a multiindex of length p. In particular, for a 1-form α,

$$d^\nabla \alpha = \left(\frac{\partial \alpha_j^A}{\partial x^i} - \frac{\partial \alpha_i^A}{\partial x^j} + (\Gamma_i(\alpha_j))^A - (\Gamma_j(\alpha_i))^A\right)(dx^i \wedge dx^j) \otimes e_A.$$

The curvature F of the connection measures the failure of d^∇ to give rise to a complex, i.e.,

$$d^\nabla d^\nabla \alpha = F \wedge \alpha$$

for any p-form α, where F is the $\mathfrak{g}$-valued two-form given by

$$F = \left(\frac{\partial \Gamma_j}{\partial x^i} - \frac{\partial \Gamma_i}{\partial x^j} + [\Gamma_i, \Gamma_j]\right) dx^i \wedge dx^j. \tag{1.1}$$

It is equation (1.1) that we will try to solve for Γ, once a section F of $\Lambda^2 T^* \otimes \mathfrak{g}$ is prescribed in advance. In other words, we will think of (1.1) as a system of partial differential equations with the Γ_i (we emphasize that each Γ_i is a matrix) as unknown functions. Note that if $n = \dim M$ and $d = \dim \mathfrak{g}$, then there are $\binom{n}{2}d$ equations for nd unknown functions (the entries of the Γ_i). Thus for $n = 2$, this is an underdetermined system (more unknowns than equations; in fact, the system is elliptic in this case: every cotangent direction is noncharacteristic for the system). If $n = 3$ there are just as many unknowns as equations (although there are no noncharacteristic directions), and if $n \geqslant 4$ the system is overdetermined.

For $n \geqslant 3$, there is a well-known condition which F (and Γ) must satisfy, namely the Bianchi identity. This identity states that $d^\nabla F = 0$, where this time d^∇ comes from the connection induced by ∇ on the vector bundle associated to the adjoint representation of $\mathfrak{g}$. Somewhat more explicitly,

$$(1.2)\quad \begin{aligned} 0 = d^\nabla F &= dF + \Gamma \wedge F \\ &= \left(\frac{\partial F_{jk}}{\partial x^i} + \frac{\partial F_{ki}}{\partial x^j} + \frac{\partial F_{ij}}{\partial x^k} + [\Gamma_i, F_{jk}] + [\Gamma_j, F_{ki}] + [\Gamma_k, F_{ij}]\right) \\ &\qquad dx^i \wedge dx^j \wedge dx^k. \end{aligned}$$

One sees that the Bianchi identity is a real obstruction to local solvability of the equation $\mathrm{Curv}(\Gamma) = F$ by considering two-forms F of the form

$$F = (F_{ij,k} x^k)\, dx^i \wedge dx^j$$

in a neighborhood of the origin, where $F_{ij,k} + F_{jk,i} + F_{ki,j} \neq 0$ for some choice of i, j, and k. It is then clear that no choice of Γ can satisfy (1.2) at the origin, where $F = 0$ but $dF \neq 0$. Of course, the condition $dF \equiv 0$ is the obvious necessary (and sufficient!) condition when the group G is abelian — we are going to consider the successively more interesting cases of nilpotent and semisimple G. A first observation is that in order for F to be the curvature of some connection, it must be true that dF (taken with respect to any connection) takes its values in the derived subalgebra $[\mathfrak{g},\mathfrak{g}]$ of $\mathfrak{g}$. Of course, this is no condition at all for semisimple G, but is quite a strong restriction on the curvature tensors of bundles with nilpotent and solvable G.

In the next two sections, we consider the problem of prescribing the curvature on a bundle with first nilpotent, and then semisimple structure group over a three-dimensional base manifold. We ignore two-dimensional bases since the problem then is elliptic (and hence trivially locally solvable) and we also ignore dimensions higher than three since then the Bianchi identity places so many algebraic conditions on the connection as to give nonexistence or determine a unique connection (which may or may not be the solution of (1.1), and one simply checks the unique candidate by plugging it into this equation. In any case, the problem is reduced in general to an algebraic computation). These considerations are explained more fully in [14]. We therefore consider only three-dimensional base manifolds for the rest of the paper.

2. Nilpotent structure groups.

In this section, we consider the problem of prescribing the curvature of a connection which takes its values in a nilpotent Lie algebra. For the sake of brevity, we restrict our attention to the case where the algebra is $\mathfrak{h}_3$, the three-dimensional Heisenberg algebra. Recall that this Lie algebra can be described as the span of $\{X, Y, Z\}$, where $[X,Z] = [Y,Z] = 0$, and $[X,Y] = Z$. For the purposes of studying equation (1.1) for the Heisenberg algebra, it will be advantageous to consider the connection Γ and the curvature F to be elements of the Lie algebra whose coefficients are differential forms rather than the other way around. Thus, we will think of the form

$$F = \alpha X + \beta Y + \gamma Z$$

as being prescribed in advance, where α, β and γ are ordinary scalar-valued two-forms, and we will search for our connection Γ in the form

$$\Gamma = pX + qY + rZ,$$

where p, q and r are ordinary scalar-valued one-forms.

The derived subalgebra of $\mathfrak{h}_3$ is the one-dimensional algebra spanned by Z alone, and so the « first observation » made at the end of the previous section tells us that dF must be a (three-form) multiple of Z in order to even have a chance to be the curvature of a connection. We can see this even more vividly if we compute the curvature of the connection Γ :

$$\begin{aligned}(2.1)\qquad \mathrm{Curv}(\Gamma) &= d\Gamma + [\Gamma,\Gamma] \\ &= dpX + dqY + (dr + p \wedge q)Z.\end{aligned}$$

Thus, p, q and r must satisfy the differential system

$$\begin{aligned} dp &= \alpha \\ dq &= \beta \\ dr + p \wedge q &= \gamma. \end{aligned} \tag{2.2}$$

This immediately implies that $d\alpha = d\beta = 0$ is a necessary condition for solvability. If this condition is satisfied, then p and q are determined to within adding the differential of a scalar function to each. We now take the exterior derivative of the third of equations (2.2), keeping the other two in mind, to find

$$\alpha \wedge q - p \wedge \beta = d\gamma. \tag{2.3}$$

If we can choose p and q so that (2.3) is satisfied, then it will be possible to use the Poincaré lemma to solve for r. But recall that we have determined $p = p_0 + df$ and $q = q_0 + dg$, where we are still free to choose f and g. If the base manifold is three-dimensional, then (2.3) becomes an underdetermined system of one linear equation for the two unknown functions f and g. This equation is easily seen to have local solutions, provided α and β are not both zero at the same point. This proves :

PROPOSITION 2.4. — *On an $\mathfrak{h}_3$-bundle over a three-dimensional base manifold, the equation* $\mathrm{Curv}(\Gamma) = \mathrm{F}$ *is generically locally solvable provided* $d\mathrm{F}$ *is a « center of* $\mathfrak{h}_3$ *»-valued three-form.*

It might be an amusing exercise for the reader to work out the situation for the $2n + 1$-dimensional Heisenberg group, or for an arbitrary two-step nilpotent Lie group. Solvable groups are somewhat harder (see [14]).

3. Semisimple structure groups.

It is for semisimple structure groups that we first encounter the new identities alluded to in the introduction. To understand how they arise, we consider the problem of constructing a (formal) power series solution of the system consisting of equations (1.1) and (1.2), that is :

$$\begin{aligned} \mathrm{Curv}(\Gamma) \equiv d\Gamma + \Gamma \wedge \Gamma &= \mathrm{F} \\ d\mathrm{F} + \Gamma \wedge \mathrm{F} &= 0. \end{aligned} \tag{3.1}$$

We remind the reader that we are working on a bundle over a three-dimensional base space. We also remark that the first of these equations is first-order in Γ and the second is only zeroth-order in Γ. Thus, to begin constructing the power series solution of (3.1), we need to consider the second equation when choosing even the constant terms in the series for Γ. When the structure group is semisimple, and F is « generic », it is always possible to choose the constant terms so that the second equation is satisfied at the origin. This is a consequence of the following.

LEMMA 3.2. — *Let $\mathfrak{g}$ be a simple Lie algebra. For the generic $\mathfrak{g}$-valued two-form* F, *the map*

$$\phi : \Lambda^1 T_0^* \otimes \mathfrak{g} \to \Lambda^3 T_0^* \otimes \mathfrak{g}$$

given by $\phi(\Gamma) = F \wedge \Gamma$ *is surjective.*

Proof. — The set of regular elements of $\mathfrak{g}$ is open and dense in $\mathfrak{g}$ (see section 103 of [16]), so the generic F (considered as a map from $\Lambda^2 T_0$ to $\mathfrak{g}$) has a regular element X in its image. Furthermore, it is generic to assume that another element Y in the image of F has a nonzero projection on each root vector E_{α_i} for some basis $\{\alpha_i\}$ of the root space of $\mathfrak{g}$ (with respect to the Cartan subalgebra $\mathfrak{h}$ generated by X). By a linear change of coordinates in T_0, we can thus assume that

$$F = X dx^2 \wedge dx^3 + Y dx^3 \wedge dx^1 + Z dx^1 \wedge dx^2.$$

If $\Gamma = \Gamma_1 dx^1 + \Gamma_2^d x^2 + \Gamma_3 dx^3$, note that

$$\phi(\Gamma) = ([X,\Gamma_1] + [Y,\Gamma_2] + [Z,\Gamma_3])\, dx^1 \wedge dx^2 \wedge dx^3 .$$

To see that ϕ is surjective, note first that the image of ad_X is precisely the space of root vectors and that its kernel is $\mathfrak{h}$. Also, the image of ad_Y projects onto $\mathfrak{h}$ (since $[E_\alpha, E_{-\alpha}] = \alpha \cdot H$ for each root α).

We now move on to the first-order terms. These terms must be chosen so that the first of equations (3.1) is satisfied at the origin, and also so that the first derivative of the second of equations (3.1) is satisfied at the origin. Again, provided F is generic, this can be done (we omit the algebraic details). This actually fulfills the first half of condition (1) of the Cartan-Kähler theorem (see the Appendix) for the system (3.1) to be involutive. It is also possible to check that conditions (2) and (3) are also satisfied, i.e., that the nonlinear map on the jet bundle corresponding to (3.1) has constant rank, and that there exists a quasiregular basis at the origin. In fact, the basis chosen for F in the proof of Lemma 3.2 is quasiregular. Next, the fact that the Bianchi identity is satisfied to zeroth order at the origin guarantees that we may choose quadratic terms for Γ so that the first of equations (3.1) is satisfied to first order. The problem comes when one tries to choose the quadratic terms in the power series for Γ so that the second derivatives of the Bianchi identity are satisfied at the origin. The reason for this is that the linear equations which the second derivatives of Γ must satisfy, namely the second prolongation of the symbol of the Bianchi identity, are not surjective, and there is nothing to force the right-hand side of the equations to lie in the image of the left. In other words, it is impossible to verify the second half of condition (1) for involutivity. The failure of the system to be involutive due to this condition indicates that there are more identities present which must be satisfied by the lower-order terms, in order that the linear polynomials we find as first-order formal solutions may all be prolonged to quadratic formal solutions. To discover the identities, we need to differentiate the equations and search for hidden relations. We proceed to do this.

We begin with the Bianchi identity. Since we are working locally, we abuse the right to take partial derivatives and pretend that they are well-defined (we could make this more « geometric » by defining a background connection on the tensor product of the tangent bundle with the various other bundles we are considering, but for now, we'll just write ∇_0 for the componentwise partial derivatives of whatever object we apply it to). Thus, we have :

$$\begin{aligned} 0 &= \nabla_0(d\mathrm{F} + \Gamma \wedge \mathrm{F}) \\ &= \nabla_0 d\mathrm{F} + \Gamma \wedge \nabla_0\mathrm{F} + \nabla_0\Gamma \wedge \mathrm{F}. \end{aligned}$$

The quantity on the right-hand side of this equation is $\mathrm{T}^* \otimes \Lambda^3\mathrm{T}^* \otimes \mathfrak{g}$-valued. We can therefore wedge the T^* part of the object with a two-form. We choose to do this with F (a $\mathfrak{g}$-valued two-form), but instead of applying the Lie bracket to the coefficients of the form, we apply the Killing form of the Lie algebra to them (recall that the Killing form K(A, B) is tr(AB)). We denote this new kind of wedge product by $\overset{\mathrm{K}}{\wedge}$, i.e., for two $\mathfrak{g}$-valued one-forms α and β, we have

$$\alpha \overset{\mathrm{K}}{\wedge} \beta = \Sigma\, \mathrm{tr}(\alpha_i \beta_j) dx^i \wedge dx^j.$$

The result of $\overset{K}{\bigwedge}$-ing F with the derivative of the Bianchi identity is

$$0 = F \overset{K}{\bigwedge} \nabla_0 dF + F \overset{K}{\bigwedge} \Gamma \wedge \nabla_0 F + F \overset{K}{\bigwedge} \nabla_0 \Gamma \wedge F. \tag{3.3}$$

The « business term » of this expression is the third term, since it involves first derivatives of Γ. This term is a sort of « triple scalar product » between a $\mathfrak{g}$-valued two form, a $\mathfrak{g}$-valued section of $T^* \otimes T^*$, and another $\mathfrak{g}$-valued two-form. The key observation that enters here is that we can replace the derivatives of Γ in this term with the values of Γ as a consequence of the following.

LEMMA 3.4. — *Let* $F \in \Lambda^2 T^* \otimes \mathfrak{g}$ *and* $Q \in S^2 T^* \otimes \mathfrak{g}$. *Then*

$$F \overset{K}{\bigwedge} Q \wedge F = 0.$$

Proof. — This is a computation (which might help elucidate the meaning of $\overset{K}{\bigwedge}$). Let $F = F_1 dx^2 \wedge dx^3 + F_2 dx^3 \wedge dx^1 + F_3 dx^1 \wedge dx^2$, and $Q = \Sigma_{i,j} Q_{ij} dx^i \otimes dx^j$, where F_i and Q_{ij} are elements of $\mathfrak{g}$. Then

$$Q \wedge F = \sum_{j=1}^{3} [Q_{ij}, F_j]\, dx^i \otimes dx^1 \wedge dx^2 \wedge dx^3,$$

and so

$$F \overset{K}{\bigwedge} Q \wedge F = \sum_{i,j=1}^{3} K(F_i, [Q_{ij}, F_j])\, dx^1 \wedge dx^2 \wedge dx^3 \otimes dx^1 \wedge dx^2 \wedge dx^3.$$

The coefficient is

$$\begin{aligned} \sum_{i,j} K(F_i, [Q_{ij}, F_j]) &= \sum_{i,j} \operatorname{tr}(F_i Q_{ij} F_j - F_i F_j Q_{ij}) \\ &= \sum_{i,j} \operatorname{tr}(F_i Q_{ij} F_j - F_j Q_{ij} F_i) \end{aligned}$$

which is clearly zero if Q is symmetric in i and j.

The upshot of Lemma 3.4 is that we may replace $\nabla_0 \Gamma$ by $d\Gamma$ in the third term of (3.3), since the symmetric part of the derivative of Γ is killed by the triple scalar product operation. But then we can express $d\Gamma$ in terms of Γ and F by using the definition of curvature (i.e., the first of equations (3.1)). Thus, equation (3.3) becomes

$$(3.5) \qquad 0 = F \overset{K}{\bigwedge} (\nabla_0 dF + \Gamma \wedge \nabla_0 F + (F - \Gamma \wedge \Gamma) \wedge F).$$

This is a scalar-valued identity (actually, it is a $\Lambda^3 T^* \otimes \Lambda^3 T^*$-valued identity) which is *zeroth order* in Γ. It will be necessary to adjoin (3.5) to (3.1) if we hope to obtain an involutive system. The advantage of this is that if (3.5) is satisfied to zeroth order, then the right-hand side of the second prolongation of the Bianchi identity will necessarily be in the image of its symbol.

As a preliminary experiment, we consider the combined system (3.1) and (3.5) for the simplest simple structure group, namely SL(2) (or SU(2), i.e., the Lie algebra should be a real form of type A_1 in Cartan's classification). Since these groups are three-dimensional, we consider a $\mathfrak{g}$-valued two form F to be generic if at each point F is a linear isomorphism from $\Lambda^2 T$ onto $\mathfrak{g}$. In this case, it is easily verified that all the conditions for involutivity are satisfied for the system consisting of (3.1) *and* (3.5), and we conclude.

PROPOSITION 3.6. — *For a generic analytic* SL(2)*-valued two-form* F, *there exists locally a connection* Γ *with* $\mathrm{Curv}(\Gamma) = F$.

The next (real) simple Lie algebra to consider is that of type D_2, namely $SO(3,1) \approx SO(3,C)$. The situation here is similar to that for A_1 algebras, except this time, we find two identities of the form (3.5), namely the real and imaginary parts of the SO(3,C) version of (3.5). By similar computations, a statement analogous to Proposition 3.6 is true for SO(3,1)-bundles.

More interesting is the case of A_2, i.e., Lie algebras of the type of SL(3) or SU(3). In this case, we find again that we cannot prolong a solution of order one to a solution of order two, even if we include (3.5) in our system. The reason for this is that there is yet another identity in force here, which arises from the *second* derivative of the Bianchi identity. Using ∇_0 as our partial derivative symbol again, we write

$$\begin{aligned} 0 &= \nabla_0 \nabla_0 (dF + \Gamma \wedge F) \\ &= \nabla_0 \nabla_0 dF + \Gamma \wedge \nabla_0 \nabla_0 F + 2\nabla_0 \Gamma \wedge \nabla_0 F + \nabla_0 \nabla_0 \Gamma \wedge F. \end{aligned}$$

To get the identity, we need to replace the second derivatives of Γ in the last term with lower derivatives. As before, we can wedge with F, but this time with a pair of F's, and instead of using the Killing form, we use the cubic Ad-invariant polynomial ($\mathrm{tr}\, X^3$) on $\mathfrak{g}$ to get another scalar-valued identity (which involves first derivatives of Γ rather than second ones because we may replace $\nabla_0\nabla_0\Gamma$ by $\nabla_0 d\Gamma$ as before and then use the curvature equation). Adjoining this new identity to (3.1) and (3.5) results in an involutive system. This proves

PROPOSITION 3.7. — *For a generic analytic* $SL(3)$*-valued two-form* F, *there exists locally a connection* Γ *with* $\mathrm{Curv}(\Gamma) = F$.

For more details on the proof of this see [14]. Extrapolating from these examples, it is easy to prove that there is a new independent identity which Γ and F must satisfy for each independent Ad-invariant polynomial on $\mathfrak{g}$. Recall that the number of such polynomials is equal to the rank of $\mathfrak{g}$, and the degrees of the polynomials are known for all of the simple Lie groups (see section 125 of [16], for example). We conjecture that for any simple Lie structure group G, it is sufficient to adjoin the identities so obtained to (3.1) and (3.5) in order to obtain an involutive system, and prove results akin to Propositions 3.6 and 3.7 for each simple group. Then, to treat semisimple groups is no problem because equations (3.1) and (3.5) decompose by direct sum the same way the structure group does, as may easily be verified.

Appendix.

The Cartan-Kähler theorem for systems of nonlinear partial differential equations.

One of the most powerful methods for proving local solvability for real analytic, nonlinear systems of partial differential equations is the so-called Cartan-Kähler theory of involutive differential systems. We shall use the form of the theory developed by Kuranishi [12] and Goldschmidt [9], [10]. See also Chapters 9 and 10 of the forthcoming book [3].

Consider a system of differential equations of order m for the unknown function u that takes its values in a vector space V:

$$\mathcal{F}(x, D^\alpha u_i) = 0 \qquad \text{(A.1)}$$

where $\mathcal{F}$ is an analytic W-valued function for some vector space W, α runs over all multiindices of weight $\leqslant m$ and $i = 1, \ldots, \dim V$. We can consider $\mathscr{F}$ to be a mapping from the jet bundle $J^m(V)$ over $\mathbf{R}^n$ to $J^0(W)$. In this sense, the left side of the equation $\mathrm{Curv}(\Gamma) - F = 0$ is a map from $J^1(T^* \otimes \mathfrak{g})$ to $J^0(\Lambda^2 T^* \otimes \mathfrak{g})$. We are interested in finding a convergent power series solution of (A.1) in a neighborhood of some point (say, the origin) of $\mathbf{R}^n$.

If $l \geqslant 0$, we say that a V-valued function u_0 defined in a neighborhood of $0 \in \mathbf{R}^n$ is an *infinitesimal solution of* (A.1) *of order* l at 0 if

$$D^\beta(\mathscr{F}(x, D^\alpha u_0))|_{x=0} = 0$$

for all multiindices β with $|\beta| \leqslant l$. Clearly, the condition for u_0 to be an infinitesimal solution of order l at 0 only restricts its Taylor coefficients at 0 up to order $m + l$. With this in mind, we define a *formal solution of order* $m + l$ at 0 to be a polynominal of degree $m + l$ which is an infinitesimal solution of order l at 0. We let R_{m+l} denote the set of all formal solutions of (A.1) of order $m + l$, and we establish the convention that if $l < 0$, then R_{m+l} consists of *all* V-valued polynomials of degree $m + l$. Note that over each point, R_{m+l} can be identified with a subset of the fiber of the jet bundle J_{m+l}. The union of all the R_{m+l}'s over all the points of $\mathbf{R}^n$ then fibers in the obvious way over $\mathbf{R}^n$. Certainly the easiest case to study (and the one we are always going to consider) is the case when this union is a submanifold of the jet bundle, and also when each R_{m+l} is a submanifold of the jet bundle fiber of the same dimension over each point. This is the reason for all of the assumptions of « constant rank » in what follows.

We will say that a formal solution p of order $m + k$ is a *prolongation* of a formal solution q of order $m + l$ if $k \geqslant l$ and all the terms of order up to and including $m + l$ of p and q agree. This notion gives rise to natural projections

$$\pi : R_{m+k} \to R_{m+l}$$

for $k \geqslant l$, where $\pi(p)$ is the unique formal solution of order $m + l$ of which p is a prolongation.

The aim of the Cartan-Kähler-Kuranishi-Goldschmidt theory is to provide formal sufficient conditions which guarantee that a polynomial infinitesimal solution can be prolonged to an analytic series solution. One such condition is *formal integrability* : With π defined as above, the surjectivity of the restriction

$$\pi : R_{m+l+1} \to R_{m+l}$$

for all $l \geq 0$ implies that a convergent series solution exists (in other words, if every formal solution of degree $m + l$ can be prolonged to a formal solution of degree $m + l + 1$, then a series solution exists). This condition is also called strong prolongability, see the Appendix of [13] for a proof of convergence. To directly prove formal integrability is generally tedious and difficult (as it is an infinitude of conditions to check).

To give a more computable criterion for solvability, we need the notion of the symbol of $\mathcal{F}$ and of its prolongations. The use of the word « symbol » in this context is slightly different from the general usage in the theory of partial differential equations. Let u_0 be any function. The usual definition of the symbol of $\mathcal{F}$ at u_0 is the map

$$\sigma(\xi, v) = \sum_{|\beta| = m} \frac{\partial \mathcal{F}}{\partial(D^\beta u_i)}(0, D^\alpha u_0)\xi^\beta v^i \in W \tag{A.2}$$

for $\xi \in T_0^*\mathbf{R}^n$ (so that $\xi^\beta = \xi_1^{\beta_1}\xi_2^{\beta_2} \ldots \xi_n^{\beta_n}$) and $v \in V$. Note that σ is linear in V and is a homogeneous polynomial of degree m in ξ. Our definition of the symbol of $\mathcal{F}$ at u_0 is the linear map naturally associated to the one above which is symmetric and multilinear of degree m on $T_0^*\mathbf{R}^n$. In other words, our symbol will be a map

$$\sigma(\mathcal{F})_{u_0} : S^m T^* \otimes V \to W.$$

If u_0 is a formal solution of order m, then the kernel g_{m,u_0} of $\sigma(\mathcal{F})_{u_0}$ parametrizes the tangent space of the set of formal solutions in R_m which agree with u_0 to order $m - 1$ (if in fact this set is a manifold). Sometimes, g_{m,u_0} is itself called the symbol of $\mathcal{F}$ at u_0.

In a similar manner, we define the *first prolongation* of the symbol

$$\sigma_1(\mathcal{F})_{u_0} : S^{m+1}T^* \otimes V \to T^* \otimes W$$

to be the linear map associated to

$$\sigma_1(\xi, v) = \sum_{|\beta| = m} \frac{\partial \mathcal{F}}{\partial(D^\beta u_i)}(0, D^\alpha u_0)v^i \left(\sum_{j=1}^{n} \xi_j \xi^\beta dx^j\right) \in T^* \otimes W.$$

We let $g_{m+1,u_0} = \ker \sigma_1(\mathcal{F})u_0$. Higher-order prolongations of the symbol are defined analogously.

Cartan ([12], see also [4]) advanced the notion of involutivity of differential systems as a sufficient condition for formal integrability. This notion has the advantage (as shown by Kuranishi) of involving only a finite (albeit undetermined) number of conditions to check. Later, it was shown [9] that involutivity is tantamount to the vanishing of all of the Spencer cohomology groups $H^{k-j,j}(g_m)$. Goldschmidt showed that, in fact, the obstructions to prolongation are contained only in certain specific cohomology groups, and proved that « 2-acyclicity » is a sufficient condition for formal integrability.

On the other hand, it is often the case that dealing with the Spencer cohomology groups is not required. In fact, one of the most useful conditions for determining that a system is involutive involves the following notion, given by Serre (cf. [11]):

A basis $\{dx_1, \ldots, dx_n\}$ of T_p^* is called *quasiregular* for g_{m,u_0} if

$$\dim g_{m+1,u_0} = \dim g_{m,u_0} + \sum_{j=1}^{n-1} \dim g_{m,j,u_0}$$

where

$$g_{m,j,u_0} = g_{m,u_0} \cap S^m \Sigma_j$$

and Σ_j is the subspace of T_p^* generated by $dx_{j+1}, \ldots, dx_n$. In other words, quasiregularity involves the behavior of the symbol mapping and its first prolongation when restricted to homogeneous polynomials that do not involve $x_1, \ldots, x_j$.

THEOREM (CARTAN-KÄHLER). — *To prove existence of local analytic solutions of a system* (A.1) *of analytic partial differential equations, it is sufficient to check that the following conditions hold :*

1. *For all x in a neighborhood of 0, there exist formal solutions of order m at x, and every solution of order m at x can be prolonged to a solution of order $m+1$.*

2. *For some specific formal solution p at 0, the derivative of the mapping $\mathscr{F}$ with respect to the variables $(x, D^\alpha u)$ for $0 \leqslant |\alpha| \leqslant k$ has constant rank in a neighborhood of $(0, D^\alpha p)$.*

3. *For all $q \in R_m$ in a neighborhood of p, the rank of the linear mapping $\sigma_1(\mathscr{F})_q$ is independent of q, and there is a quasiregular basis of T_0^* for $g_{m,p}$.*

If the conditions of the theorem are satisfied, then the system of partial differential equations is involutive. Note that the surjectivity of π need only be checked for $l = 0$, and not for all l. See Theorems 8.1 and 9.1 of [7], or [10] for more detail.

For our applications, the issue of constant rank and existence of prolongations in neighborhoods is moot, since the ranks of all of our operators and symbols are the maximum possible.

One easy consequence of this theorem is the existence of local solutions to problems for which there exist *noncharacteristic covectors* with respect to an infinitesimal solution u_0 (i.e., covectors ξ for which the linear map $\sigma(\xi,\cdot): V \to W$ of (A.2) is surjective). This is essentially the Cauchy-Kovalevska theorem, but see [8], Lemma 1 for a proof using the ideas discussed above. In fact, for such a system, it turns out that any basis of T^* with ξ as the *last* covector is quasiregular.

To give an application of the above theory, we briefly discuss the problem of prescribing the Ricci curvature of a pseudo-Riemannian metric (for more detail, see [2] or Chapter 5 of [5]). The expression for the Ricci curvature involves second derivatives of the metric, so the equation

$$\text{(A.3)} \qquad \mathrm{Ric}(g) = \mathrm{R}$$

for the unknown metric g when the tensor R is given, is a second-order system of equations. Since both g and R are symmetric covariant tensors of rank two, the map $\mathscr{F}(g) = \mathrm{Ric}(g) - \mathrm{R}$ maps $J^2(S^2T^*)$ to $J^0(S^2T^*)$. As is well-known, the Ricci tensor and the metric must satisfy the Bianchi identity, which is a first-order expression in both the Ricci tensor and the metric tensor. The Bianchi identity proves to be an obstruction to prolonging a second-order infinitesimal solution of (A.3) to a third-order one. However, if we *adjoin* (the 1-jet of) the Bianchi identity to the system (A.3) to get an overdetermined system, it turns out that, in the case when the prescribed tensor R is invertible (as a map from T_pM to T_p^*M), this enlarged, overdetermined system is in fact involutive and hence equation (A.3) is generically locally solvable (this proof is carried out in somewhat agonizing detail in [5]). On the other hand, as indicated in [2], one can take advantage of the equivariance of equation (A.3) under the action of the diffeomorphism group (i.e., that $\phi^*\mathrm{Ric}(g) = \mathrm{Ric}(\phi^*g)$ for any diffeormorphism ϕ) to prove the same result. The trick is to replace equation (A.3) by the equation

$$\text{(A.4)} \qquad \mathrm{Ric}(g) = \phi^*\mathrm{R},$$

where the unknowns are now g *and* ϕ. In contrast to the previous method, where more equations were added to the system making it overdetermined, we have this time added more unknowns to the system, making it underdetermined. For this system, it turns out that, if R is invertible, then any non-null covector (with respect to the value of g that one chooses at the initial point, recall that we are dealing with pseudo-Riemannian metrics) is noncharacteristic. Thus the Cartan-Kähler theorem applies to prove local existence of g and ϕ satisfying (A.4), and so $\phi^{-1*}g$ satisfies (A.3) (this is explained more fully in Chapter 5 of [2]).

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