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ON THE DISCREPANCY OF SEQUENCES ASSOCIATED WITH THE SUM-OF-DIGITS FUNCTION

by N. KOPECEK, G. LARCHER, R.F. TICHY and G. TURNWALD

1. Introduction.

In a series of papers J. Coquet *et al.* investigated the distribution modulo 1 of sequences $(x \cdot s_\alpha(n))_{n=0}^{\infty}$ where x is an irrational number and $s_\alpha(n)$ denotes the sum of digits in the α-adic expansion of n (cf. [1], [2], [3], [5])(*). We will give a quantitative refinement and a generalization to the multi-dimensional case.

Let $(\mathrm{y}_n)_{n=0}^{\infty}$ be a sequence of elements of $\mathbf{R}^d$ $(d \geqslant 1)$. Then the discrepancy mod 1 of (y_n) is defined by

$$\mathrm{D_N}(\mathrm{y}_n) = \sup_{\mathrm{I}} \left| \frac{\mathrm{A}(\mathrm{I}, \mathrm{N}, \mathrm{y}_n)}{\mathrm{N}} - \mathrm{vol}(\mathrm{I}) \right|, \tag{1.1}$$

where the supremum is extended over all d-dimensional subintervals of $[0,1[^d$ of the form $\mathrm{I} = \{(t_1, \ldots, t_d) : a_j \leqslant t_j < b_j \text{ for } 1 \leqslant j \leqslant d\}$, $\mathrm{vol}(\mathrm{I})$ means the volume $\prod_{j=1}^{d} (b_j - a_j)$ of I, and $\mathrm{A}(\mathrm{I}, \mathrm{N}, \mathrm{y}_n)$ denotes the number of indices n $(0 \leqslant n < \mathrm{N})$ such that the fractional part of the j-th component of y_n belongs to the interval $[a_j, b_j[$ for $j = 1, \ldots, d$. The sequence (y_n) is uniformly distributed mod 1 if and only if

$$\lim_{\mathrm{N} \to \infty} \mathrm{D_N}(y_n) = 0\,;$$

cf. the monographs [4] and [7].

(*) These investigations were initiated by M. Mendès-France [*J. Analyse Math.*, 20 (1967), 1-56].

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Let α be an irrational number with continued fraction expansion $[a_0\,; a_1\,, a_2\,, \ldots]$. Let $q_0 = 1\,, q_1 = a_1\,,$ and

$$q_{k+2} = a_{k+2}\, q_{k+1} + q_k \;(k \geqslant 0)\,.$$

We define the α-adic expansion of a positive integer by

$$n = \sum_{k=0}^{L(n)} \epsilon_k(n)\, q_k \quad (\epsilon_{L(n)}(n) \neq 0)\,, \tag{1.2}$$

where the digits $\epsilon_k(n)$ satisfy the following conditions:

(i) $0 \leqslant \epsilon_0(n) < a_1\,,$

(ii) $0 \leqslant \epsilon_k(n) \leqslant a_{k+1} \quad (k \geqslant 1)\,,$

and

(iii) $\epsilon_k(n) = a_{k+1}$ implies $\epsilon_{k-1}(n) = 0\,.$

In the following we consider the sequence $\mathbf{y}_n = \mathbf{x}\, s_\alpha(n)$ for a fixed vector $\mathbf{x} = (x_1\,, \ldots, x_d) \in \mathbf{R}^d$, where

$$s_\alpha(n) = \sum_{k=0}^{L(n)} \epsilon_k(n)\,.$$

By [2], the one-dimensional sequence $(x s_\alpha(n))$ is uniformy distributed mod 1 if x is an irrational number. In order to obtain estimates for the discrepancy $D_N(\mathbf{y}_n)$, we need information concerning the diophantine approximation properties of $\mathbf{x} = (x_1\,, \ldots, x_d)$. Let $\psi : [0\,, \infty) \longrightarrow [0\,, \infty)$ be a continuous strictly increasing function with $\psi(0) = 0$ and $\psi(t) \geqslant t$. We say that $\mathbf{x}$ is of approximation type $< \psi$ if there exists a positive constant $c = c(\mathbf{x}\,, \psi)$ such that

$$\| \mathbf{h} \cdot \mathbf{x} \| \geqslant \frac{c}{\psi(r(\mathbf{h}))} \tag{1.3}$$

for all lattice points $\mathbf{h} = (h_1\,, \ldots, h_d) \in \mathbf{Z}^d\,, \mathbf{h} \neq (0\,, \ldots, 0)\,;$ $\| t \|$ denotes the distance from the real number t to the nearest integer and $r(\mathbf{h}) = \prod_{j=1}^{d} \max(|h_j|\,, 1)$. We will prove the following results:

THEOREM 1. – *Let* $\mathbf{x} = (x_1\,, \ldots, x_d)$ *be of approximation type* $< \psi$ *and* $\alpha = [a_0\,; a_1\,; a_2\,, \ldots]$ *an irrational number. Then for every* $\epsilon > 0$ *there exists a constant* $c = c(\mathbf{x}\,, \psi\,, \epsilon\,, \alpha)$ *such that*

$$D_N(\mathbf{x}\, s_\alpha(n)) \leqslant \frac{c}{(\psi^*(L(N)^{1/2-\epsilon}))^{1/d}}$$

for all integers $N \geqslant a_1$. (ψ^* *denotes the inverse function of* ψ.)

Let $\eta \geqslant 1$ be a real number; then we say that $\mathbf{x} = (x_1, \ldots, x_d)$ is of finite approximation type η if (1.3) holds with $\psi(t) = t^{\eta+\delta}$ for every $\delta > 0$. Obviously, $1, x_1, \ldots, x_d$ must be linearly independent over the rationals; conversely, by a famous theorem of W.M. Schmidt [8], under this assumption $\mathbf{x} = (x_1, \ldots, x_d)$ is of finite approximation type $\eta = 1$, if $x_1, \ldots, x_d$ are algebraic numbers. Hence we obtain

COROLLARY. – *Let* $\mathbf{x} = (x_1, \ldots, x_d)$ *be of finite approximation type* η. *Then we have (in the notation of the theorem)*

$$D_N(\mathbf{x}\, s_\alpha(n)) \leqslant c'(\mathbf{x}, \eta, \epsilon, \alpha)\, L(N)^{-\frac{1}{2d\eta}+\epsilon} \quad \text{for every } \epsilon > 0.$$

If $1, x_1, \ldots, x_d$ *are algebraic and linearly independent over the rationals then*

$$D_N(\mathbf{x} \cdot s_\alpha(n)) \leqslant c''(\mathbf{x}, \epsilon, \alpha)\, L(N)^{-\frac{1}{2d}+\epsilon} \quad \textit{for every } \epsilon > 0.$$

At last we consider more exactly the case $d = 1$ and we show that the result of the theorem is, apart from the constant best possible, if we assume that α has bounded continued fraction coefficients.

Remark. – In [9] the authors have established a corresponding result (for dimension $d = 1$) for the sequence $(x \cdot s(q;n))$, where $s(q;n)$ denotes the sum of digits of n in the usual q-adic expansion ($q \geqslant 2$ integral).

THEOREM 2. – *Let* $x \in \mathbf{R}$, c *and* ψ *be such that*

$$\|h \cdot x\| \leqslant \frac{c}{\psi(h)}$$

for infinitely many $h \in \mathbf{N}$, *and* $\alpha = [a_0; a_1, a_2, \ldots]$ *an irrational number with* $a_i \leqslant K$ *for all* i, *then there is a constant* $c_1 = c_1(x, \psi, c, \alpha)$ *such that*

$$D_N\,(x \cdot s_\alpha\,(n)) \geqslant \frac{c_1}{(\psi^*\,(L\,(N)^{1/2})}$$

for infinitely many N.

2. Auxiliary results.

Our main tool for estimating the discrepancy of a sequence is the inequality of Erdös-Turan-Koksma ([6], cf. [7]) :

LEMMA 1. – *Let* $(y_n)_{n=0}^{\infty}$ *denote a sequence of elements of* $\mathbf{R}^d$. *Then for an arbitrary integer* $H \geqslant 1$ *we have*

$$D_N\,(y_n) \leqslant C_d \cdot \left(\frac{1}{H} + \sum_{\substack{h=(h_1,\ldots,h_d)\in \mathbf{Z}^d \\ 0<\max(|h_1|,\ldots,|h_d|)<H}} r(h)^{-1} \left|\frac{1}{N}\sum_{n=0}^{N-1} \exp\,(2\pi i\,(h \cdot y_n))\right|\right),$$

for some constant C_d *only depending on* d (exp $t := e^t$).

A useful instrument in the proof of our Theorem 1 is the following elementary inequality :

LEMMA 2. – *For non-integral* t *and integral* $n \geqslant 2$ *we have*

$$\left|\frac{1-\exp\,(2\pi int)}{1-\exp\,(2\pi it}\right| \leqslant \frac{n}{1+\pi\,\|t\|^2}\,.$$

Proof. – The left-hand side is equal to $\left|\frac{\sin n\pi t}{\sin \pi t}\right|$. Since

$$\left|\frac{\sin 2\pi t}{\sin \pi t}\right| = 2\,|\cos \pi t| \leqslant \frac{2}{2-|\cos \pi t|},$$

the inequality

$$\left|\frac{\sin n\pi t}{\sin \pi t}\right| \leqslant \frac{n}{2-|\cos \pi t|} \tag{2.1}$$

holds for $n = 2$. Suppose that (2.1) holds for some $n \geqslant 2$. Then

$$\left|\frac{\sin (n+1)\pi t}{\sin \pi t}\right| = \left|\cos n\pi t + \frac{\sin n\pi t}{\sin \pi t}\cos \pi t\right|$$

$$\leqslant 1 + \frac{n}{2 - |\cos \pi t|}\,|\cos \pi t| \leqslant \frac{n+1}{2 - |\cos \pi t|};$$

thus, by induction, (2.1) holds for all $n \geqslant 2$. Next we observe that $|\cos \pi t| = \cos \pi \| t \|$. Hence the assertion of Lemma 2 follows from (2.1) and the inequality $\cos \pi \| t \| \leqslant 1 - \pi \| t \|^2$ (which is valid since

$$\cos \pi \| t \| = 1 - \int_0^{\pi \| t \|} \sin u \, du \leqslant 1 - \int_0^{\pi \| t \|} \frac{2}{\pi} u \, du = 1 - \pi \| t \|^2).$$

In order to apply Lemma 1 we have to derive estimates for the exponential sums

$$\sum_{n=0}^{N-1} \exp (2 \pi i \mathbf{h} \cdot \mathbf{x} \, s_\alpha (n)).$$

LEMMA 3. – *Put* $\vartheta_0 = \dfrac{1}{1 + \pi \| \mathbf{h} \cdot \mathbf{x} \|^2}$, $\vartheta = \dfrac{\vartheta_0 + 4}{5}$, *and*

$$S_k = \sum_{0 \leqslant n < q_k} \exp (2 \pi i \, \mathbf{h} \cdot \mathbf{x} \, s_\alpha (n)).$$

If $\mathbf{h} \cdot \mathbf{x}$ *is non-integral, then* $|S_k| \leqslant \vartheta^{k-1} q_k$ *for* $k \geqslant 0$.

Proof. – The inequality holds for $k = 0$ since $0 < \vartheta \leqslant 1$, and is trivial for $k = 1$. For $k \geqslant 2$ we split up the range of summation $0 \leqslant n < q_k = a_k q_{k-1} + q_{k-2}$ into the intervals

$$0 \leqslant n < q_{k-1}, q_{k-1} \leqslant n < 2q_{k-1}, \ldots, (a_k - 1) q_{k-1} \leqslant n < a_k q_{k-1},$$

and $a_k q_{k-1} \leqslant n < a_k q_{k-1} + q_{k-2}$. Since

$$s_\alpha (m q_{k-1} + r) = m + s_\alpha (r)$$

for $m < a_k$ and $r < q_{k-1}$, and $s_\alpha (a_k q_k + r) = a_k + s_\alpha (r)$ for $r < q_{k-2}$, this yields

$$\begin{aligned} S_k &= (1 + \exp(2\pi i\mathbf{h}\cdot\mathbf{x}) + \dots + (\exp(2\pi i\mathbf{h}\cdot\mathbf{x}))^{a_k - 1})\, S_{k-1} \\ &\quad + (\exp(2\pi i\mathbf{h}\cdot\mathbf{x}\, a_k))\, S_{k-2} \\ &= \frac{1 - \exp(2\pi i\mathbf{h}\cdot\mathbf{x}\, a_k)}{1 - \exp(2\pi i\mathbf{h}\cdot\mathbf{x})}\, S_{k-1} + \exp(2\pi i\mathbf{h}\cdot\mathbf{x}\, a_k)\, S_{k-2}\,. \end{aligned} \tag{2.2}$$

Hence, by Lemma 2, we obtain $(k \geqslant 2)$

$$\begin{aligned} |S_k| &\leqslant \vartheta_0\, a_k\, |S_{k-1}| + |S_{k-2}| \quad \text{for} \quad a_k \neq 1\,, \\ |S_k| &\leqslant |S_{k-1}| + |S_{k-2}| \qquad\quad \text{for} \quad a_k = 1\,. \end{aligned} \tag{2.3}$$

If $k = 2$, we have

$$|S_2| \leqslant \vartheta_0\, a_2\, q_1 + 1 \leqslant \frac{1 + \vartheta_0}{2}\,(a_2\, q_1 + 1) = \frac{1 + \vartheta_0}{2}\, q_2 \leqslant \vartheta\, q_2$$

for $a_2 \neq 1$ or $a_1 \neq 1$;

$$|S_2| = |1 + \exp(2\pi i\mathbf{h}\cdot\mathbf{x})| \leqslant 2\,\vartheta_0 \leqslant 2\,\vartheta = \vartheta\, q_2$$

(by (2.2) and Lemma 2) for $a_1 = a_2 = 1$. For $k \geqslant 3$ the assertion of Lemma 3 will be proved by induction. Assume that

$$|S_m| \leqslant \vartheta^{m-1}\, q_m \tag{2.4}$$

for $0 \leqslant m < k$.

Case (i): $a_k \neq 1$. Applying (2.3) we have

$$|S_k| \leqslant \vartheta_0\, a_k\, |S_{k-1}| + |S_{k-2}|.$$

Hence by (2.4)

$$\begin{aligned} |S_k| &\leqslant \vartheta_0\, a_k\, \vartheta^{k-2}\, q_{k-1} + \vartheta^{k-3}\, q_{k-2} \\ &= \vartheta^{k-1}\,(a_k\, q_{k-1} + q_{k-2}) \\ &\quad - \vartheta^{k-3}\,((\vartheta^2 - \vartheta_0\,\vartheta)\, a_k\, q_{k-1} + (\vartheta^2 - 1)\, q_{k-2}) \\ &\leqslant \vartheta^{k-1}\, q_k - \vartheta^{k-3}\,(\vartheta^2 - \vartheta_0\,\vartheta)\, a_k \\ &\quad + (\vartheta^2 - 1))\, q_{k-1} \leqslant \vartheta^{k-1}\, q_k\,; \end{aligned}$$

the least inequality holds since

$$(\vartheta^2 - \vartheta_0 \vartheta) a_k + (\vartheta^2 - 1) \geqslant (\vartheta^2 - \vartheta_0 \vartheta) + (\vartheta^2 - 1)$$

$$= 2\vartheta^2 - (5\vartheta - 4)\vartheta - 1 = (1 - \vartheta)(3\vartheta - 1) \geqslant 0$$

$\left(\text{note that } 1 \geqslant \vartheta \geqslant \vartheta_0 \geqslant \frac{1}{2}\right)$.

Case (ii) : $a_k = 1$ and $a_{k-1} \neq 1$. By a double application of (2.3) we have

$$|S_k| \leqslant (1 + \vartheta_0 a_{k-1}) |S_{k-2}| + |S_{k-3}|.$$

Hence by (2.4)

$$|S_k| \leqslant (1 + \vartheta_0 a_{k-1}) \vartheta^{k-3} q_{k-2} + \vartheta^{k-4} q_{k-3}$$

$$= \vartheta^{k-1} ((1 + a_{k-1}) q_{k-2} + q_{k-3}) - \vartheta^{k-4} ((\vartheta^3 (1 + a_{k-1})$$

$$- \vartheta (1 + \vartheta_0 a_{k-1})) \cdot q_{k-2} + (\vartheta^3 - 1) q_{k-3})$$

$$\leqslant \vartheta^{k-1} q_k - \vartheta^{k-4} (\vartheta^3 (1 + a_{k-1}) - \vartheta (1 + \vartheta_0 a_{k-1})$$

$$+ \vartheta^3 - 1) q_{k-2} \leqslant \vartheta^{k-1} q_k ;$$

the last inequality holds since

$$\vartheta^3 (1 + a_{k-1}) - \vartheta (1 + (5\vartheta - 4) a_{k-1}) + \vartheta^3 - 1$$

$$= (\vartheta^3 - 5\vartheta^2 + 4\vartheta) a_{k-1} + (2\vartheta^3 - \vartheta - 1)$$

$$\geqslant \vartheta (1 - \vartheta)(4 - \vartheta) 2 + (2\vartheta^3 - \vartheta - 1)$$

$$= 4(1 - \vartheta) \left(\vartheta - \frac{3 - \sqrt{5}}{4}\right) \left(\frac{3 + \sqrt{5}}{4} - \vartheta\right) \geqslant 0.$$

Case (iii) : $a_k = a_{k-1} = 1$. By a double application of (2.2) we have

$$|S_k| = |(1 + \exp(2\pi i \mathrm{h} \cdot \mathrm{x})) S_{k-2} + \exp(2\pi i \mathrm{h} \cdot \mathrm{x}) S_{k-3}|$$

$$\leqslant 2\vartheta_0 |S_{k-2}| + |S_{k-3}|$$

(applying Lemma 2 for $n = 2$).

Hence by (2.4)

$$
\begin{aligned}
|S_k| &\leqslant 2\vartheta_0\, \vartheta^{k-3}\, q_{k-2} + \vartheta^{k-4}\, q_{k-3} \\
&\leqslant \vartheta^{k-1}\,(2q_{k-2} + q_{k-3}) - \vartheta^{k-4}\,((2\vartheta^3 - 2\vartheta_0\,\vartheta)\, q_{k-2} \\
&\quad + (\vartheta^3 - 1)\, q_{k-3}) \\
&\leqslant \vartheta^{k-1}\, q_k - \vartheta^{k-4}\,(3\vartheta^3 - 2\vartheta_0\,\vartheta - 1)\, q_{k-2} \leqslant \vartheta^{k-1}\, q_k\,;
\end{aligned}
$$

the last inequality holds since

$$
\begin{aligned}
&3\vartheta^3 - 2\vartheta\,(5\vartheta - 4) - 1 \\
&\qquad = 3\,(1 - \vartheta)\left(\vartheta - \frac{7 - \sqrt{37}}{6}\right)\left(\frac{7 + \sqrt{37}}{6} - \vartheta\right) \geqslant 0\,.
\end{aligned}
$$

Thus, by induction, (2.4) holds for all $m \geqslant 0$ and the proof of Lemma 3 is completed.

LEMMA 4. – *Let* ϑ *be defined as in Lemma 3. If* $\mathbf{h} \cdot \mathbf{x}$ *is non-integral then we have*

$$
\left| \sum_{0 \leqslant n < N} \exp\,(2\pi\, i\mathbf{h} \cdot \mathbf{x}\; s_\alpha\,(n)) \right| \leqslant \frac{1}{\vartheta}\left(\frac{1 + \vartheta}{2}\right)^{L(N)} N\,.
$$

Proof. – Put $u_n = \exp\,(2\pi\, i\mathbf{h} \cdot \mathbf{x}\; s_\alpha\,(n))$ and $N = \sum_{k=0}^{L} \epsilon_k\, q_k$ (compare (1.2)). Splitting up the range of sommation $0 \leqslant n < N$ into the intervals

$$
\begin{aligned}
&0 \leqslant n < \epsilon_L\, q_L\,,\ \epsilon_L\, q_L \leqslant n < \epsilon_L q_L \\
&\qquad + \epsilon_{L-1}\, q_{L-1}\,, \ldots, \epsilon_L q_L + \ldots + \epsilon_1 q_1 \leqslant n < \epsilon_L\, q_L + \ldots + \epsilon_0\, q_0
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\left| \sum_{0 \leqslant n < N} u_n \right| &= \left| \sum_{0 \leqslant n < \epsilon_L q_L} u_n \right. \\
&\quad + \exp\,(2\pi\, i\mathbf{h} \cdot \mathbf{x}\, \epsilon_L) \sum_{0 \leqslant n < \epsilon_{L-1} q_{L-1}} u_n + \ldots \\
&\quad \left. + \exp\,(2\pi\, i\mathbf{h} \cdot \mathbf{x}\,(\epsilon_L + \ldots + \epsilon_1)) \sum_{0 \leqslant n < \epsilon_0 q_0} u_n \right| \\
&\leqslant \sum_{k=0}^{L} \left| \sum_{0 \leqslant n < \epsilon_k q_k} u_n \right|
\end{aligned}
$$

(cf. the first lines of the proof of Lemma 3).

Similarly we derive

$$\left|\sum_{0 \leqslant n < \epsilon_k q_k} u_n\right| = \left|\sum_{0 \leqslant n < q_k} u_n + \ldots + \sum_{(\epsilon_k - 1) q_k \leqslant n < \epsilon_k q_k} u_n\right|$$

$$= \left|(1 + e^{2\pi i \mathbf{h} \cdot \mathbf{x}} + \ldots + e^{2\pi i \mathbf{h} \cdot \mathbf{x} (\epsilon_k - 1)}) \sum_{0 \leqslant n < q_k} u_n\right| \leqslant \epsilon_k \,|\, \mathrm{S}_k \,| \,.$$

Applying Lemma 3 thus yields

$$\left|\sum_{0 \leqslant n < \mathrm{N}} u_n\right| \leqslant \sum_{k=0}^{\mathrm{L}} \epsilon_k \, \vartheta^{k-1} \, q_k \,.$$

In order to complete the proof of Lemma 4 it remains to show

$$\sum_{k=0}^{l} \epsilon_k \, \vartheta^{k-1} \, q_k \leqslant \frac{1}{\vartheta} \left(\frac{1+\vartheta}{2}\right)^{l} \sum_{k=0}^{l} \epsilon_k \, q_k \tag{2.5}$$

for $l = \mathrm{L}$. For $l = 0$ (2.5) holds trivially ; inductively we assume that (2.5) holds for $l < \mathrm{L}$. Then

$$\sum_{k=0}^{\mathrm{L}} \epsilon_k \, \vartheta^{k-1} \, q_k = \sum_{k=0}^{\mathrm{L}-1} \epsilon_k \, \vartheta^{k-1} \, q_k + \epsilon_{\mathrm{L}} \, q_{\mathrm{L}} \, \vartheta^{\mathrm{L}-1}$$

$$\leqslant \frac{1}{\vartheta} \left(\frac{1+\vartheta}{2}\right)^{\mathrm{L}-1} \sum_{k=0}^{\mathrm{L}-1} \epsilon_k \, q_k + \epsilon_{\mathrm{L}} \, q_{\mathrm{L}} \, \vartheta^{\mathrm{L}-1}$$

$$= \frac{1}{\vartheta} \left(\frac{1+\vartheta}{2}\right)^{\mathrm{L}} \sum_{k=0}^{\mathrm{L}} \epsilon_k \, q_k$$

$$- \frac{1}{\vartheta} \left(\left(\frac{1+\vartheta}{2}\right)^{\mathrm{L}} - \left(\frac{1+\vartheta}{2}\right)^{\mathrm{L}-1}\right) \sum_{k=0}^{\mathrm{L}-1} \epsilon_k \, q_k$$

$$+ \epsilon_{\mathrm{L}} \, q_{\mathrm{L}} \left(\vartheta^{\mathrm{L}-1} - \frac{1}{\vartheta} \left(\frac{1+\vartheta}{2}\right)^{\mathrm{L}}\right) \leqslant \frac{1}{\vartheta} \left(\frac{1+\vartheta}{2}\right)^{\mathrm{L}} \sum_{k=0}^{\mathrm{L}} \epsilon_k \, q_k$$

$$- \frac{1}{\vartheta} \left(\left(\frac{1+\vartheta}{2}\right)^{\mathrm{L}} - \left(\frac{1+\vartheta}{2}\right)^{\mathrm{L}-1}\right) q_{\mathrm{L}}$$

$$+ \left(\vartheta^{\mathrm{L}-1} - \frac{1}{\vartheta} \left(\frac{1+\vartheta}{2}\right)^{\mathrm{L}}\right) q_{\mathrm{L}} \leqslant \frac{1}{\vartheta} \left(\frac{1+\vartheta}{2}\right)^{\mathrm{L}} \sum_{k=0}^{\mathrm{L}} \epsilon_k \, q_k \,;$$

the last inequality follows from

$$\vartheta^{L} - \left(\frac{1+\vartheta}{2}\right)^{L} \leqslant \vartheta \left(\frac{1+\vartheta}{2}\right)^{L-1} - \left(\frac{1+\vartheta}{2}\right)^{L}$$

$$= \left(\frac{1+\vartheta}{2}\right)^{L} - \left(\frac{1+\vartheta}{2}\right)^{L-1}.$$

Thus the proof of Lemma 4 is completed.

3. Proof of Theorem 1.

From Lemma 4 and (1.3) we obtain (with $L = L(N)$

$$\begin{aligned} \left|\frac{1}{N}\sum_{0\leqslant n<N} \exp(2\pi i \mathbf{h}\cdot \mathbf{x}\, s_\alpha(n))\right| &\leqslant 2\left(\frac{\vartheta_0+9}{10}\right)^{L} \\ &\leqslant 2\cdot\left(\frac{9c_1+10\psi(r(\mathbf{h}))^2}{10c_1+10\psi(r(\mathbf{h}))^2}\right)^{L} \\ &= 2\left(1-\frac{c_1 L/10c_1+10\psi(r(\mathbf{h}))^2}{L}\right)^{L} \\ &\leqslant 2\exp\left(-\frac{c_1 L}{10c_1+10\psi(r(\mathbf{h}))^2}\right) \end{aligned} \tag{3.1}$$

for some constant $c_1 = c_1(\mathbf{x}, \psi) > 0$; the last inequality holds since $\left(1-\frac{1}{u}\right)^{u} \leqslant \frac{1}{e}$ for $u \geqslant 1$. Hence Lemma 1 yields (for some $c_2 = c_2(d)$)

$$D_N(\mathbf{x}\, s_\alpha(n)) \leqslant c_2\left(\frac{1}{H} + (\log(H+1))^d \exp\left(-\frac{c_1 L}{10c_1+10\psi(H^d)^2}\right)\right) \tag{3.2}$$

where we have used $r(\mathbf{h}) \leqslant H^d$ and

$$\sum_{\substack{\mathbf{h}=(h_1,\dots,h_d)\in \mathbf{Z}^d \\ 0<\max(|h_1|,\dots,|h_d|)\leqslant H}} r(\mathbf{h})^{-1} = \left(1+2\sum_{h=1}^{H}\frac{1}{h}\right)^{d} - 1$$

$$\leqslant 6^d (\log(H+1))^d .$$

We put $H = [(\psi^*(L^{1/2-\epsilon}))^{\frac{1}{d}}]$ for some fixed ϵ with $0 < \epsilon < \frac{1}{2}$ ($[t]$ denotes the greatest integer $\leqslant t$). Let N be sufficiently large so that we can assume $\psi^*(L^{1/2-\epsilon}) \geqslant 2^d$; hence by (3.2)

$$D_N(x\, s_\alpha(n)) \leqslant c_2 \Big(2(\psi^*(L^{1/2-\epsilon}))^{-\frac{1}{d}}$$

$$+ (2\log((\psi^*(L^{1/2-\epsilon}))^{1/d}))^d \times \exp\Big(-\frac{c_1 L}{10c_1 + 10(L^{1/2-\epsilon})^2}\Big)\Big) .$$

Since, for sufficiently large $N \geqslant N_0 = N_0(x, \psi, \epsilon)$

$$\psi^*(L^{1/2-\epsilon})^{-1/d}(\log \psi^*(L^{1/2-\epsilon}))^{-d} \exp\Big(\frac{c_1 L}{10c_1 + 10L^{1-2\epsilon}}\Big)$$

$$\geqslant \psi^*(L^{1/2-\epsilon})^{-1/d-\epsilon} \exp(L^\epsilon)$$

$$\geqslant (L^{1/2-\epsilon})^{-1/d-\epsilon} \exp(L^\epsilon) \geqslant 1 ,$$

we have

$$D_N(x\, s_\alpha(n)) \leqslant c_3(\psi^*(L^{1/2-\epsilon}))^{-1/d} \quad \text{for} \quad N \geqslant N_0 .$$

If $N \geqslant a_1$ then $\psi^*(L^{1/2-\epsilon}) \neq 0$ (since $L = L(N) > 0$). Hence choosing $c \geqslant c_3$ such that

$$D_N(x\, s_\alpha(n)) \leqslant c(\psi^*(L^{1/2-\epsilon}))^{-1/d} \quad (c = c(x, \psi, \epsilon, \alpha)) \tag{3.3}$$

holds for the finitely many N with $a_1 \leqslant N < N_0$, (3.3) is valid for all $N \geqslant a_1$. Thus the proof of the theorem is complete.

4. Proof of Theorem 2.

In the following, we need three further Lemmas:

LEMMA 5. – *For a sequence* $(y_n)_{n=0}^{\infty}$ *in* $\mathbf{R}$, *we have for every* $h \in \mathbf{N}$:

$$D_N(y_n) \geqslant \frac{1}{2\pi \cdot h \cdot N} \cdot \left| \sum_{n=0}^{N-1} \exp(2\pi i h \cdot y_n) \right| .$$

Proof. – This is a special case of the inequality of Koksma ([7], page 142).

LEMMA 6. – *For* $t \in \mathbf{R}$ *and all integers* $n \geqslant 1$ *with* $0 < n \cdot |t| < \frac{1}{4}$ *we have*

$$\left| \frac{1 - \exp(2\pi\, int)}{1 - \exp(2\pi it)} \right| \geqslant n \cdot (1 - (n\pi t)^2) \geqslant 1 - (n\pi t)^2 .$$

Proof. – The assertion is clearly true for $n = 1$. By using the inequality

$$\cos \pi x = 1 - \int_0^{\pi x} \sin u \, du \geqslant 1 - \int_0^{\pi x} u \, du = 1 - \frac{\pi^2 x^2}{2}$$

and because $0 < |t| \leqslant (n-1) \cdot |t| < n \cdot |t| < \frac{1}{4}$ we get for $n \geqslant 2$ by induction :

$$\left| \frac{\sin \pi n t}{\sin \pi t} \right| = \left| \cos (n-1) \cdot \pi t + \frac{\sin (n-1) \pi t}{\sin \pi t} \cdot \cos \pi t \right|$$

$$= \cos (n-1) \pi t + \frac{\sin (n-1) \cdot \pi t}{\sin \pi t} \cdot \cos \pi t$$

$$\geqslant 1 - \frac{((n-1) \cdot \pi t)^2}{2} + (n-1) \cdot (1 - ((n-1) \cdot \pi t)^2)$$

$$\cdot \left(1 - \frac{(\pi t)^2}{2} \right) \geqslant n \cdot \left(1 - (\pi t)^2 \cdot \left(\frac{n}{2} + (n-1)^2 + \frac{1}{2} \right) \right)$$

$$\geqslant n \cdot (1 - (n\pi t)^2) .$$

LEMMA 7. – *Let* $z_k = v_k \cdot e^{2\pi i t_k}$, $k = 1, 2$ *be two complex numbers not equal to zero with* $|t_1 - t_2| < \frac{1}{4}$ *and* $z_1 + z_2 = v \cdot e^{2\pi i t}$; *then*

a) *If we choose* t *such, that* $-\frac{1}{2} \leqslant t_1 - t < \frac{1}{2}$, *then* :

$$|t_1 - t| \leqslant \frac{1}{1 + \dfrac{2v_1}{\pi v_2}} \cdot |t_1 - t_2|$$

b) $$v \geqslant (1 - (2\pi \cdot |t_1 - t_2|)^2) \cdot (v_1 + v_2) .$$

Proof. – a) We have $\operatorname{sgn}(t_1 - t) = -\operatorname{sgn}(t_2 - t)$, so $|t_1 - t_2| = |t_1 - t| + |t - t_2|$ and $|t_i - t| < \frac{1}{4}$.

Since $v_1 \cdot \sin(2\pi |t - t_1|) = v_2 \cdot \sin(2\pi |t - t_2|)$, we have $v_1 \cdot \frac{2}{\pi}(2\pi |t - t_1|) \leqslant v_2 \cdot 2\pi |t - t_2|$ and the assertion a) follows.

b) We have $v = v_1 \cdot \cos(2\pi(t_1 - t)) + v_2 \cdot \cos(2\pi(t_2 - t))$, $|t_i - t| \leqslant |t_1 - t_2|$ and therefore

$$\cos(2\pi(t_i - t)) \geqslant 1 - (2\pi |t_1 - t_2|)^2$$

and the assertion b) follows.

To complete the proof of Theorem 2 we proceed as follows. For a complex $z = v \cdot e^{2\pi i u}$ we define $\arg z := u$, then we take $t > 0$ so small that $K \cdot t \leqslant \frac{1}{\sqrt{648} \cdot \pi}$ and then we first show by induction that for the exponential sums

$$S_n = \sum_{0 \leqslant n < q_n} \exp(2\pi i t \cdot s_\alpha(n))$$

we have

$$\| \arg(S_{n+1}) - \arg(S_n) \| \leqslant \frac{15}{2} \cdot K \cdot t \quad \text{for} \quad n \geqslant 0 .$$

We have $S_0 = 1$, $S_1 = \frac{1 - \exp(2\pi i t a_1)}{1 - \exp(2\pi i t)}$, so $\arg(S_1) = \frac{t \cdot (a_1 - 1)}{2}$

and $\| \arg(S_1) - \arg(S_0) \| < \frac{15}{2} \cdot K \cdot t$.

Now by formula (2.2):

$$S_{k+1} = \frac{1 - \exp(2\pi i t a_{k+1})}{1 - \exp(2\pi i t)} \cdot S_k + \exp(2\pi i t a_{k+1}) \cdot S_{k-1} .$$

If we assume that our assertion is true for $k < n$ then for $k < n$:

$$\left\| \arg\left(\frac{1 - \exp(2\pi i t a_{k+2})}{1 - \exp(2\pi i t)} \cdot S_{k+1}\right) - \arg(\exp(2\pi i t a_{k+2}) \cdot S_k) \right\|$$

$$\leqslant \frac{3t}{2} a_{k+2} + \frac{15}{2} K \cdot t < 9t \cdot K < \frac{1}{4} \qquad (4.1)$$

and therefore especially because of $|z_1 + z_2| \geqslant \max(|z_1|, |z_2|)$ if $|\arg(z_1) - \arg(z_2)| < \frac{1}{4}$, and because of (2.2) and Lemma 6 we have:

$$|S_n| \geqslant \left|\frac{1 - \exp(2\pi it\, a_n)}{1 - \exp(2\pi it)}\right| \cdot |S_{n-1}| \geqslant (1 - (K\pi t)^2) \cdot |S_{n-1}|,$$

and further

$$\left|\frac{1 - \exp(2\pi\, it\, a_{n+1})}{1 - \exp(2\pi\, it)}\right| \cdot |S_n| \geqslant (1 - (K\pi t)^2)^2 \cdot |S_{n-1}|$$

$$> (1 - 2\,(K\pi t)^2) \cdot |(\exp(2\pi\, it\, a_{n+1})) \cdot S_{n-1}|,$$

and so because of (4.1) and Lemma 7a):

$$\|\arg(S_{n+1}) - \arg(S_n)\| \leqslant \frac{3t}{2} \cdot K$$

$$+ \|\arg(S_{n+1}) - \arg\left(\left(\frac{1 - \exp(2\pi\, it\, a_{n+1})}{1 - \exp(2\pi\, it)}\right) \cdot S_n\right)\|$$

$$\leqslant \frac{3t}{2} \cdot K + \frac{1}{1 + \frac{2}{\pi} \cdot (1 - 2\,(K\pi t)^2)} \cdot 9 \cdot K \cdot t\,.$$

Hence, because t is so small that $\frac{2}{\pi} \cdot (1 - 2\,(K\pi t)^2) > \frac{1}{2}$, this is less that $\frac{15\,K}{2} \cdot t$. By Lemma 7b), by (4.1) and by Lemma 6 we have:

$$|S_{n+1}| \geqslant (1 - (18\,\pi K t)^2) \cdot (a_{n+1} \cdot (1 - (\pi K t)^2) \cdot |S_n| + |S_{n-1}|)$$

$$\geqslant (1 - 648 \cdot \pi^2 K^2 \cdot t^2) \cdot (a_{n+1} \cdot |S_n| + |S_{n-1}|)\,.$$

We take $\gamma := 648\,\pi^2 \cdot K^2$ and because $t \leqslant \frac{1}{\sqrt{\gamma}}$ by induction now it is easy to show that

$$\left|\frac{S_n}{q_n}\right| \geqslant (1 - \gamma \cdot t^2)^n \quad \text{for all} \quad n\,.$$

This is true for $n = 0$ and $n = 1$ and so:

$$\left|\frac{S_{n+1}}{q_{n+1}}\right| \geqslant (1 - \gamma \cdot t^2) \cdot \left|\frac{a_{n+1} \cdot S_n + S_{n-1}}{a_{n+1} \cdot q_n + q_{n-1}}\right|$$

$$\geqslant (1 - \gamma t^2) \cdot \left|\frac{a_{n+1} \cdot q_n \cdot (1 - \gamma \cdot t^2)^n + q_{n-1} \cdot (1 - \gamma \cdot t^2)^{n-1}}{a_{n+1} \cdot q_n + q_{n-1}}\right|$$

$$\geqslant (1 - \gamma \cdot t^2)^{n+1} .$$

It we take now h such that

$$\| hx \| \leqslant \frac{c}{\psi(h)} \quad \text{and} \quad \frac{c}{\psi(h)} < \frac{1}{\sqrt{648} \cdot \pi \cdot K},$$

then by Lemma 5 we have:

$$D_{q_n}(x \cdot s_\alpha(n)) \geqslant \frac{1}{2\pi \cdot h \cdot q_n} \cdot \left|\sum_{k=0}^{q_n - 1} \exp(2\pi i h \cdot x \cdot s_\alpha(k))\right|$$

$$\geqslant \frac{1}{2\pi \cdot h \cdot q_n}(1 - \gamma \cdot \| hx \|^2)^n \cdot q_n \geqslant \frac{1}{2\pi \cdot h}\left(1 - \frac{\gamma \cdot c^2}{\psi^2(h)}\right)^n .$$

If we take $N = q_n$ and $n = L(N)$ such that $n - 1 \leqslant \psi^2(h) < n$, then $h \leqslant \psi^*(n^{1/2}) = \psi^*(L(N)^{1/2})$ and

$$D_N \geqslant \frac{1}{2\pi \cdot \psi^*(L(N)^{1/2})} \cdot \left(1 - \frac{\gamma \cdot c^2}{\psi^2(h)}\right)^{\psi^2(h)+1} \geqslant \frac{c_1(x, \psi, c, \alpha)}{\psi^*(L(N)^{1/2})} .$$

Since we can do this for infinitely many h, the proof is finished.

Remark. – Formula (2.2) yields

$$|S_n| \leqslant \left(\frac{2}{\| hx \|} + 1\right)^n ,$$

and so $|S_n| \leqslant \left(\frac{2h}{c} + 1\right)^n$, if $\| hx \| \geqslant \frac{c}{h}$ for all $h = 1, 2, \dots,$ and a $c > 0$. From the proof of Lemma 4 we have

$$\left|\sum_{n=0}^{N=1} e^{2\pi i h x s_\alpha(n)}\right| \leqslant \sum_{k=0}^{L(N)} \epsilon_k (2h + 1)^k .$$

If we choose now $\alpha = [0;1,2,3,4,\dots]$, then for every N sufficiently large, and with absolute constants c_i by Lemma 1 and by taking $H = \frac{c}{4} N^{1/(L+1)}$ we get:

$$D_N(x \cdot s_\alpha(n)) \leqslant c_0 \cdot \left(\frac{1}{H} + \sum_{\substack{h=-H \\ h \neq 0}}^{H} \frac{1}{|h|} \cdot \frac{1}{N} \cdot \sum_{k=0}^{L} (k+1)\left(\frac{2|h|}{c} + 1\right)^k\right)$$

$$\leqslant c_1 \cdot \left(\frac{1}{H} + \frac{1}{N} \cdot \sum_{k=0}^{L} (k+1) \frac{1}{(k+1)} \left(\frac{4}{c} \cdot H\right)^k\right)$$

$$\leqslant c_2 \cdot \left(\frac{1}{H} + \frac{1}{N} \cdot \left(\frac{4}{c} \cdot H\right)^L\right) \leqslant \frac{c_3}{N^{1/(L+1)}}$$

and because of $N \geqslant L!$ this is less than

$$\frac{c_3}{(L!)^{1/(L+1)}} \leqslant \frac{c_4}{L(N)},$$

and therefore it can be seen that the lower bound of Theorem 2 does not hold for every α.

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