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Monodromy representations of braid groups and Yang-Baxter equations


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INTRODUCTION

The purpose of this paper is to give a description of the monodromy of integrable connections over the configuration space arising from classical Yang-Baxter equations. These monodromy representations define a series of linear representations of the braid groups $\theta : B_n \rightarrow \text{End} (W^\otimes n)$ with one parameter, associated to any finite dimensional complex simple Lie algebra $g$ and its finite dimensional irreducible representations $\rho : g \rightarrow \text{End} (W)$. By means of trigonometric solutions of the quantum Yang-Baxter equations due to Jimbo ([10] and [11]), we give an explicit form of these representations in the case of a non-exceptional simple Lie algebra and its vector representation (Theorem 1.2.8) and in the case of $\text{sl}(2,\mathbb{C})$ and its arbitrary finite dimensional irreducible representations (Theorem 2.2.4).

Our monodromy representation $\theta$ commutes with the diagonal action of the $q$-analogue of the universal enveloping algebra of $g$ in the sense of Jimbo [9], which was discussed as quantum groups by Drinfel'd [7]. In particular, in the case $g = \text{sl}(m,\mathbb{C})$, the representation $\theta$ gives Hecke algebra representations of $B_n$ appearing in a recent work of Jones [14].

The study of these monodromy representations is motivated by a recent development of two dimensional conformal field theory initiated by Belavin, Polyakov and Zamolodchikov [5]. The importance of the two dimensional conformal field theory with gauge symmetry was

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pointed out by Knizhnik and Zamolodchikov [18]. They showed that the total differential equations defined by our connections are satisfied by $n$-point functions in these cases.

Recently Tsuchiya and Kanie [22] developed an operator formalism of two dimensional conformal field theory on $\mathbb{P}^1$ using the Kac-Moody Lie algebra of type $A_1^{(1)}$. It turns out that in the case of the vector representation of $\mathfrak{sl}(2,\mathbb{C})$, the monodromy of $n$-point functions gives a linear representation of the braid group $B_n$ factoring through the Jones algebra of index $4\cos^2\frac{\pi}{\ell + 2}$ for a positive integer $\ell$ (see [13]). In particular this representation is unitarizable. We shall extend this unitarity result to higher representations of $\mathfrak{sl}(2,\mathbb{C})$. A neat description of the monodromy of $n$-point functions in the case of simple Lie algebras of other types might be pursued from a viewpoint of Brauer's centralizer algebras, which will be discussed in the forthcoming paper.

This paper is organized in the following way. In Sect. 1.1, we explain a process to define an integrable connection associated with a simple Lie algebra and its irreducible representation. We give an explicit description of the monodromy in Sect. 1.2 and 1.3. Sect. 2.1 is devoted to a review of two dimensional conformal field theory due to Tsuchiya and Kanie [22]. We discuss the case of higher representations of $\mathfrak{sl}(2,\mathbb{C})$ in Sect. 2.2 and 2.3.

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The following notations are of frequent use:

$B_n$: braid group on $n$ strings with generators $\sigma_i$, $1 \leq i \leq n - 1$, represented by a braid interchanging strings $i$ and $i + 1$ (see [2]).

![Diagram](https://via.placeholder.com/150)

Fig. 1.
BRAID GROUPS

$P_n$ : pure braid group on $n$ strings.

$X_n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n; \ z_\alpha \neq z_\beta \ \text{if} \ \alpha \neq \beta\}$

$\mathfrak{g}$ : a simple finite dimensional complex Lie algebra.

$\{I_\mu\}$ : orthonormal basis of $\mathfrak{g}$ with respect to the Cartan-Killing form.

$t = \Sigma_\mu I_\mu \otimes I_\mu \in \mathfrak{g} \otimes \mathfrak{g}$.

For a finite dimensional vector space $V$, we let $\sigma \in \text{End} (V \otimes V)$ the transposition defined by $\sigma(x \otimes y) = y \otimes x$. For $X \in \text{End} (V \otimes V)$ we put $\tilde{X} = \sigma X$.

$\mathcal{C} \{\lambda\}$ : ring of the convergent power series.

1. MONODROMY OF INTEGRABLE CONNECTIONS ARISING FROM CLASSICAL YANG-BAXTER EQUATIONS

1.1. Construction of connections.

Let $\mathfrak{g}$ be a simple finite dimensional complex Lie algebra and let $\{I_\mu\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to the Cartan-Killing form. We put

$$t = \Sigma_\mu I_\mu \otimes I_\mu$$

which may also be expressed as

$$t = \frac{1}{2} (\Delta \Omega - \Omega \otimes 1 - 1 \otimes \Omega).$$

Here $\Omega$ is the Casimir operator $\Sigma_\mu I_\mu \otimes I_\mu$ in the universal enveloping algebra $U(\mathfrak{g})$ and $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ stands for the comultiplication as a Hopf algebra.

Associated with a simple Lie algebra $\mathfrak{g}$ and its finite dimensional irreducible representations $\rho_\alpha : \mathfrak{g} \to \text{End} (W_\alpha)$, $1 \leq \alpha \leq n$, we consider the total differential equations with a parameter $\lambda$

$$(1.1.1) \quad \partial \Phi = \Sigma_{1 \leq \alpha < \beta \leq n} \lambda \Omega_{\alpha \beta} \partial \log (z_\alpha - z_\beta). \Phi, \quad \lambda \in \mathbb{C}$$

defined over

$X_n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n; \ z_\alpha \neq z_\beta \ \text{if} \ \alpha \neq \beta\}.$
Here $\Omega_{z^\beta} \in \text{End} \ (W_1 \otimes \ldots \otimes W_n)$ are defined by

$$\Omega_{z^\beta} = \Sigma_{\mu} \rho_\mu (I_\mu) \otimes \rho_\beta (I_\mu)$$

where $\rho_\mu$ stands for the representation $\rho_\mu$ on the $\mu$-th factor acting as the identity on the other factors.

The matrix valued 1-form

$$(1.1.3) \quad \omega = \Sigma_{1 \leq z < \beta < \gamma} \lambda \Omega_{z^\beta} \log (z_\gamma - z_\beta), \quad \lambda \in \mathbb{C}$$

is considered to be a connection of the trivial vector bundle over $X_n$ with fiber $W_1 \otimes \ldots \otimes W_n$. The integrability condition for $\omega$

$$d\omega + \omega \wedge \omega = 0$$

is satisfied in our case since we have the following relations among $\Omega_{z^\beta}$:

$$(1.1.4) \quad [\Omega_{z^\beta}, \Omega_{z^\gamma} + \Omega_{z^\delta}] = [\Omega_{z^\beta} + \Omega_{z^\gamma}, \Omega_{z^\delta}] = 0 \quad \text{for} \quad \alpha < \beta < \gamma$$

$$[\Omega_{z^\mu}, \Omega_{z^\nu}] = 0 \quad \text{for distinct} \quad \alpha, \beta, \gamma, \delta.$$  

In fact the above relations are derived from the fact that the Casimir operator $Q$ lies in the center of $U(g)$. We shall call (1.1.4) the \textit{infinitesimal pure braid relations}. These relations are relevant to the classical Yang-Baxter equation in the following sense.

Let us recall that the classical Yang-Baxter equation is a functional equation for a $g \otimes g$-valued meromorphic function $r(u), \ u \in \mathbb{C}$, given by

$$(1.1.5) \quad [r_{12}(u-v), \ r_{13}(u)] + [r_{12}(u-v), r_{23}(v)] + [r_{13}(u), r_{23}(v)] = 0.$$  

Here the above triangular equality is considered in $g \otimes g \otimes g$ and $r_{ij}$ signifies the $r$ on the $i$-th and $j$-th factors acting as the identity on the other factor. Solutions of the classical Yang-Baxter equation are classified by Belavin and Drinfel’d (see [3] for a precise statement). In particular, they discovered a rational solution $r(u) = t/u$. The infinitesimal pure braid relations are obtained from the fact that $t/u$ satisfies the classical Yang-Baxter equation.

As the monodromy of the connection $\omega$ we obtain a linear representation of the pure braid group

$$\theta : P_n \to \text{End} \ (W_1 \otimes \ldots \otimes W_n)$$
depending on the parameter \( \lambda \). Let us now suppose that the representations \( \rho_\alpha, 1 \leq \alpha \leq n, \) are the same. In this case the connection \( \omega \) defined in the above way is invariant by the diagonal action of the symmetric group \( S_n \) on \( X_n \times (W_1 \otimes \cdots \otimes W_n) \), hence it defines a local system over the quotient space \( Y_n = X_n/S_n \). Considering \( \lambda \) as a parameter we obtain a linear representation of the braid group on \( n \) strings

\[
\theta : B_n \to \text{End } (W^n) \otimes C(\lambda).
\]

Here \( C(\lambda) \) denotes the ring of the convergent power series. Our main object is to give a description of this monodromy representation.

The total differential equations of the above type appear in the two dimensional conformal field theory with gauge symmetry due to Knizhnik and Zamolodchikov [18]. Although in their situation the parameter \( \lambda \) is given by \( (\ell^g)^{-1} \) where \( \ell \) is a positive integer and \( g \) is the corresponding dual Coxeter number, we shall deal with the monodromy by considering \( \lambda \) as a parameter.

**1.2. Description of the monodromy by means of solutions of quantum Yang-Baxter equations.**

Let \( W \) be a finite dimensional complex vector space. By the quantum Yang-Baxter equation written in a multiplicative form we mean the following functional equation for a meromorphic function \( R(x) \) with values in \( \text{End } (W \otimes W) \):

\[
R_{12}(x)R_{13}(xy)R_{23}(y) = R_{23}(y)R_{13}(xy)R_{12}(x).
\]

Here the equality is considered in \( \text{End } (W \otimes W \otimes W) \) and the notation \( R_{ij} \) is standard as is explained in the previous section. Let us consider the case where \( R(x) \) contains an extra parameter \( q \) so that \( R(x,q) \) has an expansion around \( q = 1 \):

\[
R(x,q) = 1 + (q-1)r(x) + \cdots
\]

In this situation we verify that \( r(x) \) is a solution of the multiplicative classical Yang-Baxter equation

\[
[r_{12}(x), r_{13}(xy)] + [r_{12}(x), r_{23}(y)] + [r_{13}(xy), r_{23}(y)] = 0.
\]

We call \( r(x) \) the *classical limit* of \( R(x,q) \). The following typical solutions of the above classical Yang-Baxter equation was discovered by Belavin.
Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $\Delta$ be the set of roots of $\mathfrak{g}$. For a root $\alpha$, we denote by $X_\alpha$ the root vector normalized by $(X_\alpha, X_-) = 1$ with respect to the Cartan-Killing form. Putting $r = \sum_{\alpha \in \Delta} \text{sgn} \alpha \cdot X_\alpha \otimes X_-\alpha$, we define a $\mathfrak{g} \otimes \mathfrak{g}$-valued function $r(x)$ by

$$r(x) = r - t + \frac{2t}{x - 1}$$

where $t$ is defined in the previous section. These solutions are called trigonometric in the sense that they are rational functions of $x = e^t$.

The quantization problem of the above solutions was treated by Jimbo. In a series of papers [9], [10] and [11], he constructed a matrix $R(x, q)$ whose expansion around $q = 1$ is given by

$$R(x, q) = f(x)\{1 + (q - 1)((\rho \otimes \rho) r(x) + x(x)\mathbf{1}) + \cdots\}$$

with some $\mathbb{C}$-valued functions $f(x)$ and $x(x)$, for the following simple Lie algebras $\mathfrak{g}$ and their representations $\rho: \mathfrak{g} \to \text{End}(W)$

1. $\mathfrak{g}$ is non-exceptional and $\rho$ is the vector representation,
2. $\mathfrak{g}$ is $\mathfrak{sl}(2, \mathbb{C})$ and $\rho$ is an arbitrary finite dimensional irreducible representation.

In this section we discuss the case 1.2.5. Our matrices $R(x, q)$ are given by formulae 3.5 and 3.6 in [10] by putting $k = q$. In the formula 1.2.4, $f(x)$ is given by $(x - 1)$ if $\mathfrak{g}$ is of type $A$ and by $(x - 1)^2$ if $\mathfrak{g}$ is of type $B$, $C$ or $D$.

We put $\tilde{R} = \sigma R$ where $\sigma \in \text{End}(W \otimes W)$ is the transposition defined by $\sigma(x \otimes y) = y \otimes x$. One of the important properties of the matrix $R(x, q)$ is that it commutes with the diagonal action of $U^\vee(\mathfrak{g})$. Here $U^\vee(\mathfrak{g})$ denotes the $q$-analogue of the corresponding Lie algebra $\mathfrak{g}$ due to Jimbo [9], which is also denoted by $U_q(\mathfrak{g})$ with $q = e^t$ by Drinfel'd [7]. Instead of giving the complete definition we recall the case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, which is originally due to Kulish and Reshetikhin (see the references of [7]). We define $U^\vee(\mathfrak{g})$ to be the $\mathbb{C}$-algebra generated by the symbols $\hat{e}, \hat{f}, q^h$ and $q^{-h}$ with relations

$$q^{h/2}q^{-h/2} = q\hat{e}, \quad q^{h/2}\hat{f}q^{-h/2} = q^{-1}\hat{f}, \quad [\hat{e}, \hat{f}] = \frac{q^h - q^{-h}}{q - q^{-1}}.$$
We define the comultiplication $\Delta : U^-(\mathfrak{g}) \to U^-(\mathfrak{g}) \otimes U^-(\mathfrak{g})$ by the algebra homomorphism characterized by

$$\Delta(q^{\pm h/2}) = q^{\pm h/2} \otimes q^{\pm h/2}, \quad \Delta(X) = X \otimes q^{-h/2} + q^{h/2} \otimes X \text{ for } X = \hat{e}, \hat{f}.$$  

With respect to the comultiplications $\Delta$ and $\bar{\Delta} = \sigma \Delta$, $U^-(\mathfrak{g})$ has a structure of a non-commutative Hopf algebra which is considered to be a deformation of the universal enveloping algebra of $sl(2,\mathbb{C})$ (see Drinfel’d [7] and Verdier [23] for a more extensive treatment).

Let us go back to the situation of the previous section. Associated with a non-exceptional simple Lie algebra $\mathfrak{g}$ and its vector representation, we consider the connection

$$\omega = \sum_{1 \leq a < b \leq n} \lambda \Omega_{ab} d \log (z_a - z_b).$$

As the monodromy of $\omega$ we get a one parameter family of linear representation $\theta : B_n \to \text{End} (W^\otimes n) \otimes \mathbb{C}[\lambda]$. To describe $\theta$ we introduce the matrix $T(q)$ by

$$T(q) = \lim_{x \to \infty} x^{-d} \bar{R}(x, q)$$

where $d$ is the degree of the corresponding $\bar{R}(x, q)$ with respect to $x$, which is given by $d = 1$ in the case $\mathfrak{g}$ is of type A and by $d = 2$ in the other cases. We put $v = \frac{m - 1}{2m}$ if $\mathfrak{g} = sl(m,\mathbb{C})$ and $v = \frac{1}{2}$ otherwise.

Our main theorem in this section is the following:

**Theorem 1.2.8.** Let $\mathfrak{g}$ be a non-exceptional complex simple Lie algebra and let $\rho : \mathfrak{g} \to \text{End} (W^\otimes n)$ be its vector representation. As the monodromy of the associated connection

$$\omega = \sum_{1 \leq a < b \leq n} \lambda \Omega_{ab} d \log (z_a - z_b)$$

we get a linear representation $\theta : B_n \to \text{End} (W^\otimes n) \otimes \mathbb{C}[\lambda]$ given by

$$\theta(\sigma_i) = q^{\nu} (1 \otimes \cdots \otimes 1 \otimes T(q) \otimes 1 \otimes \cdots \otimes 1), \quad 1 \leq i \leq n - 1.$$  

Here $q = \exp (-\pi \sqrt{-1} \lambda)$ and $T(q)$ is situated on the $i$-th and $(i+1)$-st factors. Moreover this representation commutes with the diagonal action of $U^-(\mathfrak{g})$ on $W^\otimes n$.

The action of $U^-(\mathfrak{g})$ is defined by the multi-diagonal map in the sense of [9] and [12]. In the case $\mathfrak{g} = sl(m,\mathbb{C})$, the monodromy
representation obtained above is known as the higher order Pimsner-Popa-Temperley-Lieb representation (see [17]). In fact the matrix $T(q)$ is given by

\[
T(q) = \Sigma E_{x\beta} \otimes E_{x\alpha} + q \Sigma \delta_{x\alpha} \delta_{x\beta} E_{x\alpha} \otimes E_{x\beta} + (1 - q^2) \Sigma \delta_{x\alpha} E_{x\alpha} \otimes E_{x\beta}
\]

where $E_{x\beta}$ signify $m \times m$ matrix units. In this case the matrix $T(q)$ defines a linear representation of the braid group factoring through the Iwahori's Hecke algebra of the symmetric group.

1.3. Proof of Theorem 1.2.8.

Let us start with an integrable connection $\omega$ over $X_n$ of the form \( \omega = \sum_{1 \leq i < j \leq n} M_{ij} d \log (z_i - z_j) \), $M_{ij} \in \mathfrak{gl}(m, \mathbb{C})$. The monodromy of $\omega$ is expressed by an infinite sum using Chen's iterated integrals [6].

\[
\theta(\gamma) = 1 + \int_\gamma \omega + \int_\gamma \omega \omega + \ldots
\]

for $\gamma \in \mathbb{P}_n$. Here we have used the following standard notation for the Chen's iterated integrals.

Let $X$ be a smooth manifold and let $\omega_i$, $1 \leq i \leq n$, be matrix valued 1-forms on $X$. For a path $\gamma : [0,1] \to X$, we define the iterated integral $\int_\gamma \omega_1 \omega_2 \ldots \omega_n$ by

\[
\int_\Delta A_1(t_1) A_2(t_2) \ldots A_n(t_n) dt_1 dt_2 \ldots dt_n
\]

where $\gamma^* \omega_i = A_i(t_i) \alpha t_i$ and $\Delta = \{(t_1, \ldots, t_n) ; 0 \leq t_1 \leq \ldots \leq t_n \leq 1\}$.

Let $C \langle X_{x\beta} \rangle$ denote the ring of non-commutative formal power series with indeterminates $X_{x\beta}$, $1 \leq \alpha < \beta \leq n$, and let $J$ be its two sided ideal generated by the following infinitesimal pure braid relations among $X_{x\beta}$:

\[
[ X_{x\alpha}, X_{x\gamma} + X_{x\delta} ], \quad [ X_{x\alpha} + X_{x\gamma}, X_{x\delta} ], \quad \alpha < \beta < \gamma
\]

\[
[ X_{x\alpha}, X_{x\beta} ] \text{ for distinct } \alpha, \beta, \gamma, \delta.
\]

We denote by $A$ the quotient algebra $C \langle X_{x\beta} \rangle / J$. As a universal expression of 1.3.1, we obtain a homomorphism $\theta : \mathbb{P}_n \to A$ defined by
\[ \tilde{\theta}(\gamma) = 1 + \int_\gamma \tilde{\omega} + \int_\gamma \tilde{\omega}^\omega + \cdots \quad \text{with} \]

\[ \tilde{\omega} = \sum_{1 \leq \alpha < \beta \leq n} \chi_{\alpha\beta} \otimes \lambda \log (z_\alpha - z_\beta). \]

Let \( C[P_n]^- \) denote the completion of the group ring \( C[P_n] \) with respect to the powers of the augmentation ideal and let \( j: P_n \to C[P_n]^- \) denote the natural homomorphism. We have the following assertions:

**Proposition 1.3.3.** (i) We have an isomorphism of complete Hopf algebras \( C[P_n]^- \cong A \) such that the following diagram is commutative.

\[
\begin{array}{ccc}
C[P_n]]^- & \xrightarrow{j} & C[P_n]^- \\
P_n & \xrightarrow{\delta} & A
\end{array}
\]

(ii) The universal expression of the monodromy \( \tilde{\delta}: P_n \to A \) is injective.

The assertion (i) has been discussed by several authors in a more general situation (see [1], [8] and [16]). The primitive part of \( A \) is the Malcev Lie algebra of \( P_n \), which is the dual of the Sullivan’s 1-minimal model of \( X_n \) (see [21], [19] and [16]). The assertion (ii) is proved in [17] by the induction with respect to \( n \) by using the fibration \( \pi: X_{n+1} \to X_n \). The essential points are that the monodromy of the fibration \( \pi \) is trivial on the homology and that the natural homomorphism \( j \) is injective in the case of free groups. By using the assertion (ii) we have shown in [17] the following theorem:

**Theorem 1.3.4 ([17]).** Let \( \gamma_{\alpha\beta}, 1 \leq \alpha < \beta \leq n, \) be a system of generators of \( P_n \) given by

\[ \gamma_{\alpha\beta} = \sigma_\alpha \sigma_{\alpha+1} \cdots \sigma_{\beta-1} \sigma_\beta \sigma_{\beta-1} \cdots \sigma_\alpha^{-1}. \]

If \( \theta: P_n \to \text{GL}(m, \mathbb{C}) \) is a linear representation such that \( \|\theta(\gamma_{\alpha\beta}) - I\| \) is sufficiently small for each \( 1 \leq \alpha < \beta \leq n \), then there exist constant matrices \( M_{\alpha\beta}, 1 \leq \alpha < \beta \leq n, \) close to 0, satisfying the infinitesimal pure braid relations, such that the monodromy of the connection \( \omega = \sum_{1 \leq \alpha < \beta \leq n} \chi_{\alpha\beta} \otimes \lambda \log (z_\alpha - z_\beta) \) is equivalent to \( \theta \).

To deduce Theorem 1.3.4 from Proposition 1.3.3 we used an argument due to Hain [8].

Now let us go back to the situation of Theorem 1.2.8.
Lemma 1.3.6. — We put \( \lambda = -(\pi \sqrt{-1})^{-1} \log q, \quad -\pi \leq \text{Im} \log q < \pi \). The matrix \( T(q)^2 \) has an expansion with respect to \( \lambda \) of the form
\[
T(q)^2 = 1 + 2\pi \sqrt{-1} \lambda \{(\rho \otimes \rho)(t) - 2v \cdot 1\} + O(\lambda^2).
\]
Here \( \rho \) is the vector representation as in Sect. 1.2.

Proof of Lemma 1.3.6. — Let us recall that \( T(q) \) is defined as the leading coefficient of the matrix \( R(x,q) \) with respect to \( x \). By means of the expansion 1.2.4 and the definition of \( r(x) \) (see 1.2.3), we have
\[
T'(1) = \sigma \{(\rho \otimes \rho)(t-1) + 2v \cdot 1\}.
\]
Here we have used \( 2v = \lim_{x \to \infty} x(x) \), which is verified by a direct computation. Let us now observe that \( T(1) \) is equal to the transposition \( \sigma \). By using
\[
\sigma \cdot (\rho \otimes \rho)(t) \cdot \sigma = (\rho \otimes \rho)(t) \\
\sigma \cdot (\rho \otimes \rho)(r) \cdot \sigma = - (\rho \otimes \rho)(r)
\]
we obtain the formula
\[
T(1)T'(1) + T'(1)T(1) = -2(\rho \otimes \rho)(t) + 4v \cdot 1.
\]
Our Lemma follows immediately.

It follows from the definition of the Yang-Baxter equation 1.2.1 that the matrix \( R(x,q) \) satisfies
\[
R_{12}(x)R_{23}(xy)R_{12}(y) = R_{23}(y)R_{12}(xy)R_{23}(x).
\]
This shows that the correspondence
\[
\sigma_i \to 1 \otimes \cdots \otimes T(q) \otimes \cdots \otimes 1
\]
appearing in the statement of Theorem 1.2.8 actually defines a linear representation of the braid group. In the following we denote this representation by \( \varphi \).

If \( |\lambda| \) is sufficiently small, then we may apply Theorem 1.3.4. Hence in this situation we have a matrix \( M(\lambda) \in \text{End} (W \otimes W) \) close to 0 and analytic with respect to \( \lambda \), so that the monodromy of the connection \( \Sigma_{1 < z_1 < \beta < n} M_{2\beta}(\lambda)\frac{1}{z} \log (z_2 - z_2) \) expressed by the iterated integrals 1.3.1 is equal to \( \varphi \) restricted to \( P_n \).
Let \( M(\lambda) = Z_1\lambda + Z_2\lambda^2 + \ldots \) be an expansion of \( M(\lambda) \) around \( \lambda = 0 \). By means of the expression of the monodromy using iterated integrals and Lemma 1.3.6 we have
\[
Z_1 = (\rho \otimes \rho)(t) - 2\nu \cdot 1.
\]
In the following, we denote the above matrix by \( \Omega' \).

**Lemma 1.3.11.** — If \(|\lambda|\) is sufficiently small, there exists a matrix \( P(\lambda) \in \text{End}(W^{\otimes n}) \) with \( \lim_{\lambda \to 0} P(\lambda) = 1 \) such that
\[
P(\lambda)^{-1}M_{\beta}(\lambda)P(\lambda) = \lambda\Omega'_{\beta}.
\]

**Proof of Lemma 1.3.11.** — Let \( H_{\beta} \) denote the hyperplane in \( C^n \) defined by \( z_\beta = z_\beta \). Let \( \mu : X \to C^n \) be a blowing up with exceptional divisors \( E_k, 3 \leq k \leq n \), such that \( \mu(E_k) = \bigcap_{1 \leq \alpha < \beta \leq k} H_{\alpha} \). We denote by \( E_2 \) the proper transform of \( H_1 \). Then the residue of the connection \( \mu^*\omega \) along the divisor \( E_2 \) is expressed as \( \Sigma_{1 \leq \alpha < \beta \leq k} M_{\beta}(\lambda) \). Let us observe that a normal loop around \( E_k \) is given by \( \gamma_k = (\sigma_1 \ldots \sigma_{k-1})^k \) which lies in the center of \( B_2 \). For a generic value \( \lambda \in C \), the matrix \( \varphi(\gamma_k) \) is diagonalizable, which implies that the residue \( \Sigma_{1 \leq \alpha < \beta \leq k} M_{\beta}(\lambda) \) is diagonalizable. Moreover, by means of the infinitesimal pure braid relations for \( M_{\beta}(\lambda) \) we conclude that the residues \( \Sigma_{1 \leq \alpha < \beta \leq k} M_{\beta}(\lambda) \), \( k = 2, 3, \ldots \) are diagonalized simultaneously. We have a matrix \( Q(\lambda) = Q_0 + Q_1\lambda + Q_2\lambda^2 + \ldots \) such that for \( 2 \leq k \leq n \)
\[
Q(\lambda)^{-1}(\Sigma_{1 \leq \alpha < \beta \leq k} M_{\beta}(\lambda))Q(\lambda)
\]
is diagonal. It can be shown by using the explicit form of \( T(q) \) that the eigenvalues of \( \varphi(\gamma_k) \) is of the form \( q^m \) with some integer \( m \). This implies that the matrix 1.3.12 is linear with respect to \( \lambda \). Hence it is written as \( Q_0^{-1}(\Sigma_{1 \leq \alpha < \beta \leq k} \lambda\Omega'_{\beta})Q_0 \). Putting \( P(\lambda) = Q(\lambda)Q_0^{-1} \), we obtain a desired matrix. This proves Lemma.

The proof of Theorem 1.2.8 is completed in the following way. We put \( \omega' = \Sigma \lambda\Omega'_{\beta} \omega \log (z_\beta - z_\beta) \). By Lemma 1.3.11 the expression
\[
1 + \int_\gamma \omega' + \int_\gamma \omega'\omega' + \ldots
\]
is equal to \( P(\lambda)^{-1}\varphi(\lambda)P(\lambda) \) if \(|\lambda|\) is sufficiently small. We observe that \( P(\lambda) \) is analytically continued to a meromorphic function of \( \lambda \) on the whole complex plane. Since the expression 1.3.13 is an entire function
of \( \lambda \) we conclude by an analytic continuation that 1.3.13 is expressed as \( P(\lambda)^{-1} \varphi(\lambda) P(\lambda) \) in \( \text{End} (W^{\otimes n}) \otimes \mathbb{C} \{ \lambda \} \). Thus we have shown the statement of Theorem 1.2.8 on the pure braid group \( P_n \). To extend this to the full braid group \( B_n \) it suffices to observe that both \( \theta(\sigma_i) \) and \( \varphi(\sigma_i) \) are the transposition of the \( i \)-th factor and \((i+1)\)-st factors if \( \lambda = 0 \) and that they are holomorphic with respect to \( \lambda \). This shows the first assertion of Theorem 1.2.8. The second assertion is derived from the fact that \( \tilde{R}(x, q) \) commutes with the diagonal action of \( U^-(g) \). This completes the proof of Theorem 1.2.8.

(1.3.14) Remark. — For a complex number \( \lambda \in \mathbb{C} \), the above proof implies that the correspondence described in Theorem 1.2.8 holds true if \( \varphi(q_k) \), \( 2 \leq k \leq n \), are diagonalizable. This condition is satisfied if \( \varphi \) is completely reducible.

2. MONODROMY OF \( n \)-POINT FUNCTIONS
IN TWO DIMENSIONAL CONFORMAL FIELD THEORY

2.1. Review of \( A^{(1)}_1 \) model due to Tsuchiya and Kanie.

In this section we recall briefly the operator formalism of the two dimensional conformal field theory on \( \mathbb{P}^1 \) with gauge symmetry of type \( A_1^{(1)} \) following a recent work of Tsuchiya and Kanie [22].

**Integrable highest weight modules.** — Let \( g = \mathfrak{sl}(2, \mathbb{C}) \) and let \( \hat{g} \) be the affine Lie algebra of type \( A_1^{(1)} \) which is defined by the canonical central extension of the loop algebra \( g \otimes \mathbb{C}[[t, t^{-1}]] \) (see [15]). Putting \( \mathcal{M}_\pm = \sum_{n \geq 1} g \otimes t^{\pm n} \), \( \hat{g} \) is decomposed into

\[
\hat{g} = \mathcal{M}_+ \oplus g \oplus \mathbb{C}c \oplus \mathcal{M}_-.
\]

where \( c \) is the central element. For a positive integer \( \ell \) and a half integer \( j \) such that \( 0 \leq j \leq \ell/2 \) it is known by Kac [15] that there exists a unique irreducible left \( \hat{g} \)-module \( \mathcal{H}_j(\ell) \) with a non zero vector \( |\ell, j\rangle \) such that

\[
(2.1.1) \quad \mathcal{M}_+ |\ell, j\rangle = E|\ell, j\rangle = 0, \quad H|\ell, j\rangle = 2j|\ell, j\rangle, \quad c|\ell, j\rangle = \ell|\ell, j\rangle.
\]
In the same way, we have a unique irreducible right \( g \)-module \( \mathcal{H}_{\ell}^j \) with \( \langle j, \ell \rangle \) such that

\[
\begin{align*}
\langle j, \ell \rangle \mathcal{H}_- &= \langle j, \ell \rangle \mathcal{F} = 0, \\
\langle j, \ell \rangle \mathcal{H} &= 2j \langle j, \ell \rangle, \\
\langle j, \ell \rangle c &= \ell \langle j, \ell \rangle.
\end{align*}
\]

Here \( H, E \) and \( F \) stand for the usual Chevalley basis of \( g \). In the following we fix \( \ell \) and we write \( \mathcal{H}_j \) instead of \( \mathcal{H}_{\ell}^j \). There exists a unique bilinear form \( \mathcal{H}^j \times \mathcal{H}_j \rightarrow \mathbb{C} \) such that \( \langle j, \ell, j \rangle = 1 \) and \( \langle u \alpha | v \rangle = \langle u | av \rangle \) for any \( \alpha \in \hat{g}, u \in \mathcal{H}_j^1 \) and \( v \in \mathcal{H}_j \).

**Operation of the Virasoro Lie algebra.** – For \( X \in g \), we put \( X[n] = X \otimes t^n \) and \( X(z) = \sum_{n} z^n X[n] z^{-n-1} \) with \( z \in \mathbb{C} \setminus \{0\} \). The Segal-Sugawara form \( T(z) \) is defined to be

\[
T(z) = \frac{1}{2(2 + \ell)} \sum_{\mu} I_{\mu}(z) I_{\mu}(z),
\]

Here \( \{I_{\mu}\} \) denotes an orthonormal basis of \( g \) and \( : \) stands for the usual normal order product defined by

\[
X[m] Y[n] = \begin{cases} 
X[m] Y[n] & \text{if } m < n \\
\frac{1}{2} \{X[m] Y[n] + Y[n] X[m]\} & \text{if } m = n \\
Y[n] X[m] & \text{if } m > n.
\end{cases}
\]

We define \( L_m, m \in \mathbb{Z} \) as the coefficients of the expansion

\[
T(z) = \sum_{m \in \mathbb{Z}} L_m z^{-m-2}.
\]

We may also express \( L_m \) as

\[
L_m = \frac{1}{2(2 + \ell)} \sum_{\ell \in \mathbb{Z}} \sum_{\mu} I_{\mu}(z) I_{\mu}(m + k).
\]

These \( L_m, m \in \mathbb{Z} \), satisfy the fundamental relations of the Virasoro Lie algebra:

\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12} \delta_{m+n,0} c'.
\]

Here \( c' = \frac{3\ell}{\ell + 2} \text{id} \), which we shall call the central charge. With respect to the operation of \( L_0, \mathcal{H}_j \) is decomposed into finite dimensional subspaces

\[
\mathcal{H}_j = \bigoplus_{d \geq 0} \mathcal{H}_{j,d}.
\]
where $\mathcal{H}_{j,d}$ is the eigenspace with the eigenvalue $\frac{j^2 + j + d}{\ell + 2}$. In particular, $\mathcal{H}_{j,0}$ is identified with the spin $j$ representation of $g$, which is denoted by $V_j$.

**Definition of primary fields.** We are interested in operators on the space $\mathcal{H} = \bigoplus_{j=0}^{\ell/2} \mathcal{H}_j$. The basic operators are so called primary fields. A primary field of spin $j$ is defined to be a bilinear form $\phi(u,z) : \mathcal{H}^n \times \mathcal{H} \to \mathbb{C}$ parametrized by $u \in V_j$ and $z \in \mathbb{C}\{0\}$ in such a way that

(i) $\phi(u,z)$ is linear with respect to $u$

(ii) $\langle v|\phi(u,z)|w \rangle$ is a multivalued holomorphic function of $z$ for any $v \in \mathcal{H}^n$ and $w \in \mathcal{H}$, satisfying the following conditions:

\begin{align*}
(2.1.8) \quad [X \otimes t^n, \phi(u,z)] &= z^n\phi(Xu,z) \quad \text{(gauge condition)} \\
(2.1.9) \quad [L_m, \phi(u,z)] &= z^n\left\{z \frac{\partial}{\partial z} + (m + 1)\Delta_j\right\}\phi(u,z)
\end{align*}

where $\Delta_j = \frac{j^2 + j}{\ell + 2}$, which we shall call the conformal dimension of $\phi$.

**Existence of vertex operators.** Given a primary field of spin $j$, we associate to the triple $v = (j_1, j_2)$ the $(j_1, j_2)$ component of $\phi(u,z)$ with respect to the decomposition 2.1.7, which we denote by $\phi_v(u,z)$. This operator is called a vertex operator of type $v$. We have a Laurent series expansion $\phi_v(u,z) = \sum_{n} z^n \phi_n(u)z^{-n-\Delta}$ with $\Delta = \Delta_j + \Delta_{j_1} - \Delta_{j_2}$ ([22] Prop. 2.1.). This gives a $g$ invariant trilinear form $\varphi : V_{j_1}^* \otimes V_{j_2}^* \otimes V_{j_3} \to \mathbb{C}$ defined by $\varphi(u,v,w) = \langle u|\phi_0(v)|w \rangle$, which we shall call the initial form.

**Theorem 2.1.10 ([22] Th. 2.2.).** (i) A non trivial vertex operator of type $v$ exists if and only if the following conditions are satisfied:

\begin{align*}
(2.1.11) \quad |j_1 - j_2| \leq j \leq j_1 + j_2, \quad j_1 + j + j_2 \in \mathbb{Z} \quad \text{(Clebsch-Gordan condition)} \\
(2.1.12) \quad j_1 + j + j_2 \leq \ell
\end{align*}

(ii) Under the above conditions, a vertex operator of type $v$ is unique up to scalar and is determined by its initial form.
Differential equation of n-point functions. For an operator $A$ on $\mathcal{H}$, we denote by $\langle A \rangle$ its vacuum expectation defined by $\langle \text{vac}|A|\text{vac}\rangle = \langle 0, \epsilon|A|\epsilon, 0 \rangle$. Our purpose is to give a description of n-point functions $\langle \phi_1(u_1, z_1), \ldots, \phi_n(u_n, z_n) \rangle$ for primary fields $\phi_i$. A main tool to deduce differential equations satisfied by n-point functions is the following operator product expansions:

\begin{align*}
(2.1.13) \quad X(\zeta)\phi(u, z) &= \frac{1}{\zeta - z} \phi(Xu, z) + \text{(regular terms)} \\
(2.1.14) \quad T(\zeta)\phi(u, z) &= \left(\frac{\Delta_j}{(\zeta - z)^2} + \frac{1}{\zeta - z} \frac{\partial}{\partial z}\right) \phi(u, z) + \text{(regular terms)}
\end{align*}

for a primary field $\phi$ of spin $j$. Here the meaning of the compositions of operators is justified by the use of the decomposition 2.1.7 (see [22] for a precise definition). Following [18], we define the operation of $g$ on vertex operators by:

\begin{align*}
(2.1.15) \quad [X[m]](u, z) &= \frac{1}{2\pi\sqrt{-1}} \int_C d\zeta (\zeta - z)^m X(\zeta)\phi(u, z) \\
(2.1.16) \quad [L^\pm](\phi)(u, z) &= \frac{1}{2\pi\sqrt{-1}} \int_C d\zeta (\zeta - z)^{m \pm 1} T(\zeta)\phi(u, z)
\end{align*}

for a positively oriented small contour $C$ around $z$. Combining with the operator product expansions, we obtain

\begin{align*}
(2.1.17) \quad [X[0]](\phi)(u, z) &= \phi(Xu, z), \\
[X[m]](\phi)(u, z) &= 0 \quad \text{for} \quad m > 0
\end{align*}

\begin{align*}
(2.1.18) \quad [L_{-1}](\phi)(u, z) &= \frac{\partial}{\partial z} \phi(u, z), \quad [L_0](\phi)(u, z) = \Delta_j \phi(u, z), \\
[L_m](\phi)(u, z) &= 0 \quad \text{for} \quad m > 0.
\end{align*}

Starting from a primary field $\phi$ of spin $j$, we get new operators by the iterations of the operations of $X[m]$ and $L_m$, $m \leq 0$, of type 2.1.15 and 16. They are classified into the levels by the eigenvalues of the operator $L_0$, e.g., $L_{-n_1}L_{-n_2} \ldots L_{-n_k}\phi$ has an eigenvalue $\Sigma_{j=1}^k n_k + \Delta_j$ with respect to the operation of $L_0$. This is the whole spectrum of our operators. From the operator product expansions, we deduce the
following local Ward identities:

\[(2.1.19) \quad \left< X(\xi)\phi_1(z_1) \ldots \phi_n(z_n) \right> = \sum_{z=1}^n \frac{1}{\xi - z} \left< \phi_1(z_1) \ldots \left[ X(0)\phi_3 \right](z_2) \ldots \phi_n(z_n) \right> \]

\[(2.1.20) \quad \left< T(\xi)\phi_1(z_1) \ldots \phi_n(z_n) \right> = \sum_{z=1}^n \left( \frac{\Delta_v}{(\xi - z)^2} + \frac{1}{\xi - z} \frac{\partial}{\partial z} \right) \left< \phi_1(z_1) \ldots \phi_n(z_n) \right>. \]

Here \(\phi_x\) is supposed to be a primary field of spin \(j_x\).

**Theorem 2.1.21 (Knizhnik and Zamolodchikov [18]):** The \(n\)-point function \(\Phi = \left< \phi_1(z_1) \ldots \phi_n(z_n) \right>\) satisfies the total differential equation

\[\partial \Phi = \sum_{1 \leq \alpha < \beta \leq n} \frac{1}{\ell + 2} \Omega_{\alpha \beta} \partial \log (z_\alpha - z_\beta) \Phi.\]

Here \(\phi_x\) is a primary field of spin \(j_x\) and \(\Omega_{\alpha \beta} \in \text{End} (V_{j_1} \otimes \ldots \otimes V_{j_n})\) is determined by 1.1.2 via spin \(j_x\) representations of \(sl(2,\mathbb{C})\).

**Proof.** — Let \(\phi(u,z)\) be a primary field. By the expression of \(L_0\) given in 2.1.5 and the identities 2.1.17 and 18 we have

\[(\ell + 2) \frac{\partial}{\partial z} \phi(u,z) = [\Sigma_{\mu} I_\mu [-1] I_\mu [0] \phi](u,z). \]

The RHS turns out to be the constant term of the operator product expansion of \(\Sigma_{\mu} I_\mu(\zeta)\phi[I_\mu u, z]\). This implies that

\[(\ell + 2) \frac{\partial}{\partial z} \phi(u,z) = \lim_{\zeta \to z} \left[ \Sigma_{\mu} I_\mu(\zeta)\phi[I_\mu u, z] - \frac{1}{\zeta - z} \phi[\Omega u, z] \right] \]

where \(\Omega\) denotes the Casimir operator. Combining with the local Ward identity 2.1.19 we have

\[(\ell + 2) \frac{\partial}{\partial z_x} \Phi = \Sigma_{\beta \neq x} \frac{\Omega_{\alpha \beta}}{z_x - z_\beta} \Phi \]

which proves our Theorem.
2.2. Monodromy associated with higher representations of \( \mathfrak{sl}(2, \mathbb{C}) \).

For a half integer \( j \geq 0 \), we denote by \( V_j \) the irreducible left \( \mathfrak{sl}(2, \mathbb{C}) \) module of spin \( j \), which is an irreducible representation of dimension \( 2j + 1 \). We now proceed to discuss the monodromy representation \( \theta : B_n \to \text{End}(V_j^\otimes m) \) of the connection associated with the spin \( j \) representation of \( \mathfrak{sl}(2, \mathbb{C}) \) in the sense of Sect. 1.1. For this purpose we first recall a «fusion» process for solutions of Yang-Baxter equations due to Jimbo [11]. Let us start with the matrix \( T(q) \) given in 1.2.9 with \( m = 2 \). We put

\[
\tilde{R}(x, q) = xq^{-1}T(q) - x^{-1}qT(q)^{-1}.
\]

The matrix \( R(x, q) = \sigma \tilde{R}(x, q) \) is a solution of the quantum Yang-Baxter equation. We have an expansion of the form

\[
(2.2.1) \quad R(x, q) = (x - x^{-1}) (1 + r(x)(q - 1) + \cdots)
\]

with its classical limit \( r(x) \). We put

\[
R_k(x, q) = R_{k, 2m}(x, q)R_{k, 2m-1}(xq, q) \cdots R_{k, m+1}(xq^{m-1}, q)
\]

which is considered to be an element of \( \text{End}(V^\otimes m \otimes V^\otimes m) \). Here \( R_\alpha \beta \) stands for the matrix \( R \) acting on the \( \alpha \)-th and \( \beta \)-th factors and \( V = \mathbb{C}^2 \). We now define \( R^{(m)}(x, q) \) as

\[
R^{(m)}(x, q) = R_1(x, q)R_2(xq, q) \cdots R_m(xq^{m-1}, q).
\]

Let us regard \( V \) as a \( U^-(\mathfrak{sl}(2, \mathbb{C})) \) module and we denote by \( V_j \) the irreducible \( U^-(\mathfrak{sl}(2, \mathbb{C})) \) module of spin \( j \) considered as a subspace of \( V^\otimes 2j \). This is denoted by \( L_{2j} \) in [9] Sect. 3. The matrix \( R^{(m)}(x, q) \) defined above determines an endomorphism of \( V_j \otimes V_j \) with \( j = m/2 \). Let us define the matrix \( T^{(m)}(q) \) by

\[
(2.2.2) \quad T^{(m)}(q) = \lim_{x \to \infty} x^{-m^2} \tilde{R}^{(m)}(x, q).
\]

This matrix is also expressed explicitly as

\[
(2.2.3) \quad T^{(m)}(q) = q^{-m^3}(T_mT_{m-1} \cdots T_1)(T_{m+1}T_m \cdots T_2) \cdots \cdots (T_{2m-1}T_{2m-2} \cdots T_m)
\]

where \( T_i \) denotes the matrix \( T(q) \) on the \( i \)-th and \( (i+1) \)-st factors.
THEOREM 2.2.4. — As the monodromy of the connection associated with the spin \( j = m/2 \) representation of \( \mathfrak{sl}(2, \mathbb{C}) \), we get a one parameter family of linear representations \( \theta : B_n \to \text{End} (W^\otimes n) \otimes \mathbb{C} \{ \lambda \} \) with \( W = V_j \) defined by

\[
\theta(\sigma_i) = q^{-1/4} (1 \otimes \cdots \otimes T^{(m)}(q) \otimes \cdots \otimes 1), \quad 1 \leq i \leq n-1,
\]
where \( q = \exp (-\pi \sqrt{-1} \lambda) \) and \( T^{(m)}(q) \) is on the \( i \)-th and \((i+1)\)-st factors.

Let \( \iota : B_n \to B_{mn} \) be a homomorphism defined by

\[
(2.2.5) \quad \iota(\sigma_i) = (\sigma_{i+m} \sigma_{i+m-1} \cdots \sigma_{i+1}) \cdot (\sigma_{i+m+1} \sigma_{i+m} \cdots \sigma_{i+2}) \cdots
\]
\[
\cdots \cdot (\sigma_{i+2m-1} \sigma_{i+2m-2} \cdots \sigma_{i+m})
\]
with \( \alpha = (i-1)m \). This «parallel» embedding is illustrated in the following picture:

![Fig. 2.](image)

By means of this homomorphism our monodromy representation \( \theta \) is also expressed in the following manner:

COROLLARY 2.2.6. — Let \( \varphi : B_{mn} \to \text{End} (V^\otimes mn) \otimes \mathbb{C} \{ \lambda \} \) be the Pimsner-Popa-Temperley-Lieb representation defined by \( \varphi(\sigma_i) = 1 \otimes \cdots \otimes T(q) \otimes \cdots \otimes 1 \) (see 1.2.9.). Then the composition \( \varphi \circ \iota : B_n \to \text{End} (V^\otimes n) \otimes \mathbb{C} \{ \lambda \} \) leaves invariant the subplace \( (V^\otimes n) \) and the monodromy representation \( \theta \) is given by

\[
\theta(\sigma_i) = q^{-m^3 - \frac{1}{4}} \varphi \circ \iota(\sigma_i), \quad 1 \leq i \leq n-1.
\]

It turns out that our monodromy representation is the same as that studied by Murakami [20] up to a scalar representation.
Proof of Theorem 2.2.4. - We put \( m = 2j \). It follows from the fact that \( R^{(m)}(x,q) \) is a solution of the Yang-Baxter equation ([11] Th. 2) that the correspondence in the statement of Theorem 2.2.4 actually defines a linear representation of \( B_n \). Let \( \rho \) denote the spin \( j \) representation of \( \mathfrak{sl}(2,\mathbb{C}) \). By using the classical limit \( r(x) \) of \( R(x,q) \), we have an expansion

\[
(2.2.6) \quad R^{(m)}(x,q) = (x-x^{-1})^{m^2}\{ 1 + (\rho \otimes \rho)(r(x))(q-1) + \cdots \}.
\]

By the definition of \( T^{(m)}(q) \) and the above formula we have

\[
\frac{d}{dq} T^{(m)}(q) = (\rho \otimes \rho)\left( r - t - \frac{1}{2} \right).
\]

Here \( r \) and \( t \) are defined in Sect. 1.2. As a consequence we have

\[
\frac{d}{dq} T^{(m)}(q)^2|_{q=1} = (\rho \otimes \rho)(-2t - 1).
\]

This implies that \( T^{(m)}(q)^2 \) has an expansion

\[
1 + 2\pi \sqrt{-1} \lambda \left\{ (\rho \otimes \rho)(t) + \frac{1}{2} \right\} + \mathcal{O}(\lambda^2)
\]

with \( \lambda = -(\pi \sqrt{-1})^{-1} \log q, -\pi \leq \text{Im} \log q < \pi \). Let us observe that the eigenvalues of \( \varphi(\gamma_k), 2 \leq k \leq n - 1 \), are of the form \( q^\alpha \) with some integer \( \alpha \). Hence the same argument as in the proof of Theorem 1.2.8 can be applied to our Theorem.

2.3. Unitarity of the monodromy of \( n \)-point functions.

Let us now apply the fusion process introduced in the previous section to a description of the monodromy of \( n \)-point functions when \( \phi_\alpha, 1 \leq \alpha \leq n \), are vertex operators of spin \( j \). For a pair of half integers \((j,t)\), we denote by \( \Gamma_{n,t}^j \) the set defined by

\[
\Gamma_{n,t}^j = \{ (p_0, p_1, \ldots, p_n); p_i \in \frac{1}{2} \mathbb{Z}_{>0} \text{ such that } p_0 = 0, p_n = t \text{ and each triple } v_i = (p_{i-1}, i, p_i) \text{ satisfies the conditions 2.1.11 and 12} \}.
\]
We fix a positive integer \( \ell \). To each element of \( \Gamma_{n,t}^j \), we associate the composition of vertex operators of type \( v_i, 1 \leq i \leq n \). This defines the \( n \)-point function

\[
\phi_{v_1 \ldots v_n}(z_1, \ldots, z_n) = \langle \text{vac} | \phi_{v_1}(z_1) \ldots \phi_{v_n}(z_n) | v \rangle
\]

for \( v \in V_j \). It is shown in [22] that this is a holomorphic function in the region \( |z_1| > \cdots > |z_n| \) and is analytically continued to a multi-valued holomorphic function on \( X_n \). Moreover, they showed that the monodromy of the \( n \)-point functions associated with \( \Gamma_{n,t}^j \) defines a linear representation of the braid group \( B_n \), which we denote by \( \theta : B_n \to \text{End}(W_{n,t}^j) \). Our main object is to describe this representation.

Let us remark that the above composition of vertex operators is illustrated by the lattice obtained from the decomposition of \( V_j \otimes \cdots \otimes V_j \) into simple \( \mathfrak{sl}(2,\mathbb{C}) \) modules. Here are some examples.

(i) \( j = \frac{1}{2}, \ell = 3 \)

(ii) \( j = 1, \ell = 4 \)

We denote by \( \langle \alpha, \beta \rangle \) the atom corresponding to \( n = \alpha \) and spin \( \beta \). The composition of vertex operators defined by \((p_0, \ldots, p_n) \in \Gamma_{n,t}^j \) is represented by the path connecting \( \langle 1, p_1 \rangle, \ldots, \langle n-1, p_{n-1} \rangle \). By means of an explicit computation of the 4-point functions Tsuchiya and Kanie showed that in the case \( j = 1/2 \) the monodromy \( \theta : B_n \to \text{End}(W_{n,t}^{1/2}) \) factors through the Jones algebra with index \( \tau^{-1} = 4 \cos^2 \frac{\pi}{\ell + 2} \) (see [13]) and is equivalent to an irreducible unitarizable representation of \( B_n \) obtained by Wenzl [24]. Here we may identify the lattice illustrated in Fig. 3 (i) to the Bratteli diagram of the corresponding Jones algebra.
Our result is as follows:

**Theorem 2.3.1.** — For any positive half integer \( j \), the monodromy of \( n \)-point functions \( \theta : B_n \to \text{End}(W_{n,1}^j) \) is unitarizable.

**Outline of Proof.** — Let us first recall the differential equation satisfied by the \( n \)-point functions (Th. 2.1.21). Let \( \psi : B_n \to B_{2jn} \) be the homomorphism defined by 2.2.5 with \( m = 2j \). Let \( \theta_0 : B_{2jn} \to \text{End}(W_{2jn}^{1/2}) \) be the monodromy of \( 2jn \)-point functions with spin \( 1/2 \). It follows from [22] Th. 5.2 that \( \theta_0 \) is unitarizable. In particular, the matrices

\[
(\theta_0 \circ \psi)(\sigma_1 \ldots \sigma_{k-1})^k, \quad 1 \leq k \leq n,
\]

are diagonalizable. Hence we may apply an argument of the proof of Theorem 2.2.5 and Corollary 2.2.6 to our situation (see also Remark 1.3.14). This implies that the monodromy representation \( \theta : B_n \to \text{End}(W_{n,1}^j) \) is equivalent to a subrepresentation of the representation given by the correspondence

\[
\sigma_i \to q^\mu \theta_0 \circ \psi(\sigma_i)
\]

with some constant \( \mu \). Combining with the fact that \( \theta_0 \) is unitarizable we obtain our Theorem.

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