ANNALES DE L’INSTITUT FOURIER

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Annales de l’institut Fourier, tome 37, nº 4 (1987), p. 161-166
<http://www.numdam.org/item?id=AIF_1987__37_4_161_0>

Ann. Int. Fourier, Grenoble
37, 4 (1987), 161-166.

GRADED MORPHISMS OF G-MODULES

by H. KRAFT and C. PROCESI

1. Introduction.

During the 1987 meeting in honor of J. K. Koszul, Steve Halperin explained to us the following conjecture (motivated by the study of the spectral sequence associated to a homogeneous space).

1.1. CONJECTURE. — *If $f_1, f_2, \ldots, f_n$ is a regular sequence in the polynomial ring $\mathbf{C}[x_1, x_2, \ldots, x_n]$, the connected component of the automorphism group of the (finite dimensional) algebra $\mathbf{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ is solvable.*

In this paper we prove a weak form of this (Corollary 4.3) which implies the conjecture at least when the f_i's are homogeneous (Remark 4.4).

2. Preliminaries.

Our base field is $\mathbf{C}$, the field of complex numbers, or any other algebraically closed field of characteristic zero.

2.1. DEFINITION. — *A morphism $\varphi : V \to W$ between finite dimensional vector spaces V and W is called* graded *if there is a basis of W such that the components of φ are all homogeneous polynomials.*

Let us denote by $\mathcal{O}(V)$, $\mathcal{O}(W)$ the ring of regular functions on V and W. These $\mathbf{C}$-algebras are naturally graded by degree: $\mathcal{O}(V) = \bigoplus_i \mathcal{O}(V)_i$. A subspace $S \subset \mathcal{O}(V)$ is called *graded* if $S = \bigoplus_i S \cap \mathcal{O}(V)_i$.

Key-words : Automorphism group of an algebra - G-module - Equivariant graded morphism - Regular sequence.

If $\varphi : V \to W$ is a morphism and $\varphi^* : \mathcal{O}(W) \to \mathcal{O}(V)$ the corresponding comorphism we have the following equivalence:

$$\varphi \text{ is graded} \Leftrightarrow \varphi^*(W^*) \text{ is a graded subspace of } \mathcal{O}(V).$$

2.2. LEMMA. — *For any graded morphism* $\varphi : V \to W$ *there is a unique decomposition* $W = \bigoplus_{\nu \geqslant 0} W_\nu$ and homogeneous morphisms $\varphi_\nu : V \to W_\nu$ *of degree* ν *such that*

$$\varphi = (\varphi_0, \varphi_1, \varphi_2, \dots) : V \to W_0 \oplus W_1 \oplus W_2 \oplus \cdots .$$

(This is clear from the definitions.)

2.3. *Remark.* — Let G be an algebraic group. Assume that V and W are G-modules and that $\varphi : V \to W$ is graded and G-equivariant. Then in the notations of lemma 2.2 all W_ν are submodules and all components φ_ν are G-equivariant.

2.4. *Remark.* — If $\varphi : V \to W$ is graded and dominant with $\varphi^{-1}(0) = \{0\}$, then φ is a finite surjective morphism. In fact given a finitely generated graded algebra $A = \bigoplus_{i \geqslant 0} A_i$ with $A_0 = \mathbf{C}$ and a graded subspace $S \subset A$ such that the radical $\operatorname{rad}(S)$ of the ideal generated by S is the homogeneous maximal ideal $\bigoplus_{i > 0} A_i$ of A, then A is a finitely generated module over the subalgebra $\mathbf{C}[S]$ generated by S (see [1, II.4.3 Satz 8]).

3. The Main Theorem.

3.1. THEOREM. — *Let G be a connected reductive algebraic group and let* V, W *be two G-modules. Assume that* V *and* W *do not contain 1-dimensional submodules. Then any graded G-equivariant dominant morphism with finite fibres is a linear isomorphism.*

We first prove this for $G = SL_2$ and then reduce to this situation.

For any $\mathbf{C}^*$-module V we have the weight decomposition

$$V = \bigoplus_j V_j, \qquad V_j := \{v \in V \mid t(v) = t^j \cdot v\}.$$

We say that V has *only positive weights* if $V = \bigoplus_{j > 0} V_j$.

3.2. Lemma. — *Let* V, W *be two* **C***-*modules with only positive weights, and let* $\varphi : V \to W$ *be a* **C***-*equivariant graded morphism with finite fibres. For all* $k \geqslant 0$ *we have*

$$\varphi^{-1}\left(\bigoplus_{j\leqslant k} W_j\right) \subseteq \bigoplus_{j\leqslant k} V_j,$$

and the inclusion is strict for at least one k *in case* φ *is not linear.*

Proof. — By lemma 2.2 and remark 2.3 we have $\varphi = \sum_{\nu\geqslant 1} \varphi_\nu$ where $\varphi_\nu : V \to W_\nu$ is homogeneous of degree ν and **C***-equivariant. Let $v = \sum_{j=1}^{k} v_j \in \bigoplus_{j>0} V_j = V$ with $v_k \neq 0$. Then

$$\lim_{\lambda\to 0} \lambda^k \cdot t_\lambda^{-1}(v) = v_k .$$

(Here t_λ denotes the action of **C***.) Since φ_ν is homogeneous of degree ν and **C***-equivariant we obtain

$$\lim_{\lambda\to 0} \lambda^{\nu k} \cdot t_\lambda^{-1}(\varphi_\nu(v)) = \varphi_\nu(v_k) . \tag{1}$$

This implies that $\varphi_\nu(v) \in \bigoplus_{j\leqslant \nu k} W_i$ for all ν, proving the first claim.

If φ is not linear, i.e. $\varphi \neq \varphi_1$, then there is a $\nu > 1$, an index k and an element $v \in V_k$ such that $\varphi_\nu(v) \neq 0$. But $\varphi_\nu(v) \in W_{\nu k}$ by (1) and so $v \notin \varphi^{-1}\left(\sum_{j\leqslant k} W_j\right)$. □

3.3. Corollary. — *Under the assumptions of lemma* 3.2 *suppose that* φ *is surjective. Put* $\lambda_j := \dim V_j$ *and* $\mu_j := \dim W_j$. *Then for all* $k \geqslant 1$ *we have*

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k \geqslant \mu_1 + \mu_2 + \cdots \mu_k . \tag{2}$$

If φ *is not linear the inequality is strict for at least one* k.

(This is clear.)

3.4. Proposition. — *Let* V, W *be two* SL_2-*modules containing no fixed lines. Let* $\varphi : V \to W$ *be a graded* SL_2-*equivariant morphism, which is dominant and has finite fibres. Then* φ *is a linear isomorphism.*

Proof. — Consider the maximal unipotent subgroup

$$U := \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \subset SL_2$$

and the maximal torus

$$T := \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in \mathbf{C}^* \right\} \simeq \mathbf{C}^* .$$

By assumption φ is finite and surjective (Remark 2.4), and $\varphi^{-1}(W^U) = V^U$. Hence the induced morphism

$$\varphi|_{V^U} : V^U \to W^U$$

is graded, T-equivariant, finite and surjective too. Furthermore all weights λ_j of V^U and μ_j of W^U are positive. It follows from (2) that

$$\lambda_k + \lambda_{k+1} + \cdots \leqslant \mu_k + \mu_{k+1} + \cdots$$

for all k, because $\sum_j \lambda_j = \dim V^U = \dim W^U = \sum_j \mu_j$. From this we get

$$\begin{aligned} \dim V &= 2\lambda_1 + 3\lambda_2 + \cdots + (n+1)\lambda_n \\ &\leqslant 2\mu_1 + 3\mu_2 + \cdots + (n+1)\mu_n = \dim W \end{aligned}$$

for all n which are big enough. (Remember that an irreducible SL_2-module of highest weight j is of dimension $j+1$). If φ is not linear this inequality is strict (Corollary 3.3), contradicting the fact that φ is finite and surjective. □

3.5. *Proof of the Theorem.* — Assume that $\varphi : V \to W$ is not linear, i.e. there is a $\nu_0 > 1$ such that the component $\varphi_{\nu_0} : V \to W_{\nu_0}$ is non-zero. Then there is a homomorphism $SL_2 \to G$ and a non-trivial irreducible SL_2-submodule $M \subseteq V$ such that $\varphi_j|_M \neq 0$. (In fact the intersection of the fixed point sets $V^{\iota(SL_2)}$ for all homomorphisms $\iota : SL_2 \to G$ is zero.) Now consider the G-stable decompositions $V = V^{SL_2} \oplus V'$ and $W = W^{SL_2} \oplus W'$ and the following morphism :

$$\varphi' : V' \hookrightarrow V \xrightarrow{\varphi} W \xrightarrow{pr} W' .$$

Since V' and W' are sums of isotypic components the morphism φ' is again graded. Furthermore $\varphi^{-1}(W^{SL_2}) = V^{SL_2}$, hence $\varphi^{-1}(0) = V^{SL_2} \cap V' = \{0\}$. This implies that $\varphi' : V' \to W'$ is dominant with finite fibres and satisfies therefore the assumptions of proposition 3.4. As a consequence φ' is linear. Since $\varphi|_{V'} : V' \to W$ is graded too we have $\varphi_\nu|_{V'} = 0$ for all $\nu > 1$. This contradicts the facts that $M \subseteq V'$ and $\varphi_{\nu_0}|_M \neq 0$ (see the construction above).

4. Some Consequences.

We add some corollaries of the theorem. Let G be a connected reductive group. For every G-module V we have the canonical G-stable decomposition $V = V^\circ \oplus V'$ where V° is the sum of all 1-dimensional representations (i.e. $V^\circ = V^{(G,G)}$) and V' the sum of all others. The proof of the theorem above easily generalizes to obtain the following result :

4.1. THEOREM. — *Let* $\varphi : V \to W$ *be a graded G-equivariant dominant morphism with finite fibres. Then* φ *induces a linear isomorphism*

$$\varphi|_{V'} : V' \xrightarrow{\sim} W' .$$

4.2. COROLLARY. — *Let* $\mathcal{O}(V)$ *be the ring of regular functions on a G-module* V, *and let* $f_1, \ldots, f_n$ *be a regular sequence of homogenous elements of* $\mathcal{O}(V)$ *such that the linear span* $\langle f_1, \ldots, f_n \rangle$ *is G-stable. Then* $\langle f_1, \ldots, f_n \rangle$ *contains all non-trivial representations of* (G,G) *in* $\mathcal{O}(V)_1$, *the linear part of* $\mathcal{O}(V)$.

Proof. — The regular sequence $f_1, \ldots, f_n$ defines a G-equivariant finite morphism $\varphi : V \to W$, $W := \langle f_1, \ldots, f_n \rangle^*$. By the theorem above the restriction $\varphi'|_{V'} : V' \to W'$ is a linear isomorphism which means that every non-trivial (G,G)-submodule of $\langle f_1, \ldots, f_n \rangle$ is contained in the linear part $\mathcal{O}(V_1)$ of $\mathcal{O}(V)$. □

4.3. Recall that a finite dimensional **C**-algebra is called a *complete intersection* if it is of the form $\mathbf{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ with a regular sequence $f_1, \ldots, f_n$.

COROLLARY. — *Let* A *be a finite dimensional local* **C**-*algebra with maximal ideal* **m** *and let* $\mathrm{gr}_{\mathbf{m}}A$ *be the associated graded algebra (with respect to the* **m**-*adic filtration). If* $\mathrm{gr}_{\mathbf{m}}A$ *is a complete intersection then the connected component of the automorphism group of* A *is solvable.*

Proof. – Let G and $\bar{G}$ be the connected components of the automorphism groups of A and of $gr_{\mathbf{m}}A$ respectively. Since the **m**-adic filtration of A is G-stable we have a canonical homomorphism $\rho : G \to \bar{G}$. It is easy to see that ker ρ is unipotent, so it remains to show that $\bar{G}$ is solvable.

Assume that $\bar{G}$ is not solvable. Then $\bar{G}$ contains a (non-trivial) semisimple subgroup H. By assumption we have an isomorphism

$$gr_{\mathbf{m}}A \simeq \mathbf{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

with a regular sequence $f_1, \ldots, f_n$ where all f_i are homogeneous of degree $\geqslant 2$. Clearly the action of $\bar{G}$ on $gr_{\mathbf{m}}A$ is induced from a (faithful) linear representation on $\mathbf{C}[x_1, \ldots, x_n]_1 \subset \mathbf{C}[x_1, \ldots, x_n]$. Hence it follows from corollary 4.2 that $\langle f_1, \ldots, f_n \rangle$ contains all non-trivial H-submodules of $\mathbf{C}[x_1, \ldots, x_n]_1$, contradicting the fact that all f_i have degree $\geqslant 2$. □

4.4. *Remark.* – The corollary above implies that conjecture 1.1 is true in case all f_i are homogeneous, i.e. if the algebra

$$A = \mathbf{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

is finite dimensional and graded with all x_i of degree 1.

4.5. *Remark.* – Another formulation of our result is the following: Let V be a representation of a connected algebraic group G and $Z \subset V$ a G-stable graded subscheme, which is a complete intersection supported in $\{0\}$. Then (G,G) acts trivially on Z.

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