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VANISHING THEOREMS ON COHOMOLOGY ASSOCIATED TO HERMITIAN SYMMETRIC SPACES

by Shingo MURAKAMI

This is a survey of the development of research on certain cohomology groups attached to Hermitian symmetric spaces which Matsushima and myself began to study by the papers [11, 12]. As a main purpose, we introduce vanishing theorems obtained recently by Floyd L. Williams.

1. Definitions.

Let X be a Hermitian symmetric space of non-compact type represented as $X = G/K$ where G is a connected semisimple Lie group and K is a maximal compact subgroup of G. Let Γ be a discrete subgroup of G with compact quotient $\Gamma\backslash G$ and which acts freely on X. Put $M = \Gamma\backslash X$. Then M is a compact Kähler manifold admitting X as universal covering manifold.

Now let V be a finite-dimensional complex vector space and let $j : G \times X \to GL(V)$ be an automorphic factor, namely a C^∞-mapping such that

1) $j(st,x) = j(s,tx)j(t,x)$ for $s, t \in G$ and $x \in X$,

2) $j(s,x)$ is holomorphic in $x \in X$ for each $s \in G$.

Such a factor j defines an action of Γ on $X \times V$ by the rule $\gamma(x,v) = (\gamma x, j(\gamma,x)v)((x,v) \in X \times V, \gamma \in \Gamma)$, and the quotient $\Gamma\backslash(X \times V)$ is a holomorphic vector bundle, denoted by $E(j)$, over M with typical fibre V. The cohomology groups we are concerned are those of M with coefficients in the sheaf $\mathbf{E}(j)$ of germs of holomorphic sections of $E(j)$. The q − th cohomology group is denoted by $H^q(M,\mathbf{E}(j))$. We

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have the well-known isomorphism:

$$H^q(M,E(j)) \cong H^{0,q}_{d''}(E(j)),$$

where the right hand side is the q-th cohomology group of the complex $A(E(j)) = \sum_{q\geqslant 0} A^{0,q}(E(j))$ equipped with the differential operator d'', $A^{0,q}(E(j))$ being the space of $E(j)$-valued differential forms of type $(0,q)$. For this reason, we call $H^q(M,E(j))$ the $(0,q)$-cohomology of $E(j)$.

We consider exclusively the cohomology for the case where j is the so-called canonical automorphic factor. Observe that a GL(V)-valued automorphic factor j defines a representation τ of K on V such that $\tau(t) = j(t,x_0)$ $(t \in K)$ where $x_0 = K \in X = G/K$. We know that any representation τ of K may be defined by the canonical automorphic factor of type τ, denoted by J_τ, and if τ is irreducible, any automorphic factor j defining τ is equivalent to J_τ, namely $E(j)$ and $E(J_\tau)$ are isomorphic holomorphic vector bundles over M ([11] and [15, Appendix], [16]).

2. Classical approaches.

In our old works [11, 12], applying the harmonic theory we reduced the study of $(0,q)$-cohomology to the so-called d-cohomology. Let us briefly sketch the mechanism.

A representation ρ of G on a complex vector space F defines a GL(F)-valued automorphic factor j such that $j(s,x) = \rho(s)$ for $(s,x) \in G \times X$. The vector bundle $E(\rho)$ is a locally constant vector bundle. Therefore the space $A(E(\rho)) = \sum_{r\geqslant 0} A^r(E(\rho))$ where $A^r(E(\rho))$ is the space of $E(\rho)$-valued differential forms of degree r is a complex with the differential operator d, from which we obtain the d-cohomology group $H(E(\rho)) = \sum_{r\geqslant 0} H^r(E(\rho))$. Then we have the decomposition:

$$H^r(E(\rho)) = \sum_{p+q=r} H^{p,q}(E(\rho))$$

where $H^{p,q}(E(\rho))$ is the subgroup of $H^r(E(\rho))$ consisting of cohomology classes represented by d-closed forms of type (p,q) $(p+q=r)$. Assuming that the representation τ of K is the irreducible component of the restriction $\rho|K$ of ρ to K having the same highest weight (relative to the order explained later in § 3), we have

$$H^{0,q}_{d''}(E(J_\tau)) \cong H^{0,q}(E(\rho)). \tag{2.1}$$

(A slightly simplified proof of this isomorphism is reported in [18]). The Bochner technique allows us to obtain vanishing theorems for d-cohomology groups and, via (2.1), also for the $(0,q)$-cohomology groups.

It is needless to say that the d-cohomology groups can be defined in a more general setting, i.e., without the assumption that the Riemannian symmetric space G/K is Hermitian. Our study of d- and d''-cohomology groups was motivated by the works of Weil, Calabi-Vesentini, Matsushima ([8, 9]) and others. We intended by [11, 12] to treat their results in a general situation; for example, a vanishing theorem obtained there reduces to the famous vanishing theorem of Calabi-Vesentini in the case τ is the holomorphic isotropic representation of K. We refer to my report [14] for a summary of this classical approach with a review of applications and references, and for unified presentations of it we cite [15] and Koszul's lecture at Urbino [6].

Some years ago, Zucker [25] discusses our subjects in the frame work of locally homogeneous variation of Hodge structure, and in particular he explains in this way some important relations between various laplacians which we have obtained by calculation. Also Faltings [3] treats the subjects and derives, among others, some of main results in [12] (independently of us).

3. Dimension formula.

An important formula giving the dimension of the cohomology groups was discovered by Matsushima [10] for a special but distinguished case and later in the joint work [12] for general case.

The formula is formulated in a general setting. Namely, let Γ be a discrete subgroup of a semisimple Lie group G, K a maximal compact subgroup of G, τ a representation of K on a finite-dimensional complex vector space V, and λ a complex constant. By an automorphic form of type (Γ,τ,λ), we mean a V-valued smooth function f on G such that

1) $f(st) = \tau(t)^{-1}f(s)$ for $s \in G$, $t \in K$,
2) $f(\gamma s) = f(s)$ for $\gamma \in \Gamma$, $s \in G$, and
3) $Cf = \lambda f$,

where C is the Casimir operator, a left invariant differential operator on G defined in terms of the Lie algebra of G. The space $A(\Gamma,\tau,\lambda)$

of all automorphic forms of type (Γ,τ,λ) is of finite dimension provided that $\Gamma\backslash G$ is compact which we now assume. Let U be the right regular representation of G on the Hilbert space $L_2(\Gamma\backslash G)$. We know that U is a unitary representation of G and decomposes into sum of countable number of irreducible representations among which each irreducible representation π enters with a finite multiplicity $m(\pi)$. Now, the formula is :

$$\dim A(\Gamma,\tau,\lambda) = \sum_{\pi} m(\pi)(\pi|K : \tau^*) \tag{3.1}$$

where π runs over the set of irreducible representations of G such that $C_\pi\varphi = \lambda\varphi$ for all φ in the domain of the operator C_π representing C under π; moreover, $(\pi|K : \tau^*)$ denotes the intertwinning number between the restriction $\pi|K$ and the representation τ^* of K contragradient to τ.

The dimension of the $(0,q)$-cohomology group of $E(J_\tau)$ is expressed by means of this formula. To explain this, let $\mathfrak{g}$ be the Lie algebra of G and k the subalgebra corresponding to K. The superscript C designating the complexification, the complex structure of G/K gives rise to a vector space decomposition of $\mathfrak{g}^C$:

$$\mathfrak{g}^C = k^C + \mathfrak{n}^+ + \mathfrak{n}^-,$$

where $\mathfrak{n}^+ (\mathfrak{n}^-)$ consists of those elements of $\mathfrak{n}^C$ which project to complex tangent vectors of type (1,0) (resp. (0,1)) at the point $x_0 \in X$. The adjoint action of K on $\mathfrak{g}^C$ leaves stable the subspace $\mathfrak{n}^+$ and so we get the representation ad_+ of K on the complex vector space $\mathfrak{n}^+$. We denote by ad_+^q the representation of K on the exterior product space $\wedge^q\mathfrak{n}^+$ induced from ad_+. On the other hand, since G/K is a Hermitian symmetric space, k contains a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let Δ be the rootsystem of $\mathfrak{g}^C$ with respect to the Cartan subalgebra $\mathfrak{h}^C$, and let

$$\mathfrak{g}^C = \mathfrak{h}^C + \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$$

be the decomposition of $\mathfrak{g}^C$ with $\mathfrak{g}_\alpha$ the eigenspace of the root $\alpha\in\Delta$. Then k^C and $\mathfrak{n}^+$, $\mathfrak{n}^-$ are expressed as

$$k^C = \mathfrak{h}^C + \sum_{\alpha\in\Delta_k} \mathfrak{g}_\alpha; \qquad \mathfrak{n}^+ = \sum_{\alpha\in\Delta_n^+} \mathfrak{g}_\alpha, \qquad \mathfrak{n}^- = \sum_{\alpha\in\Delta_n^+} \mathfrak{g}_{-\alpha}$$

for certain subsets Δ_k and Δ_n^+. We can introduce a linear ordering in the set of weights on $\mathfrak{h}^C$ with the following property. Any root $\alpha\in\Delta_n^+$ is totally positive, i.e., $\alpha+\beta\in\Delta_n^+$ for every $\beta\in\Delta_k$ such that $\alpha+\beta\in\Delta$. We denote by Δ^+ the set of all positive roots, and put

$\Delta_k^+ = \Delta^+ \cap \Delta_k$. This notation being settled, let τ be an irreducible representation of K. Then the space of harmonic forms belonging to the space $A^{0,q}(E(J_\tau))$ can be identified with the space of automorphic forms $A(\Gamma, \tau \otimes \mathrm{ad}_+^q, \lambda)$ where $\lambda = \langle \Lambda, \Lambda + 2\delta \rangle$. Here Λ is the highest weight of τ relative to the positive root system Δ_k^+ of k^C and $\langle\, ,\, \rangle$ is the inner product defined by the Killing form of $\mathfrak{g}^C$. By the dimension formula (3.1), we get then

$$\dim H^q(M, E(J_\tau)) = \sum_\pi m(\pi)(\pi \mid K : \tau \otimes \mathrm{ad}_+^q) \tag{3.2}$$

where π runs over the irreducible components of the right regular representation of G on $L_2(\Gamma\backslash G)$ for which $C_\pi = \langle \Lambda, \Lambda + 2\delta \rangle$ (cf. [5]).

4. Vanishing theorems.

The theory of unitary representations of semisimple Lie groups has greatly developed in 1960's, so that one can apply the results obtained in this field to effective use of the formula (3.1). Hotta-Wallach [5] used for the first time the Matsushima's formula to obtain a vanishing theorem for $(0,q)$ Betti numbers. In this direction, further profound studies have been made at Princeton in the year 1976/1977 ([2]).

Now we discuss the vanishing of $(0,q)$-cohomology of $E(J_\tau)$ assuming always that τ is an irreducible representation of K. We retain the notation introduced in § 3.

THEOREM 1 ([4]). – *Suppose that the highest weight Λ of τ relative to Δ_k^+ is a dominant integral form of $\mathfrak{g}^C$ relative to Δ^+. Then*

$$H^q(M, E(J_\tau)) = (0)$$

for q satisfying one of the following conditions :

(I) $q < |\{\alpha \in \Delta_n^+ \langle \Lambda, \alpha \rangle > 0\}|$, *where* $|\ |$ *denotes the cardinality.*

(II) $q < r$, *if* X *is an irreducible Hermiltian symmetric space of rank* r *and unless* $q = 0$ *nor* $\Lambda = 0$.

The proof of this theorem is a generalization of that of Hotta-Wallach's and is based on a technique found originally by Partasarathy [19]. The vanishing for the case (I) has been proved in [11] and [12] by more elementary methods.

Recently, Williams has given sharper vanishing theorems of $(0,q)$-cohomology. Here is applied (3.2) together with Parthasarathy's criteria for the unitarizability of some highest weight modules [20]. Parthasarathy himself applies the criteria to show vanishing theorems for $(0,q)$ Betti numbers, and the following theorem is a direct generalization of his theorem.

THEOREM 2. – *Under the same assumption as in Theorem 1, if* $H^q(M,E(J_\tau)) \neq (0)$, *then there exists a parabolic subalgebra* $\mathfrak{q}$ *of* $\mathfrak{g}^C$ *containing the Borel subalgebra* $\mathfrak{h}^C + \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ *such that if* $\mathfrak{q} = \mathfrak{m} + \mathfrak{n}$ *is the decomposition into sum of reductive part* $\mathfrak{m}$ *and nilradical* $\mathfrak{n}$ *we have*

(i) $q = |\{\alpha \in \Delta_n^+ \,;\, \mathfrak{g}_\alpha \subset \mathfrak{n}\}|$.

(ii) $\langle \Lambda,\alpha \rangle = 0$ *for every root* α *such that* $\mathfrak{g}_\alpha \subset \mathfrak{m}$.

In particular the vanishing for the case (I) of Theorem 1 follows.

Under the assumption that the Hermitian symmetric space $X = G/K$ is irreducible, or equivalently that the group G is simple, Parthasarathy shows that there exists no parabolic subalgebra $\mathfrak{q}$ containing the Borel subalgebra $\mathfrak{h}^C + \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ for which the number q in (i) is less than the rank of X. Therefore the vanishing for the case (II) of Theorem 1 is contained in the assertion of Theorem 2. In more detail, we can calculate the numbers in (i) of Theorem 2 for all parabolic subalgebras to each type of irreducible Hermitian space X. The result is listed on the following table.

TABLE

Type	G	Set of numbers q
$I_{n,m}$	$SU(n,m)$, $n \geqslant m$	$\{nm - n'm' \mid 0 \leqslant n' \leqslant n,\ 0 \leqslant m' \leqslant m\}$
II_n	$SO^*(2n)$, $n > 3$	$\left\{\frac{n(n-1)}{2} - \frac{j(j-1)}{2} \,\middle|\, j = 3,\ldots,n\right\} \cup \left\{\frac{n(n-1)}{2} - i \,\middle|\, i = 0,1,\ldots,n-1\right\}$
III_n	$Sp(n,R)$	$\{0\} \cup \{n + (n-1) + \cdots + (n-j) \mid j = 0,1,\ldots,n-1\}$
IV_n	$SO_0(n,2)$, $n > 2$	$\{0\} \cup \left\{\left[\frac{n+1}{2}\right], \ldots, n\right\}$
V_6	real form of E_6	$\{0, 8, 11, 12, 13, 14, 15, 16\}$
VI_7	real from of E_7	$\{0, 17, 21, 22, 23, 24, 25, 26, 27\}$

Theorem 1 was given by Williams as a corollary of the following general theorem. To state this, we need some notation. Denote by $\mathscr{F}'_0$ the set of all integral linear forms on $\mathfrak{h}^C$ such that

$$\langle \Lambda+\delta,\alpha\rangle \neq 0 \quad \text{for all} \quad \alpha \in \Delta,$$

and

$$\langle \Lambda+\delta,\alpha\rangle > 0 \quad \text{for all} \quad \alpha \in \Delta_k^+ .$$

For a form $\Lambda \in \mathscr{F}'_0$ the set

$$P^{(\Lambda)} = \{\alpha \in \Delta; \langle \Lambda+\delta,\alpha\rangle > 0\}$$

is a system of positive roots. Denoting Δ_n the set of $\pm\, \alpha(\alpha \in \Delta_n^+)$, put

$$P_k^{(\Lambda)} = P^{(\Lambda)} \cap \Delta_k, \qquad P_n^{(\Lambda)} = P^{(\Lambda)} \cap \Delta_n$$

and let $2\delta^{(\Lambda)}$ $(2\delta_k^{(\Lambda)}, 2\delta_n^{(\Lambda)})$ be the sum of roots belonging to $P^{(\Lambda)}$ (resp. $P_k^{(\Lambda)}$, $P_n^{(\Lambda)}$). Set

$$Q_\Lambda = \{\alpha \in \Delta_n^+; \langle \Lambda+\delta,\alpha\rangle > 0\}$$

and

$$Q'_\Lambda = \Delta_n^+ - Q.$$

THEOREM 3 [22]. – *The notation being as above. Suppose that the highest weight Λ of τ relative to Δ_k^+ belongs to $\mathscr{F}'_0$. Then, if $H^q(M,E(J_\tau)) \neq (0)$, there exists a parabolic subalgebra $\mathfrak{q} = \mathfrak{m} + \mathfrak{n}$, with reductive part $\mathfrak{m}$ and nilradical $\mathfrak{n}$, containing the Borel subalgebra $\mathfrak{h}^C + \sum_{\alpha \in P^{(\Lambda)}} \mathfrak{g}_\alpha$ such that if $\theta_{u,n}$ is the set of roots $\alpha \in \Delta_n$ for which $\mathfrak{g}_\alpha \subset \mathfrak{n}$*

(i) $q = 2|Q_\Lambda \cap \theta_{u,n}| + |Q'_\Lambda| - |\theta_{u,n}|$, *and*

(ii) $\langle \Lambda+\delta-\delta^{(\Lambda)},\alpha\rangle = 0$ *for every root α such that $\mathfrak{g}_\alpha \subset \mathfrak{m}$.*

Moreover, we can choose $\mathfrak{q}$ to be stable under the Cartan involution of $\mathfrak{g}$ in the sense of Vogan-Zuckerman [21].

Williams proves this theorem first in [22] under the assumption that every root in $P_n^{(\Lambda)}$ is totally positive. Recently, he improves it in this form depending on the results of Kumarsan [7] and [21]. (The proof is not yet published.)

Before concluding, we mention about the *non-vanishing* theorems of $(0,q)$-cohomology. Following up some preceding works of Kazdhan, Shimura and others, Anderson [1] has constructed cocompact discrete subgroups Γ for which $(0,q)$ Betti number of $M = \Gamma\backslash X$ is non-zero for each of irreducible Hermitian symmetric spaces of type $I_{n,m}$, II_n, III_n and for each possible q in the Table (except for some values of q for the case II_n). Applying the isomorphism (2.1), we see also that the example constructed by Borel-Wallach [2, Chap. VIII, 5.10] is an example of Γ with non-vanishing $(0,q)$-cohomology for the general case (as pointed out to me by Y. Konno).

BIBLIOGRAPHY

[1] G. W. Anderson, Theta functions and holomorphic differential forms on compact quotients of bounded symmetric domains, *Duke Math. J.*, 50 (1983), 1137-1170.

[2] A. Borel and N. Wallach, Continuous cohomology, discrete subgroups, and representations of reductive groups, Princeton, Princeton University Press, 1980. (*Annals of mathematical studies*, 94.)

[3] G. Faltings, On the cohomology of locally symmetric Hermitian symmetric spaces, Paul Dubriel and Marie-Paul Malliavin algebra seminar, 25th year (Paris, 1982), 55-98, *Lecture Notes in Math.*, 1029 (1983).

[4] R. Hotta and S. Murakami, On a vanishing theorem for certain cohomology groups, *Osaka J. Math.*, 12 (1975), 555-564.

[5] R. Hotta and N. Wallach, On Matsushima's formula for the Betti numbers of a locally symmetric space, *Osaka J. Math.*, 12 (1975), 419-431.

[6] J. L. Koszul, Formes harmoniques vectorielles sur espaces localement symétriques, Geometry of Homogeneous Bounded Domains, *C.I.M.E. 3 Ciclo*, 1967, 197-260.

[7] S. Kumaresan, On the canonical k-types in the irreducible unitary g-modules with non-zero relative cohomology, *Inventiones Math.*, 59 (1980), 1-11.

[8] Y. Matsushima, On the first Betti number of compact quotient spaces of higher dimensional symmetric spaces, *Ann. of Math.*, 75 (1962), 312-330.

[9] Y. Matsushima, On Betti numbers of compact, locally symmetric Riemannian manifolds, *Osaka J. Math.*, 14 (1982), 312-330.

[10] Y. Matsushima, A formula for the Betti numbers of locally symmetric Riemannian manifolds, *J. Differential Geometry*, 1 (1987), 99-109.

[11] Y. Matsushima and S. Murakami, On vector bundle valued harmonic forms and automorphic forms on symmetric Riemannian manifolds, *Ann. of Math.*, 78 (1963), 365-416.

[12] Y. Matsushima and S. Murakami, On certain cohomology groups attached to hermitian symmetric spaces, *Osaka J. Math.*, 2 (1965), 1-35.

[13] Y. Matsushima and S. Murakami, On certain cohomology groups attached to Hermitian symmetric spaces (II), *Osaka J. Math.*, 5 (1968), 223-241.

[14] S. Murakami, Cohomology of vector-valued forms on compact, locally symmetric Riemannian manifolds, *Proceeding Symposia in Pure Mathematics*, vol. 9, Algebraic groups and discontinuous subgroups, 1966, 387-399.

[15] S. Murakami, Cohomology groups of vector-valued forms on symmetric spaces, *Lecture Notes*, University of Chicago, 1966.
[16] S. Murakami, Facteur d'automorphie associé à un espace hermitien symétrique, Geometry of Homogeneous Bounded Domains, *C.I.M.E. 3 Ciclo*, (1967), 281-287.
[17] S. Murakami, Certain cohomology groups attached to Hermitian symmetric spaces and unitary representations, *Southeast Asian Bull. Math.*, 5 (1981), 39-44.
[18] S. Murakami, Laplacians and cohomologies associated to locally symmetric Hermitian manifolds, Spectra of Riemannian Manifolds (Proceedings of the France-Japan Seminar, Kyoto, 1981), *Kaigai Publ. Tokyo*, (1983), 73-78.
[19] R. Parthasarathy, A note of the vanishing of L^2-cohomologies, *J. Math. Soc. Japan*, 22 (1971), 1-30.
[20] R. Parthasarathy, Criteria for the unitarizability of some highest weight modules, *Proc. Indian Acad. Sci.*, 89 (1980), 1-24.
[21] D. A. Vogan and G. J. Zuckerman, Unitary representations with non-zero cohomology, *Compositio Mathematica*, 53 (1984), 51-90.
[22] F. L. Williams, Vanishing theorems for type $(0,q)$-cohomology of locally symmetric spaces, *Osaka J. Math.*, 18 (1981), 147-160.
[23] F. L. Williams, Remarks on the unitary representations appearing in the Matsushima-Murakami formula, Proceeding of the Conference on Non-commutative Harmonic Analysis, Marseille-Luminy, France, *Lecture Notes in Math.*, 880 (1981), 536-553.
[24] F. L. Williams, Vanishing theorems for type $(0,q)$-cohomology of locally symmetric spaces II, *Osaka J. Math.*, 20 (1983), 95-108.
[25] S. Zucker, Locally homogeneous variations of Hodge structure, *L'Enseignement Mathématiques*, II^e série, 27 (1981), 243-276.

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