

ANNALES DE L'INSTITUT FOURIER

GENEVIÈVE POURCIN

Deformations of coherent foliations on a compact normal space

Annales de l'institut Fourier, tome 37, n° 2 (1987), p. 33-48

<http://www.numdam.org/item?id=AIF_1987__37_2_33_0>

© Annales de l'institut Fourier, 1987, tous droits réservés.

L'accès aux archives de la revue « Annales de l'institut Fourier » (<http://annalif.ujf-grenoble.fr/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

*Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>*

DEFORMATIONS OF COHÉRENT FOLIATIONS ON A COMPACT NORMAL SPACE

by Geneviève POURCIN

Introduction.

Let X be a normal reduced compact analytic space with countable topology. Let Ω_X^1 be the coherent sheaf of holomorphic 1-forms on X and $\Theta_X = \text{Hom}_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X)$ its dual sheaf. The bracket of holomorphic vector fields on the smooth part of X induces a C -bilinear morphism $m : \Theta_X \times \Theta_X \rightarrow \Theta_X$ (section 1); therefore, for any open subset U of X , m defines a map $m_U : \Theta_X(U) \times \Theta_X(U) \rightarrow \Theta_X(U)$ which is continuous for the usual topology on $\Theta_X(U)$.

We shall study coherent foliations on X (section 1 definition 2), using the definition given in [2], this notion generalizes the notion of analytic foliations on manifolds introduced by P. Baum ([1]) (see also [8]). A coherent foliation on X defines a quotient \mathcal{O}_X -module of Θ_X by a m -stable submodule (condition (i) of definition 2), this quotient being a non zero locally free \mathcal{O}_X -module outside a rare analytic subset of X (condition (ii) of definition (ii)).

Then the set of the coherent foliations on X is a subset of the universal space H of all the quotient \mathcal{O}_X -modules of Θ_X ; the analytic structure of H has been constructed by A. Douady in [4].

The aim of this paper is to prove that the set of the quotient \mathcal{O}_X -modules of Θ_X which satisfy conditions (i) and (ii) of definition 2 is an analytic subspace \mathcal{H} of an open set of H and that \mathcal{H} satisfies a universal property (Theorem 2). Any coherent foliation gives a point of \mathcal{H} , any point of \mathcal{H} defines a coherent foliation but two different points of \mathcal{H} can define the same foliation (cf. section 1, remark 3).

Key-words: Singular holomorphic foliations - Deformations.

In section 2 one proves that, in the local situation, m -stability is an analytic condition on a suitable Banach analytic space (of infinite dimension).

In section 3 we follow the construction of the universal space of A. Douady and we get the analytic structure of \mathcal{H} .

Notations :

- For any analytic space Y and any analytic space not necessarily of finite dimension Z let us denote $p_Z : Z \times Y \rightarrow Y$ the projection.
- For any $O_{Z \times Y}$ -module \mathcal{F} and any $z \in Z$ let us denote $\mathcal{F}(z)$ the O_Y -module which is the restriction to $\{z\} \times Y$ of \mathcal{F} , by definition we have for any $y \in Y$

$$\mathcal{F}(z)_y = \mathcal{F}_{(z,y)} \otimes_{O_{Z,z}} O_{Z,z}/m_z.$$

1. Coherent foliations.

Let X be a reduced connected normal analytic space with countable topology; let Ω_X^1 be the coherent sheaf of holomorphic differential 1-forms on X and

$$(*) \quad \Theta_X = \text{Hom}_{O_X}(\Omega_X^1, O_X)$$

Θ_X is called the tangent sheaf on X . Let S be the singular locus of X , then S is at least of codimension two and the restriction of Θ_X to $X - S$ is the sheaf of holomorphic vector fields on the manifold $X - S$.

Bracket of two sections of Θ_X .

The bracket of two holomorphic vector fields on the manifold $X - S$ is well-defined; recall that, if $z = (z_1, \dots, z_p)$ denotes the coordinates on C^p , if U is an open set in C^p and if a and b are two holomorphic vector fields on U , with

$$a = \sum_{i=1}^p a_i(z) \frac{\partial}{\partial z_i}, \quad b = \sum_{i=1}^p b_i(z) \frac{\partial}{\partial z_i}$$

then we have $[a,b] = c$ with

$$c = \sum_{i=1}^p c_i \frac{\partial}{\partial z_i} \text{ where } c_i = \sum_{j=1}^p \left(a_j \frac{\partial b_i}{\partial z_j} - b_j \frac{\partial a_i}{\partial z_j} \right).$$

Let $m_U : O(U)^p \times O(U)^p \rightarrow O(U)^p$ be the \mathbb{C} -bilinear map which sends $((a_1, \dots, a_p), (b_1, \dots, b_p))$ onto (c_1, \dots, c_p) ; the Cauchy majorations imply the continuity of m_u for the Frechet topology of uniform convergence on compacts of U .

PROPOSITION 1. — *For every open subset U of X the restriction homomorphism*

$$\rho : H^0(U, \Theta_X) \rightarrow H^0(U - U \cap S, \Theta_X)$$

is an isomorphism of Frechet spaces.

Proof. — One knows that ρ is continuous; by the open mapping theorem it is sufficient to prove that ρ is bijective.

Now we may suppose that X is an analytic subspace of an open set V in \mathbb{C}^n ; let I be the coherent ideal sheaf defining X in V ; one has an exact sequence

$$(1) \quad O \rightarrow \Theta_X \rightarrow O_X^n \xrightarrow{\alpha} \text{Hom}_{O_U}(I/I^2, O_X)$$

where the map α is defined by

$$\alpha(a_1, \dots, a_n)(f) = \sum_{i=1}^n a_i \frac{\partial f}{\partial z_i} \Big|_X$$

z_1, \dots, z_n being the coordinates in \mathbb{C}^n .

Because the complex space X is reduced and normal it follows from the second removable singularities theorem two isomorphisms

$$(2) \quad \begin{aligned} O_X(V) &\approx O_X(V - S) \\ I(V) &\approx I(V - S). \end{aligned}$$

Then the proposition 1 follows from (1) and (2). As an immediate consequence of proposition 1 we obtain the following corollary :

COROLLARY AND DEFINITION. — *It exists a unique homomorphism of sheaves of \mathbb{C} -vector spaces*

$$m : \Theta_X \times \Theta_X \rightarrow \Theta_X$$

extending the bracket defined on $X - S$. Therefore, for every open subset U

of X , the induced map

$$m_U : H^0(U, \Theta_X) \times H^0(U, \Theta_X) \rightarrow H^0(U, \Theta_X)$$

is \mathbb{C} -bilinear and continuous for the Fréchet topology on $H^0(U, \Theta_X)$. We call bracket-map the sheaf morphism $m : \Theta_X \times \Theta_X \rightarrow \Theta_X$.

Coherent foliations.

DEFINITION 1. — A coherent \mathcal{O}_X -submodule T of Θ_X is said to be maximal if for any open $U \subset X$, any section $s \in \Theta_X(U)$ and any nowhere dense analytic set A in U

$$s \in T(U - A) \Rightarrow s \in T(U)$$

holds.

Because X is reduced and normal, then locally irreducible, T is maximal if and only if Θ_X/T has no \mathcal{O}_X -torsion.

DEFINITION 2 [2]. — A coherent foliation on X is a coherent \mathcal{O}_X -submodule T of Θ_X such that:

- (i) Θ_X/T is non zero locally free outside a nowhere dense analytic subset of X ;
- (ii) T is a subsheaf of Θ_X stable by the bracket-map;
- (iii) T is maximal.

Remarks. — 1) A coherent foliation induces a classical smooth holomorphic foliation outside a nowhere dense analytic subset of $X - S$.

2) If T is maximal the stability of T by the bracket-map on X is equivalent to the stability of T on $X - A$, for any rare analytic subset A .

3) A coherent foliation on a connected reduced normal complex space X is characterized by a quotient module F of Θ_X , without \mathcal{O}_X -torsion, such that $\ker[\Theta_X \rightarrow F]$ is stable by the bracket-map and which is a non zero locally free \mathcal{O}_X -module outside a rare analytic subset of X .

4) Let T be a coherent \mathcal{O}_X -submodule of Θ_X satisfying conditions (i) and (ii) of definition 2; then T is included in a maximal coherent sheaf \hat{T} which is equal to T outside a rare analytic subset of X ([7] 2.7); the conditions (i) and (ii) are also fulfilled for \hat{T} , hence one can associate to T a maximal foliation on X . But two different T for which (i) and (ii) hold may give the same maximal sheaf \hat{T} .

We suppose X compact.

The purpose of this paper is to put an analytic structure on the set of all subsheaves of Θ_X satisfying conditions (i) and (ii) of Definition 2 (Theorem 2 below), that gives a versal family of holomorphic singular foliations for which a coherent extension exists.

First we have the following proposition :

PROPOSITION 2. — *Let X be an irreducible complex space; let Z be a complex space and F a coherent $\mathcal{O}_{Z \times X}$ -module. Let F be Z -flat.*

Let Z_1 be the set of points $z \in Z$ such that $F(z)$ is a non-zero locally free \mathcal{O}_X -module outside a rare analytic subset of X .

Then Z_1 is an open subset of Z .

Proof. — For every $z \in Z$ let σ_z be the analytic subset of points $x \in X$ where $F(z)$ is not locally free ([3]). Put $z_0 \in Z_1$. The irreducibility of X implies that G_{z_0} is nowhere dense; fix $x_0 \in X - S \cap \sigma_{z_0}$ and denote $r > 0$ the rank of the \mathcal{O}_{X,x_0} -module $F(z_0)$. The Z -flatness of F implies that F is $\mathcal{O}_{Z \times X}$ -free of rank r in an open neighborhood V of (z_0, x_0) . Let U be the projection of V on Z . For any point z of the open set U the Z -flatness of F implies that $F(z, x_0)$ is \mathcal{O}_{X,x_0} -free of rank r ; then the support of the sheaf $F(z)$ contains a neighborhood of x_0 ; hence the irreducibility of X implies

$$\text{support } F(z) = X$$

and the proposition.

For any analytic space S $m_S : p_S^* \Theta_X \times p_S^* \Theta_X \rightarrow p_S^* \Theta_X$ denotes the pull back of m by the projection $p_S : S \times X \rightarrow X$ (i.e. the bracket map in the direction of the fibers of the projection $S \times X \rightarrow S$). Our aim is the proof of the following theorem :

THEOREM 1. — *Let X be a compact connected normal space. There exist an analytic space \tilde{H} and a coherent $\mathcal{O}_{\tilde{H} \times X}$ -submodule \tilde{T} of $p_{\tilde{H}}^* \Theta_X$ such that :*

- (i) $p_{\tilde{H}}^* \Theta_X / \tilde{T}$ is \tilde{H} -flat;
- (iii) \tilde{T} is a $m_{\tilde{H}}$ -stable submodule of $p_{\tilde{H}}^* \Theta_X$;
- (iii) (\tilde{H}, \tilde{T}) is universal for properties (i) and (ii).

As a corollary of proposition 2 and theorem 1 we obtain :

THEOREM 2. — Let X be a compact connected normal space and r a positive integer. There exist an analytic space \mathcal{H} and a coherent $O_{\mathcal{H} \times X}$ -submodule \mathcal{T} of $p_{\mathcal{H}}^* \Theta_X$ such that :

- (i) $p_{\mathcal{H}}^* \Theta_X / \mathcal{T}$ is \mathcal{H} -flat;
- (ii) \mathcal{T} is $m_{\mathcal{H}}$ -stable and for any $h \in \mathcal{H} \Theta_X / \mathcal{T}(h)$ is a locally free O_X -module of rank r outside a rare analytic subset of X ;
- (iii) $(\mathcal{H}, \mathcal{T})$ is universal, i.e. for any analytic space S and any coherent $O_{S \times X}$ -submodule \mathcal{F} of $p_S^* \Theta_X$ such that
 - $p_S^* \Theta_X / \mathcal{F}$ is S -flat;
 - \mathcal{F} is m_S -stable and for any $s \in S$ $\Theta_X / \mathcal{F}(s)$ is a locally free O_X -module of rank r outside a rare analytic subset of X then it exists a unique morphism $f: S \rightarrow \mathcal{H}$ satisfying

$$(f \times I_X)^*(p_{\mathcal{H}}^* \Theta_X / \mathcal{T}) = p_S^* \Theta_X / \mathcal{F}.$$

We shall use the following theorem and Douady ([4]) :

THEOREM. — Let X be a compact analytic space and \mathcal{E} a coherent O_X -module; there exist an analytic space H and a quotient $O_{H \times X}$ -module \mathcal{R} of $p_H^* \mathcal{E}$ such that :

- (i) \mathcal{R} is H -flat;
- (ii) for any analytic space S and any quotient $O_{S \times H}$ -module \mathcal{F} of $p_S^* \mathcal{E}$ which is S -flat, it exists a unique morphism $f: S \rightarrow H$ satisfying

$$(f \times I_X)^* \mathcal{R} = \mathcal{F}.$$

2. Local deformations.

One uses notations and results of [4]; the notions of infinite dimensional analytic spaces, called Banach analytic spaces, and of anaflatness are defined respectively in ([4] § 3) and in ([4] § 8).

In this section we fix an open subset U of C^n , two compact polycylinders of non-empty interior K and K' satisfying

$$K' \subset \mathring{K} \subset K \subset U$$

and a reduced normal analytic subspace X of U . Let $B(K)$ be the Banach algebra of those continuous functions on K which are analytic on the interior \mathring{K} of K ; one defines $B(K')$ in an analogous way.

For every coherent sheaf \mathcal{F} on U , one knows that it exists finite free resolutions of \mathcal{F} in a neighborhood of K ; for such a resolution

$$(L.) \quad O \rightarrow L_n \rightarrow L_{n-1} \rightarrow \cdots \rightarrow L_0$$

let us consider the complex of Banach spaces

$$B(K, L.) = B(K) \otimes_{O(K)} H^0(K, L.)$$

and the vector space

$$B(K, \mathcal{F}) = \text{coker } [B(K; L_1) \rightarrow B(K, L_0)].$$

DEFINITION 1 ([4] §7, [5]). — K is \mathcal{F} -privileged if and only if it exists a finite free resolution $L.$ of \mathcal{F} on a neighborhood of K such that the complex $B(K, L.)$ is direct exact.

Then this is true for every finite free resolution; therefore $B(K, \mathcal{F})$ is a Banach space which does not depend of the resolution; \mathcal{F} -privileged polycylinders give fundamental systems of neighborhoods at every point of U . For a more geometric definition of privilege, the reader can refer to ([6]).

In the following, we always suppose that the two polycylinders K and K' are Θ_X -privileged, Θ_X being the tangent sheaf defined by 1 – (*).

Let G_K be the Banach analytic space of those $B(K)$ -submodules Y of $B(K, \Theta_X)$ (or equivalently of quotient modules) for which it exists an exact sequence of $B(K)$ -modules

$$O \rightarrow B(K)^{r_n} \rightarrow \cdots \rightarrow B(K)^{r_0} \rightarrow B(K, \Theta_X) \rightarrow B(K, \Theta_X)/Y \rightarrow O$$

which is a direct sequence of Banach vector spaces.

A universal sheaf R_X on $G_X \times K$ is constructed in [4]; R_K satisfies the following proposition :

PROPOSITION 1 ([4] § 8 n° 5). — (i) R_K is G_K -anaflat.

(ii) For every Banach analytic space Z and for every Z -anaflat quotient \mathcal{F} of $p_Z^* \Theta_X$ it exists a natural morphism $\varphi : Z \rightarrow G_K$ such that

$$(\varphi \times I_K)^* R_K = \mathcal{F}_{S \times K}.$$

Recall that the Z -anaflatness generalizes to the infinite dimensional space Z the notion of flatness; pull back preserves anaflatness.

Let $G_{K,K'}$ be the set of the $B(K)$ -submodules E of $B(K,\Theta_X)$, element of G_K , such that $E \otimes_{B(K)} B(K')$ gives an element of $G_{K'}$.

PROPOSITION 2. — (i) $G_{K,K'}$ is an open subset of G_K .

(ii) Let \mathcal{R} be the pull back of R_K by the inclusion $G_{K,K'} \hookrightarrow G_K$. Then the map from $G_{K,K'}$ to $G_{K'}$ which maps every $B(K)$ -module E element of $G_{K,K'}$ onto the $B(K')$ -module $E \otimes_{B(K)} B(K')$ is given by a unique morphism

$$\rho_{K,K'} : G_{K,K'} \rightarrow G_{K'}$$

satisfying

$$\rho_{K,K'}^* R_{K'} = \mathcal{R}.$$

Proof. — Proposition 2 follows from ([4] 14 prop. 4).

Let $\rho_1 : B(K,\Theta_X) \times B(K,\Theta_X) \rightarrow \Theta_X(\dot{K}) \times \Theta_X(\dot{K})$ and

$$\rho_2 : \Theta_X(\dot{K}) \rightarrow B(K',\Theta_X)$$

be the restriction homomorphisms and

$$m : \Theta_X(\dot{K}) \times \Theta_X(\dot{K}) \rightarrow \Theta_X(\dot{K})$$

the bracket map.

Let

$$m_{K,K'} : B(K,\Theta_X) \times B(K,\Theta_X) \rightarrow B(K',\Theta_X)$$

be the continuous C -bilinear map defined by

$$m_{K,K'} = \rho_2 \circ m \circ \rho_1.$$

DEFINITION 2. — A $B(K)$ -submodule Y of $B(K,\Theta_X)$ is said to be $m_{K,K'}$ -stable if it verifies :

- (i) Y is an element of $G_{K,K'}$,
- (ii) for every f and g in Y one has

$$m_{K,K'}(f,g) \in \rho_{K,K'}(Y).$$

Then, if \mathcal{T} is a m -stable Θ_X -submodule of Θ_X such that K and K' are \mathcal{T} -privileged, $B(K,\mathcal{T})$ is $m_{K,K'}$ -stable ; the converse is not necessarily true ; however we have the following proposition :

PROPOSITION 3. — *Let Y be a $m_{K,K}$ -stable $B(K)$ -submodule of $B(K, \Theta_X)$; then Y defines in a natural way a coherent \mathcal{O}_X -submodule of Θ_X on \mathring{K} , the restriction to \mathring{K}' of which is m -stable (i.e. stable by the bracket-map).*

Proof. — Let B_Y be the privileged B_K -module given by Y ([6]); the restriction to \mathring{K} of B_Y is a coherent sheaf; therefore one has ([6] th. 2.3 (ii) and prop. 2.11)

$$Y = \mathring{H}(K, B_Y)$$

and the restriction homomorphism

$$i : Y = H^0(K, B_Y) \rightarrow H^0(\mathring{K}, B_Y)$$

is injective and has dense image; therefore the restriction $B_{Y|K}$ is a submodule of Θ_X ([4] § 8 lemme 1(b)), hence $H^0(\mathring{K}', B_Y)$ is a closed subspace of the Frechet space $H^0(\mathring{K}', \Theta_X)$.

Let us show that $m_{K,K'}$ induces a \mathbb{C} -bilinear continuous map

$$\mathring{m} : H^0(\mathring{K}, B_Y) \times H^0(\mathring{K}, B_Y) \rightarrow H^0(\mathring{K}', B_Y).$$

Take t_1, t_2 two elements of $H^0(\mathring{K}, B_Y)$ and (t_1^n) and (t_2^n) two sequences of elements of Y with

$$\lim_{n \rightarrow \infty} t_i^n = t_i, \quad i = 1, 2.$$

Because the bracket-map $m : H^0(\mathring{K}, \Theta_X) \times H^0(\mathring{K}, \Theta_X) \rightarrow H^0(\mathring{K}, \Theta_X)$ is continuous one has

$$\lim_{n \rightarrow \infty} m(t_{1|K}^n, t_{2|K}^n) = m(t_1, t_2) \in H^0(\mathring{K}, \Theta_X).$$

Therefore the $m_{K,K}$ -stability of Y implies for every m

$$m_{K,K'}(t_1^n, t_2^n) \in B(K', B_Y) \subset H^0(\mathring{K}', B_Y)$$

then $m(t_1, t_2)|_{\mathring{K}'} \in H^0(\mathring{K}', B_Y)$ follows.

In order to prove the proposition it is sufficient to remark that, for every polycylinder $K'' \subset \mathring{K}'$, the restriction homomorphism

$$H^0(\mathring{K}', B_Y) \rightarrow H^0(\mathring{K}'', B_Y)$$

has a dense image. Q.E.D.

Recall some properties of infinite dimensional spaces : let V be an open subset of a Banach C -vector space; let F be a Banach vector space and $f: V \rightarrow F$ an analytic map. Let \mathcal{X} the Banach analytic space defined by the equation $f = 0$; \mathcal{X} is a local model of general Banach analytic space; the morphisms from \mathcal{X} into a Banach vector space G extend locally in analytic maps on open subsets of V ; for such a morphism $\varphi: \mathcal{X} \rightarrow G$ the equation $\varphi = 0$ defines in a natural way a Banach analytic subspace of \mathcal{X} ; the morphisms from a Banach analytic space \mathcal{Y} into \mathcal{X} are exactly the morphisms $\psi: \mathcal{Y} \rightarrow V$ such that $f \circ \psi = 0$.

PROPOSITION 4. — *Let $S_{K,K'}$ be the subset of elements of $G_{K,K'}$ which are $m_{K,K'}$ -stable. Then $S_{K,K'}$ is a Banach analytic subspace of $G_{K,K'}$.*

Proof. — Let $Y_0 \in S_{K,K'}$ and $Y'_0 = \rho_{K,K'}(Y_0)$; let G_0 (resp. G'_0) a closed C -vector subspace of $B(K, \Theta_X)$ (resp. $B(K', \Theta_X)$) which is a topological supplementary of Y_0 (resp. Y'_0). Let U_0 (resp. U'_0) the set of closed C -vector subspaces of $B(K, \Theta_X)$ (resp. $B(K', \Theta_X)$) which are topological supplementaries of G_0 (resp. G'_0); we identify U_0 and $L(Y_0, G_0)$, hence $U_0 \cap G_K$ is a Banach analytic subspace of U_0 ([4] § 4).

For every Y in U_0 one denotes $p_Y: B(K, \Theta_Y) = Y \oplus G_0 \rightarrow G_0$ the projection and $j_Y: Y_0 \rightarrow Y \subset B(K, \Theta_X)$ the reciprocal map of the restriction to Y of the projection $B(K, \Theta_X) = Y_0 \oplus G_0 \rightarrow Y_0$.

Then the two maps

$$\begin{aligned} p^K: G_K &\rightarrow L(B(K, \Theta_X), G_0) \\ j^K: G_K &\rightarrow L(Y_0, B(K, \Theta_X)) \end{aligned}$$

defined by $p^K(Y) = p_Y$ and $j^K(Y) = j_Y$ are induced by morphisms ([4] § 4, n° 1); associated to the polycylinder K' we have in the same way morphisms $p^{K'}$ and $j^{K'}$. Put $W_0 = G_{K,K'} \cap U_0 \cap \rho_{K,K'}^{-1}(U'_0)$; W_0 is an open subset of $G_{K,K'}$. Let be

$$\varphi_1 = p^{K'} \circ \rho_{K,K'}: W_0 \rightarrow L(B(K', \Theta_X), G'_0)$$

and $\Delta: G_K \rightarrow L(Y_0 \underset{\pi}{\otimes} Y_0, B(K', \Theta_X))$ the morphism defined by

$$\Delta(Y) = m_{K,K'} \circ (j_Y \times j_Y).$$

Let be $\varphi_2 = \Delta \circ j^K: W_0 \rightarrow L(Y_0 \underset{\pi}{\otimes} Y_0, B(K', \Theta_X))$; φ_1 and φ_2 are

morphisms; let

$$\varphi : W_0 \rightarrow L(Y_0 \underset{\pi}{\otimes} Y_0, G'_0)$$

be the morphism defined by

$$\varphi(Y) = \varphi_2(Y) \circ \varphi_1(Y).$$

We have $W_0 \cap S_{K,K'} = \varphi^{-1}(0)$, hence $S_{K,K'} \cap W_0$ is a Banach analytic subspace of W_0 ; following ([4] § 4, n° 1 (i) and (ii)) one easily proves that the analytic structures obtained in the different charts of G_K and $G_{K'}$ patch together in an analytic structure on $S_{K,K'}$; that proves proposition 4.

Remark 1. — With the previous notations the morphisms of Banach analytic spaces $g : Z \rightarrow S_{K,K'} \cap W_0$ are the morphisms $g : Z \rightarrow W_0$ satisfying $\varphi \circ g = 0$.

Let $i : S_{K,K'} \rightarrow G_K$ be the inclusion and $R_{K,K'}$ the pullback of R_K by i ; $R_{K,K'}$ is $S_{K,K'}$ -anaflat; by construction $R_{K,K'}$ is a quotient of $p_{S_{K,K'}}^* \Theta_X$, then put

$$R_{K,K'} = p_{S_{K,K'}}^* \Theta_X / T_{K,K'}.$$

By anaflatness one obtains for every $s \in S_{K,K'}$ an exact sequence of coherent sheaves on \hat{K} :

$$0 \rightarrow T_{K,K'}(s) \rightarrow \Theta_X \rightarrow R_{K,K'}(s) \rightarrow 0.$$

From the definition of the analytic structure of $S_{K,K'}$ and from proposition 3 one deduces the following theorem :

THEOREM 3. — (i) For every $s \in S_{K,K'}$ the restriction to \hat{K}' of the coherent subsheaf $T_{K,K'}(s)$ of Θ_X is stable by the bracket-map.

(ii) For every Banach analytic space Z and every quotient $\mathcal{F} = p_Z^* \Theta_X / T$ of $p_Z^* \Theta_X$ by a $O_{Z \times X}$ -submodule T such that

- \mathcal{F} is Z -anaflat.
- T is m_Z -stable and for any $z \in Z$ the polycylinders K et K' are $\mathcal{F}(z)$ -privileged;

then the unique morphism $g : Z \rightarrow G_K$ satisfying

$$(g \times I_K)^* R_K = \mathcal{F}$$

factorizes through $S_{K,K'}$ (i.e. it exists a unique morphism $f : Z \rightarrow S_{K,K'}$ with $r \circ f = g$).

Remark 2. — We don't know if the restriction of $R_{K,K'}$ to $S_{K,K'} \times \hat{K}'$ is $m_{S_{K,K'}}$ -stable; but if S is a finite dimensional analytic space then the pull back of $R_{K,K'}$ by any morphism $S \rightarrow S_{K,K'}$ is m_S -stable.

3. Proof of theorem 1.

In this section X denotes a compact reduced normal space and Θ_X its tangent sheaf. Let H be the universal space of quotient O_X -modules of Θ_X and \mathcal{R} the H -flat universal sheaf on $H \times X$ ([4]). Put $\mathcal{R} = p_H^* \Theta_X / \mathcal{C}$, \mathcal{C} being a coherent submodule of $p_H^* \Theta_X$; for any $h \in H$ $\mathcal{C}(h)$ is a coherent submodule of Θ_X . We shall construct the space \tilde{H} as an analytic subspace of an open subset of H .

1. Refining of a privileged « cuirasse ».

Let M be a Θ_X -privileged « cuirasse » ([4] § 9, n° 2); M is given by,

- (i) a finite family $(\varphi_i)_{i \in I}$ of charts of X , i.e. for every $i \in I$ φ_i is an isomorphism from an open set $X_i \subset X$ onto a closed analytic subspace of an open set U_i in C^{n_i} ,
- (ii) for every $i \in I$ a Θ_X -privileged polycylinder $K_i \subset U_i$ (i.e. a $\varphi_{i*} \Theta_X$ -privileged polycylinder), and an open set $V_i \subset X_i$ satisfying

$$\bar{V}_i \subset \varphi_i^{-1}(\hat{K}_i) \subset X_i$$

$$X = \bigcup_{i \in I} V_i$$

- (iii) for every $(i,j) \in I \times J$ a chart φ_{ij} defined on $X_i \cap X_j$ with values in an open $U_{ij} \subset C^{n_{ij}}$ and a finite family $(K_{ij\alpha})$ of Θ_X -privileged polycylinders in U_{ij} such that conditions

$$\begin{aligned} \bar{V}_i \cap \bar{V}_j &\subset \bigcup_{\alpha} \psi_{ij}^{-1}(K_{ij\alpha}) \\ \varphi_{ij}^{-1}(K_{ij\alpha}) &\subset \varphi_i^{-1}(\hat{K}_i) \cap \varphi_j^{-1}(\hat{K}_j) \end{aligned}$$

are fulfilled.

As in ([4]) let us denote H_M the open subset of the elements F of H for which M is F -privileged (i.e. all the polycylinders K_i , $K_{ij\alpha}$ are F -privileged); we shall construct \tilde{H} as union of open subsets $\tilde{H} \cap H_M$.

— For any Θ_X -privileged polycylinder K let us denote G_K (§ 2) the Banach analytic space of quotients of $B(K, \Theta_X)$ with finite direct resolution.

For every $i \in I$ let G_i be the open subset of G_{K_i} on which, for any α , the restriction homomorphisms $B(K_i) \rightarrow B(K_{ij\alpha})$ induce morphisms $G_i \rightarrow G_{K_{ij\alpha}}$. The Douady construction of H_M gives a natural injective morphism

$$i: H_M \rightarrow \prod_{i \in I} G_i.$$

DEFINITION 5. — *A refining of the « cuirasse » M is given by a family $(K'_i)_{i \in I}$ of polycylinders satisfying :*

- (i) *for every i $\phi_i(V_i) \subset K'_i \subset K_i \subset \hat{K}_i$,*
- (ii) *for every i, j, α $\phi_{ij}^{-1}(K_{ij\alpha}) \subset \phi_i^{-1}(\hat{K}_i) \cap \phi_j^{-1}(\hat{K}_j)$,*
- (iii) *for every i K'_i is Θ_X -privileged.*

We denote by $M((K'_i))$ such a refining; for any coherent sheaf \mathcal{F} on X we shall say that $M((K'_i))$ is \mathcal{F} -privileged if M is \mathcal{F} -privileged and if, for every i , K'_i is \mathcal{F} -privileged.

LEMMA 1. — (i) *Let \mathcal{F} be a coherent sheaf such that M is \mathcal{F} -privileged; then it exists a \mathcal{F} -privileged refining of M .*

(ii) *Let $M((K'_i))$ a refining of M ; then the set of quotient \mathcal{F} of Θ_X such that $M((K'_i))$ is \mathcal{F} -privileged is open in H_M .*

Proof. — (i) follows from ([4] § 7, n° 3 corollary of prop. 6) and (ii) is an immediate consequence of flatness and privilege.

2. Now we fix a Θ_X -privileged « cuirasse » $M = M(I, (K_i), (V_i), (K_{ij\alpha}))$ and a Θ_X -privileged refining $M((K'_i))$ of M .

LEMMA 2. — *Let H'_M be the subset of H_M the points of which are quotients Θ_X/T satisfying :*

- (i) $M((K'_i))$ is Θ_K/T -privileged,
- (ii) T is a subsheaf of Θ_X stable by the bracket-map.

Then H'_M is an analytic subspace of an open subset of H_M .

Proof. — Using notations of section 2 one puts for every $i \in I$

$$G'_i = G_{K_i, K'_i} \cap G_i$$

G'_i is an open subset of G_i and G_{K_i} ; put $S_i = S_{K_i, K'_i} \cap G'_i$.

One knows that the category of Banach analytic spaces has finite products, kernel of double arrows and hence fiber products (for all this notions the reader can refer to ([4] § 3, n° 3). Then $\prod_{i \in I} S_i$ is a Banach analytic subspace of $\prod_{i \in I} G'_i$; since $\prod_{i \in I} G'_i$ is an open subset of $\prod_{i \in I} G_i$ it follows from (§ II Theorem 3)

$$H'_M = H_M \times_{\prod_{i \in I} G_i} \prod_{i \in I} S_i$$

and the lemma is proved.

— Now let R'_M (resp. T'_M) be the pull back of \mathcal{R} (resp. \mathcal{T}) by the inclusion morphism $H'_M \times X \rightarrow H \times X$; R'_M is the quotient of $p_{H'_M}^* \Theta_X$ by T'_M (the sheaves T'_M and $\ker [p_{H'_M}^* \Theta_X \rightarrow R'_M]$ are H'_M -flat and equal on the fibers $\{h\} \times X$).

LEMMA 3. — T'_M is a $m_{H'_M}$ -stable submodule of $p_{H'_M}^* \Theta_X$.

The proof follows immediately of the remark 2 of paragraph 2 and of

$$X = \bigcup_{i \in I} V_i = \bigcup_{i \in I} \varphi_i^{-1}(\hat{K}_i).$$

— Using the universal property of H_M , Theorem 3 § 2 and the commutative diagram

$$\begin{array}{ccc} H'_M \times X & \rightarrow & H_M \times X \\ \downarrow & & \downarrow \\ \left(\prod_{i \in I} G'_i \right) \times X & \rightarrow & \left(\prod_{i \in I} G_i \right) \times X \end{array}$$

we obtain the following proposition :

PROPOSITION 1. — Let Z be an analytic space and T_Z a coherent subsheaf of $p_Z^* \Theta_X$ satisfying :

- (i) $p_Z^* \Theta_X / T_Z$ is Z -flat.
- (ii) For every $z \in Z$ the cuirasse $M((K_i))$ is $\Theta_X / T_Z(z)$ -privileged.
- (iii) T_Z is a m_Z -stable submodule of $p_Z^* \Theta_X$.

Then the unique morphism $g : Z \rightarrow H$ such that

$$(g \times I_X)^* \mathcal{R} = p_Z^* \Theta_X / T_Z$$

factorizes through H'_M and verifies

$$(g \times I_X)^* T'_M = T_Z.$$

3. End of the proof of Theorem 1.

Notations are those of the previous proposition; the unicity of g implies the unicity of its factorization through the subspace H'_M of H . Hence, when the refinings of a given M are varying, one obtains analytic spaces H'_M which patch together in an analytic subspace of an open subset of H_M .

When the « cuirasse » M varies in the family of all the Θ_X -privileged « cuirasse » the spaces H_M form an open covering of H ; then the universal property of the H_M 's implies that $\tilde{H} = \bigcup_M H'_M$ is an analytic subspace of an open subset of H . Theorem 4 is proved.

Remark. — More generally if X is not compact, let Θ be a coherent sheaf on X and $m : \Theta \times \Theta \rightarrow \Theta$ a sheaf morphism inducing for each open set U a continuous C -bilinear map $m_U : \Theta(U) \times \Theta(U) \rightarrow \Theta(U)$; let H be the Douady space of the coherent quotients of Θ with compact support ([4]). We get a universal analytic structure on the subset of those quotients which are m -stable.

BIBLIOGRAPHY

- [1] P. BAUM, Structure of foliation singularities, *Advances in Math.*, 15 (1975), 361-374.
- [2] G. BOHNHORST and H. J. REIFFEN, Holomorphe blätterungen mit singularitäten, *Math. Gottingensis*, heft 5 (1985).
- [3] H. CARTAN, *Faisceaux analytiques cohérents*, C.I.M.E., Edizioni Cremonese, 1963.
- [4] A. DOUADY, Le problème des modules pour les sous-espaces analytiques..., *Ann. Inst. Fourier*, XVI, Fasc. 1 (1966), 1-96.
- [5] B. MALGRANGE, Frobenius avec singularités-codimension 1, *Pub I.H.E.S.*, n° 46 (1976).

- [6] G. POURCIN, Sous-espaces privilégiés d'un polycylindre, *Ann. Inst. Fourier*, XXV, Fasc. 1 (1975), 151-193.
- [7] Y. T. SIU et G. TRAUTMANN, Gap-sheaves and extension of coherent analytic subsheaves, *Lect. Notes*, 172 (1971).
- [8] T. SUWA, Singularities of complex analytic foliations, *Proceedings of Symposia in Pure mathematics*, Vol. 40 (1983), Part. 2.

Manuscrit reçu le 12 mars 1986.

Geneviève POURCIN,
Département de Mathématiques
Faculté des Sciences
2 Bd Lavoisier
49045 Angers Cedex (France).