

ANNALES DE L'INSTITUT FOURIER

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Embeddability of abstract CR structures and integrability of related systems

Annales de l'institut Fourier, tome 37, n° 3 (1987), p. 131-141

http://www.numdam.org/item?id=AIF_1987__37_3_131_0

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EMBEDDABILITY OF ABSTRACT CR STRUCTURES AND INTEGRABILITY OF RELATED SYSTEMS

by

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1. Introduction.

Let M be a smooth real manifold of dimension N , and \mathcal{V} a subbundle of the complex tangent bundle, \mathbf{CTM} , with $\dim \mathcal{V} = n$. We shall say that \mathcal{V} is *integrable* at a point $p_0 \in M$ if there exists a neighborhood Ω_0 of p_0 and smooth functions $\zeta_1, \dots, \zeta_{N-n}$ defined on Ω_0 with linearly independent differentials and satisfying

$$(1.1) \quad L \zeta_k = 0 \quad \text{in } \Omega_0, \quad k = 1, \dots, N - n,$$

for all $L \in \mathbf{L}_0$, where $\mathbf{L}_0 = C^\infty(\Omega_0, \mathcal{V})$, the space of smooth sections of \mathcal{V} over Ω_0 . In this paper we shall give a criterion for local integrability.

We call \mathcal{V} *formally integrable* if

$$(1.2) \quad [\mathcal{V}, \mathcal{V}] \subset \mathcal{V},$$

i.e. if for any sections $L, L' \in \mathbf{L}$, we have $[L, L'] \in \mathbf{L}$, where $\mathbf{L} = C^\infty(M, \mathcal{V})$. The Frobenius theorem then says that formal integrability implies integrability if \mathcal{V} is real (resp. real analytic), i.e. if \mathbf{L} has a basis of real (resp. real analytic) sections. In the general case it is easy to check by dimension that formal integrability is a necessary condition for integrability.

If, in addition, \mathcal{V} satisfies

$$(1.3) \quad \mathcal{V} \cap \overline{\mathcal{V}} = (0)$$

then \mathcal{V} is called an *abstract CR bundle*, and M an *abstract CR manifold*. In this case we have $N = 2n + \ell$ with $\ell \geq 0$. We say that \mathcal{V} is of *codimension* ℓ .

Key-words: Embeddability – CR structures – Complex Lie algebra.

A submanifold of $\mathbf{C}^{n+\ell}$ is a *generic CR manifold* if it is locally given by $\rho_j = 0, j = 1, \dots, \ell$, with ρ_j real valued, smooth, and satisfying $\partial\rho_1, \dots, \partial\rho_\ell$ linearly independent. It can be easily shown that an abstract CR manifold is integrable at p_0 if and only if near p_0 , M can be embedded as a generic CR manifold in $\mathbf{C}^{n+\ell}$, with the image of \mathcal{V} equal to the induced CR bundle i.e. the bundle whose sections are tangential, antiholomorphic vector fields.

For this reason an integrable CR structure is also called *embeddable* or *realizable*. The first example of a nonembeddable strictly pseudoconvex abstract hypersurface was given by Nirenberg [8]. (See also Jacobowitz-Treves [5]).

Our main result is the following :

THEOREM. — *Let M be a smooth manifold and $\mathcal{V} \subset \mathbf{CTM}$ a subbundle satisfying*

$$[\mathbf{L}, \mathbf{L}] \subset \mathbf{L},$$

where $\mathbf{L} = \mathbf{C}^\infty(M, \mathcal{V})$. Then \mathcal{V} is locally integrable at $p_0 \in M$ if and only if there exist $\Omega_0 \subset M$, an open neighborhood of p_0 in M , and smooth complex vector fields R_1, \dots, R_ℓ defined in Ω_0 spanning a complex Lie algebra i.e.

$$(1.4) \quad [R_i, R_j] = \sum_{k=1}^{\ell} a_{ijk} R_k, \quad a_{ijk} \in \mathbf{C},$$

and satisfying

$$(1.5) \quad [\mathbf{L}_0, R_j] \subset \mathbf{L}_0, \quad j = 1, \dots, \ell,$$

with $\mathbf{L}_0 = \mathbf{C}^\infty(\Omega_0, \mathcal{V})$, and for every $p \in \Omega_0$

$$(1.6) \quad \mathcal{V}_p + \overline{\mathcal{V}}_p + \mathcal{R}_p + \overline{\mathcal{R}}_p = \mathbf{CT}_p \Omega_0,$$

where \mathcal{V}_p is the fiber of \mathcal{V} at p , and \mathcal{R}_p is the span of the R_j at p . More precisely, if \mathcal{V} is integrable, we may find R_j so that $a_{ijk} = 0$ for all i, j, k and replace (1.6) by

$$(1.7) \quad \mathcal{V}_p + \overline{\mathcal{V}}_p \oplus \mathcal{R}_p = \mathbf{CT}_p \Omega_0.$$

For an integrable structure, the existence of vector fields R_j satisfying conditions similar to (1.4) with $a_{ijk} = 0$, (1.5) and (1.6) was proved and used in Baouendi-Treves [2]. However, the proof we

give here is more natural to the embedding and is used to establish the result for the general case.

For the case where \mathcal{V} is an abstract CR structure, the integrability result generalizes a theorem of Jacobowitz [4] where \mathcal{V} is of codimension one, and a theorem of the authors and Treves [1] for the case where the R_j are real independent vector fields. As in [1], the proof of integrability depends, in the CR case, on the Newlander-Nirenberg theorem [6], and in the general case on a corollary of Nirenberg [7], (see also Hörmander [3] and Treves [9]) which states that \mathcal{V} is integrable if $\mathcal{V} + \overline{\mathcal{V}} = \mathbf{CTM}$; we reprove this result by methods in the spirit of this paper.

Remark. – Note that we do not require the vector fields R_j satisfying (1.4), (1.5) and (1.6) to be linearly independent at every point of Ω_0 . However, when \mathcal{V} is integrable, we may choose them linearly independent, and such that the subbundle \mathcal{R} whose sections are spanned by them is totally real i.e.

$$\overline{\mathcal{R}} = \mathcal{R}.$$

2. Proof of the existence of the R_j .

We assume first that \mathcal{V} is CR i.e. $\mathcal{V} \cap \overline{\mathcal{V}} = \{0\}$. Assume M is integrable at p_0 , so that M may be regarded as a submanifold of $\mathbf{C}^{n+\ell}$ given by

$$(2.1) \quad \rho_j = 0, \quad j = 1, \dots, \ell$$

and $\partial\rho_1, \dots, \partial\rho_\ell$ linearly independent.

By relabeling the coordinates in $\mathbf{C}^{n+\ell}$ we may take $(z, w) \in \mathbf{C}^{n+\ell}, w \in \mathbf{C}^\ell$, and assume that

$$(2.2) \quad \rho_w = \begin{pmatrix} \frac{\partial\rho_1}{\partial w_1} & \dots & \frac{\partial\rho_1}{\partial w_\ell} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{\partial\rho_\ell}{\partial w_1} & \dots & \frac{\partial\rho_\ell}{\partial w_\ell} \end{pmatrix}$$

is invertible near the origin. Similarly, we let

$$(2.3) \quad \rho_z = \begin{pmatrix} \frac{\partial \rho_1}{\partial z_1} & \cdots & \frac{\partial \rho_1}{\partial z_n} \\ \vdots & & \vdots \\ \frac{\partial \rho_\ell}{\partial z_1} & \cdots & \frac{\partial \rho_\ell}{\partial z_n} \end{pmatrix}$$

be an $\ell \times n$ matrix. Then a local basis for $C^\infty(M, \mathcal{V})$ is obtained

$$\text{as } (L) = \begin{pmatrix} L_1 \\ \vdots \\ L_n \end{pmatrix}, \text{ with}$$

$$(L) = \left(\frac{\partial}{\partial \bar{z}} \right) - {}^t \rho_z {}^t \rho_w^{-1} \left(\frac{\partial}{\partial \bar{w}} \right),$$

where we have written $\left(\frac{\partial}{\partial \bar{z}} \right)$ for $\begin{pmatrix} \frac{\partial}{\partial \bar{z}_1} \\ \vdots \\ \frac{\partial}{\partial \bar{z}_n} \end{pmatrix}$ and similarly for $\frac{\partial}{\partial \bar{w}}$.

We have

$$(2.4) \quad \text{PROPOSITION. — Set } (R) = \begin{pmatrix} R_1 \\ \vdots \\ R_\ell \end{pmatrix} \text{ where}$$

$$(R) = \left(\frac{\partial}{\partial w} \right) - {}^t \rho_w {}^t \rho_w^{-1} \left(\frac{\partial}{\partial w} \right).$$

Then the R_j are tangent to M , commute, and satisfy (1.5), and (1.7).

Proof. — Since $R_j \rho_k = 0$ by construction, the R_j are tangent to M . To prove (1.7) we observe that since $N = 2n + \ell$, and the L_j, \bar{L}_j and R_k are all linearly independent, the result holds by dimension.

For (1.4) and (1.5) we calculate $[L_j, R_k]$ and $[R_j, R_k]$. Each is again tangent to M , and from the form of the L 's and R 's, they contain only $\frac{\partial}{\partial \bar{w}_k}$, and hence are antiholomorphic. Since the L_j form a basis for the tangential antiholomorphic vector fields to M , each $[L_j, R_k]$ and $[R_j, R_k]$ is a linear combination of the L_j 's with smooth coefficients. These coefficients must be zero, since neither commutator contains a term of the form $\frac{\partial}{\partial \bar{z}_p}$. This proves (1.4) (with $a_{ijk} = 0$) and (1.5), and hence Proposition (2.4). □

We now assume that \mathcal{V} is integrable but not necessarily CR. We shall construct the R_j by adding variables in order to reduce to the case of a CR bundle. Let Ω be a small neighborhood of p_0 in M . First choose a basis L_j of $C^\infty(\Omega, \mathcal{V})$ and coordinates (x, y, t, s) in Ω vanishing at p_0 ,

$$x, y \in \mathbf{R}^r, t \in \mathbf{R}^{n-r}, s \in \mathbf{R}^{\varrho}$$

with $\varrho = N - n - r$, such that

$$(2.5) \quad L_j|_{p_0} = \frac{\partial}{\partial z_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right), \quad 1 \leq j \leq r,$$

and

$$(2.6) \quad L_{j+r}|_{p_0} = \frac{\partial}{\partial t_j}, \quad r+1 \leq j \leq n.$$

We introduce $n-r$ new variables t'_1, \dots, t'_{n-r} and define new vector fields \tilde{L}_j in $\Omega' = \Omega \times \mathbf{R}^{n-r}$ by

$$\tilde{L}_j = L_j, \quad 1 \leq j \leq r,$$

and for $r+1 \leq j \leq n$, \tilde{L}_j is obtained from L_j by replacing $\frac{\partial}{\partial t_j}$ by $\frac{\partial}{\partial t_j} + i \frac{\partial}{\partial t'_j}$. Let \mathcal{V}' be the bundle with sections spanned by the \tilde{L}_j on Ω' . If $\xi_1, \dots, \xi_{r+\varrho}$ is a set of independent solutions for \mathcal{V} , then $\xi_1, \dots, \xi_{r+\varrho}, t_1 + it'_1, \dots, t_{n-r} + it'_{n-r}$ is a set of independent solutions for \mathcal{V}' . Since $\mathcal{V}' \cap \bar{\mathcal{V}}' = \{0\}$, we have proved

(2.7) LEMMA. — \mathcal{V}' is an integrable CR bundle on Ω' .

Let $\tau_j = t_j + it'_j$, $\tau = (\tau_1, \dots, \tau_{n-r})$ and $\zeta = (\zeta_1, \dots, \zeta_{r+\ell})$. The mapping

$$(x, y, t, t', s) \mapsto (\zeta(x, y, t, s), \tau)$$

is an embedding of Ω' onto a CR generic submanifold of $\mathbf{C}^{n+\ell}$. Therefore there exist real smooth functions $\rho_j(Z, \bar{Z})$ in $\mathbf{C}^{n+\ell}$ so that locally the image of Ω' is given by $\rho_j = 0, j = 1, \dots, \ell$, with $\partial\rho_1, \dots, \partial\rho_\ell$ linearly independent. Hence we have for $j = 1, \dots, \ell$

$$(2.8) \quad \rho_j(\zeta(x, y, t, s), \overline{\zeta(x, y, t, s)}, \tau) \equiv 0$$

in Ω' .

We may assume that $\zeta(0) = 0$. If $Z_1, \dots, Z_{n+\ell}$ are the variables in $\mathbf{C}^{n+\ell}$, we write τ_k for $Z_{k+r+\ell}, k = 1, \dots, n-r$.

(2.9) LEMMA. — *We may assume that the ρ_j are independent of t'_k . Also we have for $j = 1, \dots, \ell$ and $k = 1, \dots, n-r$*

$$\frac{\partial \rho_j}{\partial \tau_k}(0) = 0.$$

Proof. — It suffices to differentiate (2.8) with respect to t_k and t'_k , and to use (2.6) and the fact that the ζ_j satisfy the equations

$$L_p \zeta_k = 0 \quad 1 \leq p \leq n, \quad 1 \leq k \leq r + \ell.$$

This proves the lemma. □

Since the ρ_j have independent complex differentials, the matrix

$$\begin{bmatrix} \rho_{1z_1} & \dots & \rho_{1z_{\ell+r}} & \rho_{1\tau_1} & \dots & \rho_{1\tau_{n-r}} \\ \rho_{2z_1} & \dots & \rho_{2z_{\ell+r}} & \rho_{2\tau_1} & \dots & \rho_{2\tau_{n-r}} \end{bmatrix}$$

has rank ℓ , therefore by Lemma (2.9) the submatrix

$$\left[\frac{\partial \rho_j}{\partial z_k} \right]_{1 \leq j \leq \ell, 1 \leq k \leq \ell+r}$$

must have rank ℓ at 0 . Hence we may find new coordinates $(z, w) \in \mathbf{C}^r \times \mathbf{C}^{\ell}$ such that the matrix $\begin{bmatrix} \frac{\partial \rho}{\partial w} \end{bmatrix}$ is invertible at 0 . In these coordinates we may find a basis for \mathfrak{V}' in the form $(\tilde{\mathbf{L}}) = \begin{pmatrix} \tilde{\mathbf{L}}' \\ \tilde{\mathbf{L}}'' \end{pmatrix}$, where

$$(2.11) \quad (\tilde{\mathbf{L}}') = \left(\frac{\partial}{\partial \bar{z}} \right) - {}^t \rho_{\bar{z}} {}^t \rho_{\bar{w}}^{-1} \left(\frac{\partial}{\partial \bar{w}} \right),$$

and

$$(2.12) \quad (\tilde{\mathbf{L}}'') = \left(\frac{\partial}{\partial \bar{\tau}} \right) - {}^t \rho_{\bar{\tau}} {}^t \rho_{\bar{w}}^{-1} \left(\frac{\partial}{\partial \bar{w}} \right),$$

where we use the notation conventions of § 2. Restricting to $t' = 0$ we find a basis (\mathbf{L}) for \mathfrak{V} given by $(\mathbf{L}) = \begin{pmatrix} \mathbf{L}' \\ \mathbf{L}'' \end{pmatrix}$:

$$(2.13) \quad (\mathbf{L}') = \left(\frac{\partial}{\partial \bar{z}} \right) - {}^t \rho_{\bar{z}} {}^t \rho_{\bar{w}}^{-1} \left(\frac{\partial}{\partial \bar{w}} \right),$$

and

$$(2.14) \quad (\mathbf{L}'') = \left(\frac{\partial}{\partial t} \right) - {}^t \rho_t {}^t \rho_{\bar{w}}^{-1} \left(\frac{\partial}{\partial \bar{w}} \right).$$

Now put

$$(\mathbf{R}) = \left(\frac{\partial}{\partial w} \right) - {}^t \rho_w {}^t \rho_{\bar{w}}^{-1} \left(\frac{\partial}{\partial \bar{w}} \right)$$

as before. □

3. Proof of Integrability.

We now assume $\{R_j\}$ exist satisfying (1.4), (1.5), and (1.6) and prove \mathfrak{V} is integrable. First we give a new proof of the following result of Nirenberg [7].

(3.1) PROPOSITION. — *If \mathfrak{V} is a formally integrable subbundle of CTM for which*

$$(3.2) \quad \mathfrak{R}^{\mathfrak{Q}} + \overline{\mathfrak{R}^{\mathfrak{Q}}} = \mathbf{CTM},$$

then $\mathfrak{R}^{\mathfrak{Q}}$ is locally integrable.

Proof. — Let Ω be a small neighborhood of $p_0 \in \mathbf{M}$, and V_1, V_2, \dots, V_n be a commuting basis for $C^\infty(\Omega, \mathfrak{R}^{\mathfrak{Q}})$. After renumbering and multiplication by complex numbers we may assume V_1, \dots, V_r is a maximal set for which $V_1, \dots, V_r, \bar{V}_1, \dots, \bar{V}_r$ is linearly independent at p_0 , and that these, together with $\operatorname{Re} V_j, j > r$, span the section of $\mathbf{CT}\Omega$. Now let $\tilde{\mathfrak{R}}^{\mathfrak{Q}}$ be the bundle over $\Omega \times \mathbf{R}^{n-r}$ whose sections are spanned by $\tilde{V}_j = V_j, 1 \leq j \leq r$, and $\tilde{V}_j = V_j + i \frac{\partial}{\partial t_{r-j}}, j = r + 1, \dots, n$. Then $\tilde{\mathfrak{R}}^{\mathfrak{Q}}$ satisfies the conditions of the Newlander-Nirenberg theorem [6] since

$$\tilde{\mathfrak{R}}^{\mathfrak{Q}} \cap \overline{\tilde{\mathfrak{R}}^{\mathfrak{Q}}} = (0).$$

Hence there exist n solutions $f_1(u, t), \dots, f_n(u, t)$ for $\tilde{\mathfrak{R}}^{\mathfrak{Q}}$, where (u) is a coordinate system near p_0 in Ω vanishing at p_0 , and t is in a neighborhood of 0 in \mathbf{R}^{n-r} . We may assume $f_j(0) = 0, j = 1, \dots, n$.

We shall obtain solutions for $\mathfrak{R}^{\mathfrak{Q}}$ in the form

$$\zeta_k = F_k(f_1, \dots, f_n),$$

where each $F_k(Z)$ is holomorphic and satisfies

$$(3.3) \quad \frac{\partial}{\partial t_j} [F_k(f_1(u, t), \dots, f_n(u, t))] \equiv 0, \quad j = 1, \dots, n - r.$$

We shall prove that there exist F_1, \dots, F_r holomorphic satisfying (3.3) with linearly independent differentials. Indeed, for F holomorphic

$$(3.4) \quad \frac{\partial}{\partial t_j} F(f_1, \dots, f_n) = \sum_{p=1}^n \frac{\partial f_p}{\partial t_j} \frac{\partial F}{\partial Z_p}(f_1, \dots, f_n).$$

Since we may choose a basis for $\tilde{\mathfrak{R}}^{\mathfrak{Q}}$ taking vector fields with coefficients independent of the $t_j, \frac{\partial f_p}{\partial t_j}$ is again a solution for $\tilde{\mathfrak{R}}^{\mathfrak{Q}}$.

Hence there exists a holomorphic function H_{pj} such that

$$(3.5) \quad \frac{\partial f_p}{\partial t_j} = H_{pj}(f_1, \dots, f_n), \quad 1 \leq p \leq n, \quad 1 \leq j \leq n - r.$$

Substituting (3.4) and (3.5) into (3.3) we obtain the system

$$(3.6) \quad \sum_{p=1}^n H_{pj}(Z) \frac{\partial F}{\partial Z_p}(Z) = 0, \quad j = 1, \dots, n - r.$$

Since $df_1, \dots, df_n, d\bar{f}_1, \dots, d\bar{f}_n$ are linearly independent we conclude that the matrix

$$\left(\frac{\partial f_p}{\partial t_j} \right), \quad 1 \leq p \leq n, \quad 1 \leq j \leq n - r,$$

is of rank $n - r$. Therefore by (3.5) the same is true for the matrix (H_{pj}) at the origin. It follows by the Cauchy-Kovalevsky Theorem that there are $n - (n - r) = r$ linearly independent solution F_k of (3.6) near 0. Hence the functions

$$\zeta_k(u) = F_k(f_1(u, t), \dots, f_n(u, t)), \quad 1 \leq k \leq r,$$

provide a system of solutions for $\mathfrak{R}^{\mathfrak{Q}}$, proving integrability. □

We may now complete the proof of the theorem. We assume we are given the R_j satisfying (1.4), (1.5) and (1.6). We let S_1, \dots, S_ϱ be a basis for an abstract complex Lie algebra satisfying the same commutation relations as the R_j i.e.

$$(3.7) \quad [S_i, S_j] = \sum_{k=1}^{\varrho} a_{ijk} S_k.$$

By introducing local exponential coordinates on any corresponding connected complex Lie group we may find coordinates in an open neighborhood \mathfrak{O} of 0 in \mathbf{C}^ϱ near 0 in which we may represent the S_j as holomorphic vector fields with holomorphic coefficients i.e.

$$(3.8) \quad S_j = \sum_{k=1}^{\varrho} a_{jk}(t) \frac{\partial}{\partial t_k}$$

with $t_k = t'_k + it''_k \in \mathbf{C}$ and the matrix (a_{jk}) is invertible. Now we let $R'_j = R_j + S_j$. We claim that the bundle $\tilde{\mathfrak{V}}$ over $\Omega \times \mathfrak{O}$

spanned by \mathfrak{V} , $\{R'_j\}_{1 \leq j \leq \ell}$ and $\left\{ \frac{\partial}{\partial \bar{t}_k} \right\}_{1 \leq k \leq \ell}$ satisfies the condition of Proposition (3.1) for integrability.

Indeed, note that the S_j commute with $\frac{\partial}{\partial \bar{t}_j}$, as well as the R_j and L_0 . Hence

$$(3.9) \quad [R_i + S_i, R_j + S_j] = \sum a_{ijk} (R_k + S_k),$$

which proves that $\tilde{\mathfrak{V}}$ is formally integrable. Also, the span of the $\tilde{R}_j, \bar{\tilde{R}}_j, \frac{\partial}{\partial t_j}$ and $\frac{\partial}{\partial \bar{t}_j}$ is the same as that of the $R_j, \bar{R}_j, \frac{\partial}{\partial t_j}$ and $\frac{\partial}{\partial \bar{t}_j}$. Hence $\tilde{\mathfrak{V}}$ satisfies condition (3.2). By Proposition (3.1) there exist $N - n = N + 2\ell - (n + 2\ell)$ solutions $f_k(u, t', t'')$ which have linearly independent differentials.

Now let $\zeta_k(u) = f_k(u, 0, 0), k = 1, \dots, N - n$. Since the coefficients of elements of L are independent of (t', t'') , it is clear that the ζ_k are solutions of (1.1). It suffices to check that the ζ_k have linearly independent differentials. This will follow if the matrix $\left(\frac{\partial f_i}{\partial u_k} \right)_{\substack{1 \leq j \leq N-n \\ 1 \leq k \leq N}}$ has rank $N - n$. By the linear independence of the

f_k in the (u, t', t'') variables, it suffices to show that $\frac{\partial f_k}{\partial t'_j}$ and $\frac{\partial f_k}{\partial t''_j}$ are linear combinations of $\frac{\partial f_k}{\partial u}$. Since $\frac{\partial f_k}{\partial \bar{t}_i} = 0$ and $(R_j + S_j) f_k = 0, 1 \leq j \leq \ell$, this follows, and hence the proof of the theorem is complete.

□

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Manuscrit reçu le 23 septembre 1986
révisé le 12 novembre 1986.

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