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Deformations of coherent foliations on a compact normal space


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DEFORMATIONS OF COHERENT FOLIATIONS
ON A COMPACT NORMAL SPACE

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Introduction.

Let $X$ be a normal reduced compact analytic space with countable
topology. Let $\Omega^1_X$ be the coherent sheaf of holomorphic 1-forms on $X$ and
$\Theta_X = \text{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ its dual sheaf. The bracket of holomorphic vector
fields on the smooth part of $X$ induces a $\mathbb{C}$-bilinear morphism
$m : \Theta_X \times \Theta_X \to \Theta_X$ (section 1); therefore, for any open subset $U$ of $X$, $m$
defines a map $m_U : \Theta_X(U) \times \Theta_X(U) \to \Theta_X(U)$ which is continuous for the
usual topology on $\Theta_X(U)$.

We shall study coherent foliations on $X$ (section 1 definition 2), using
the definition given in [2], this notion generalizes the notion of analytic
foliations on manifolds introduced by P. Baum ([1]) (see also [8]). A
coherent foliation on $X$ defines a quotient $\mathcal{O}_X$-module of $\Theta_X$ by a $m$-stable
submodule (condition (i) of definition 2), this quotient being a non zero
locally free $\mathcal{O}_X$-module outside a rare analytic subset of $X$ (condition (ii) of
definition (ii)).

Then the set of the coherent foliations on $X$ is a subset of the universal
space $\mathcal{H}$ of all the quotient $\mathcal{O}_X$-modules of $\Theta_X$; the analytic structure of $\mathcal{H}$
has been constructed by A. Douady in [4].

The aim of this paper is to prove that the set of the quotient $\mathcal{O}_X$-modules
of $\Theta_X$ which satisfy conditions (i) and (ii) of definition 2 is an analytic
subspace $\mathcal{H}$ of an open set of $\mathcal{H}$ and that $\mathcal{H}$ satisfies a universal property
(Theorem 2). Any coherent foliation gives a point of $\mathcal{H}$, any point of $\mathcal{H}$
defines a coherent foliation but two different points of $\mathcal{H}$ can define the
same foliation (cf. section 1, remark 3).

Key-words: Singular holomorphic foliations - Deformations.
In section 2 one proves that, in the local situation, $m$-stability is an analytic condition on a suitable Banach analytic space (of infinite dimension).

In section 3 we follow the construction of the universal space of A. Douady and we get the analytic structure of $\mathcal{H}$.

Notations:

- For any analytic space $Y$ and any analytic space not necessarily of finite dimension $Z$ let us denote $p_Z : Z \times Y \to Y$ the projection.

- For any $\mathcal{O}_{Z,Y}$-module $\mathcal{F}$ and any $z \in Z$ let us denote $\mathcal{F}(z)$ the $\mathcal{O}_Y$-module which is the restriction to $\{z\} \times Y$ of $\mathcal{F}$, by definition we have for any $y \in Y$

$$\mathcal{F}(z)_y = \mathcal{F}(z,y) \otimes_{\mathcal{O}_{Z,z}} \mathcal{O}_{Z,z}/m_z.$$

1. Coherent foliations.

Let $X$ be a reduced connected normal analytic space with countable topology; let $\Omega^1_X$ be the coherent sheaf of holomorphic differential 1-forms on $X$ and

$$(*) \quad \Theta_X = \operatorname{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$$

$\Theta_X$ is called the tangent sheaf on $X$. Let $S$ be the singular locus of $X$, then $S$ is at least of codimension two and the restriction of $\Theta_X$ to $X - S$ is the sheaf of holomorphic vector fields on the manifold $X - S$.

Bracket of two sections of $\Theta_X$.

The bracket of two holomorphic vector fields on the manifold $X - S$ is well-defined; recall that, if $z = (z_1, \ldots, z_p)$ denotes the coordinates on $\mathbb{C}^p$, if $U$ is an open set in $\mathbb{C}^p$ and if $a$ and $b$ are two holomorphic vector fields on $U$, with

$$a = \sum_{i=1}^p a_i(z) \frac{\partial}{\partial z_i}, \quad b = \sum_{i=1}^p b_i(z) \frac{\partial}{\partial z_i},$$

then we have $[a, b] = c$ with

$$c = \sum_{i=1}^p c_i \frac{\partial}{\partial z_i} \text{ where } c_i = \sum_{j=1}^p \left( a_j \frac{\partial b_i}{\partial z_j} - b_j \frac{\partial a_i}{\partial z_j} \right).$$
Let $m_U : \mathcal{O}(U)^p \times \mathcal{O}(U)^p \to \mathcal{O}(U)^p$ be the $\mathbb{C}$-bilinear map which sends $(a_1, \ldots, a_p), (b_1, \ldots, b_p)$ onto $(c_1, \ldots, c_p)$; the Cauchy majorations imply the continuity of $m_u$ for the Frechet topology of uniform convergence on compacts of $U$.

**Proposition 1.** — For every open subset $U$ of $X$ the restriction homomorphism

$$\rho : H^0(U, \Theta_X) \to H^0(U - U \cap S, \Theta_X)$$

is an isomorphism of Frechet spaces.

**Proof.** — One knows that $\rho$ is continuous; by the open mapping theorem it is sufficient to prove that $\rho$ is bijective.

Now we may suppose that $X$ is an analytic subspace of an open set $V$ in $\mathbb{C}^n$; let $I$ be the coherent ideal sheaf defining $X$ in $V$; one has an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X^e \to \text{Hom}_{\mathcal{O}_U}(I/I^2, \mathcal{O}_X)$$

where the map $\alpha$ is defined by

$$\alpha(a_1, \ldots, a_n)(f) = \sum_{i=1}^n a_i \frac{\partial f}{\partial z_i}|_X$$

$z_1, \ldots, z_n$ being the coordinates in $\mathbb{C}^n$.

Because the complex space $X$ is reduced and normal it follows from the second removable singularities theorem two isomorphisms

$$O_X(V) \cong O_X(V - S) \quad I(V) \cong I(V - S).$$

Then the proposition 1 follows from (1) and (2). As an immediate consequence of proposition 1 we obtain the following corollary:

**Corollary and definition.** — It exists a unique homomorphism of sheaves of $\mathbb{C}$-vector spaces

$$m : \Theta_X \times \Theta_X \to \Theta_X$$

extending the bracket defined on $X - S$. Therefore, for every open subset $U$
of $X$, the induced map

$$m_U : H^0(U, \Theta_X) \times H^0(U, \Theta_X) \to H^0(U, \Theta_X)$$

is $C$-bilinear and continuous for the Frechet topology on $H^0(U, \Theta_X)$. We call bracket-map the sheaf morphism $m : \Theta_X \times \Theta_X \to \Theta_X$.

Coherent foliations.

**Definition 1.** — A coherent $\mathcal{O}_X$-submodule $T$ of $\Theta_X$ is said to be maximal if for any open $U \subset X$, any section $s \in \Theta_X(U)$ and any nowhere dense analytic set $A$ in $U$

$$s \in T(U-A) \Rightarrow s \in T(U)$$

holds.

Because $X$ is reduced and normal, then locally irreducible, $T$ is maximal if and only if $\Theta_X/T$ has no $\mathcal{O}_X$-torsion.

**Definition 2** [2]. — A coherent foliation on $X$ is a coherent $\mathcal{O}_X$-submodule $T$ of $\Theta_X$ such that:

- (i) $\Theta_X/T$ is non zero locally free outside a nowhere dense analytic subset of $X$;
- (ii) $T$ is a subsheaf of $\Theta_X$ stable by the bracket-map;
- (iii) $T$ is maximal.

**Remarks.** — 1) A coherent foliation induces a classical smooth holomorphic foliation outside a nowhere dense analytic subset of $X - S$.

2) If $T$ is maximal the stability of $T$ by the bracket-map on $X$ is equivalent to the stability of $T$ on $X - A$, for any rare analytic subset $A$.

3) A coherent foliation on a connected reduced normal complex space $X$ is characterized by a quotient module $F$ of $\Theta_X$, without $\mathcal{O}_X$-torsion, such that $\text{ker} [\Theta_X \to F]$ is stable by the bracket-map and which is a non zero locally free $\mathcal{O}_X$-module outside a rare analytic subset of $X$.

4) Let $T$ be a coherent $\mathcal{O}_X$-submodule of $\Theta_X$ satisfying conditions (i) and (ii) of definition 2; then $T$ is included in a maximal coherent sheaf $\hat{T}$ which is equal to $T$ outside a rare analytic subset of $X$ ([7] 2.7); the conditions (i) and (ii) are also fulfilled for $\hat{T}$, hence one can associate to $T$ a maximal foliation on $X$. But two different $T$ for which (i) and (ii) hold may give the same maximal sheaf $\hat{T}$. 
We suppose $X$ compact.

The purpose of this paper is to put an analytic structure on the set of all subsheaves of $\Theta_x$ satisfying conditions (i) and (ii) of Definition 2 (Theorem 2 below), that gives a versal family of holomorphic singular foliations for which a coherent extension exists.

First we have the following proposition:

**Proposition 2.** — Let $X$ be an irreducible complex space; let $Z$ be a complex space and $F$ a coherent $O_{Z \times X}$-module. Let $F$ be $Z$-flat.

Let $Z_1$ be the set of points $z \in Z$ such that $F(z)$ is a non-zero locally free $O_x$-module outside a rare analytic subset of $X$.

Then $Z_1$ is an open subset of $Z$.

**Proof.** — For every $z \in Z$ let $\sigma_z$ be the analytic subset of points $x \in X$ where $F(z)$ is not locally free ([3]). Put $z_0 \in Z_1$. The irreducibility of $X$ implies that $G_{z_0}$ is nowhere dense; fix $x_0 \in X - S \cap \sigma_{z_0}$ and denote $r > 0$ the rank of the $O_{X, x_0}$-module $F(z_0)$. The $Z$-flatness of $F$ implies that $F$ is $O_{Z \times X}$-free of rank $r$ in an open neighborhood $V$ of $(z_0, x_0)$. Let $U$ be the projection of $V$ on $Z$. For any point $z$ of the open set $U$ the $Z$-flatness of $F$ implies that $F(z)_{x_0}$ is $O_{X, x_0}$-free of rank $r$; then the support of the sheaf $F(z)$ contains a neighborhood of $x_0$; hence the irreducibility of $X$ implies

$$\text{support } F(z) = X$$

and the proposition.

For any analytic space $S m_S : p^*_S O_X \times p^*_S O_X \to p^*_S O_X$ denotes the pull back of $m$ by the projection $p_S : S \times X \to X$ (i.e. the bracket map in the direction of the fibers of the projection $S \times X \to S$). Our aim is the proof of the following theorem:

**Theorem 1.** — Let $X$ be a compact connected normal space. There exist an analytic space $\tilde{A}$ and a coherent $O_{\tilde{A} \times X}$-submodule $\tilde{T}$ of $p^{\tilde{A}} O_X$ such that:

(i) $p^{\tilde{A}} O_X / \tilde{T}$ is $\tilde{A}$-flat;

(ii) $\tilde{T}$ is a $m_{\tilde{A}}$-stable submodule of $p^{\tilde{A}} O_X$;

(iii) $(\tilde{A}, \tilde{T})$ is universal for properties (i) and (ii).

As a corollary of proposition 2 and theorem 1 we obtain:
THEOREM 2. — Let $X$ be a compact connected normal space and $r$ a positive integer. There exist an analytic space $\mathcal{H}$ and a coherent $\mathcal{O}_{X \times X}$-submodule $\mathcal{E}$ of $p_X^*\Theta_X$ such that:

(i) $p_X^*\Theta_X/\mathcal{E}$ is $\mathcal{H}$-flat;

(ii) $\mathcal{E}$ is $m_\mathcal{H}$-stable and for any $h \in \mathcal{H}\Theta_X/\mathcal{E}(h)$ is a locally free $\mathcal{O}_X$-module of rank $r$ outside a rare analytic subset of $X$;

(iii) $(\mathcal{H}, \mathcal{E})$ is universal, i.e. for any analytic space $S$ and any coherent $\mathcal{O}_{S \times X}$-submodule $\mathcal{F}$ of $p_S^*\Theta_X$ such that

- $p_S^*\Theta_X/\mathcal{F}$ is $S$-flat;
- $\mathcal{F}$ is $m_S$-stable and for any $s \in S\Theta_X/\mathcal{F}(s)$ is a locally free $\mathcal{O}_X$-module of rank $r$ outside a rare analytic subset of $X$ then it exists a unique morphism $f : S \to \mathcal{H}$ satisfying

$$(f \times 1_X)^* (p_X^*\Theta_X/\mathcal{E}) = p_S^*\Theta_X/\mathcal{F}.$$ 

We shall use the following theorem and Douady ([4]):

THEOREM. — Let $X$ be a compact analytic space and $\mathcal{E}$ a coherent $\mathcal{O}_X$-module; there exist an analytic space $\mathcal{H}$ and a quotient $\mathcal{O}_H$-module $\mathcal{R}$ of $p_H^*\mathcal{E}$ such that:

(i) $\mathcal{R}$ is $\mathcal{H}$-flat;

(ii) for any analytic space $S$ and any quotient $\mathcal{O}_{S \times H}$-module $\mathcal{F}$ of $p_S^*\mathcal{E}$ which is $S$-flat, it exists a unique morphism $f : S \to \mathcal{H}$ satisfying

$$(f \times 1_X)^* \mathcal{R} = \mathcal{F}.$$ 

2. Local deformations.

One uses notations and results of [4]; the notions of infinite dimensional analytic spaces, called Banach analytic spaces, and of anaflatness are defined respectively in ([4] § 3) and in ([4] § 8).

In this section we fix an open subset $U$ of $\mathbb{C}^n$, two compact polycylinders of non-empty interior $K$ and $K'$ satisfying

$$K' \subseteq \bar{K} \subseteq K \subseteq U$$

and a reduced normal analytic subspace $X$ of $U$. Let $B(K)$ be the Banach algebra of those continuous functions on $K$ which are analytic on the interior $\bar{K}$ of $K$; one defines $B(K')$ in an analogous way.
For every coherent sheaf $\mathcal{F}$ on $U$, one knows that it exists finite free resolutions of $\mathcal{F}$ in a neighborhood of $K$; for such a resolution

$$(L.) \quad 0 \rightarrow L_n \rightarrow L_{n-1} \rightarrow \cdots \rightarrow L_0$$

let us consider the complex of Banach spaces

$$B(K,L.) = B(K) \otimes_{O(K)} H^0(K,L.)$$

and the vector space

$$B(K,\mathcal{F}) = \text{coker} \ [B(K;L_1) \rightarrow B(K,L_0)].$$

**Definition 1** ([4] §7, [5]). — $K$ is $\mathcal{F}$-privileged if and only if it exists a finite free resolution $L.$ of $\mathcal{F}$ on a neighborhood of $K$ such that the complex $B(K,L.)$ is direct exact.

Then this is true for every finite free resolution; therefore $B(K,\mathcal{F})$ is a Banach space which does not depend of the resolution; $\mathcal{F}$-privileged polycylinders give fundamental systems of neighborhoods at every point of $U$. For a more geometric definition of privilege, the reader can refer to ([6]).

In the following, we always suppose that the two polycylinders $K$ and $K'$ are $\Theta_x$-privileged, $\Theta_x$ being the tangent sheaf defined by $1 - (\ast)$.

Let $G_K$ be the Banach analytic space of those $B(K)$-submodules $Y$ of $B(K,\Theta_x)$ (or equivalently of quotient modules) for which it exists an exact sequence of $B(K)$-modules

$$0 \rightarrow B(K)^\ast \rightarrow \cdots \rightarrow B(K)^0 \rightarrow B(K,\Theta_x) \rightarrow B(K,\Theta_x)/Y \rightarrow 0$$

which is a direct sequence of Banach vector spaces.

A universal sheaf $R_X$ on $G_X \times \mathring{K}$ is constructed in [4]; $R_K$ satisfies the following proposition:

**Proposition 1** ([4] § 8 n° 5). — (i) $R_K$ is $G_K$-anaflat.

(ii) For every Banach analytic space $Z$ and for every $Z$-anaflat quotient $\mathcal{F}$ of $p^*\Theta_x$ it exists a natural morphism $\varphi : Z \rightarrow G_K$ such that

$$(\varphi \times I_K)^*R_K = \mathcal{F}_{s \times \mathring{K}}.$$ 

Recall that the $Z$-anaflatness generalizes to the infinite dimensional space $Z$ the notion of flatness; pull back preserves anaflatness.
Let $G_{K,K'}$ be the set of the $B(K)$-submodules $E$ of $B(K,\Theta_X)$, element of $G_K$, such that $E \otimes_{B(K)} B(K')$ gives an element of $G_K$.

**Proposition 2.** — (i) $G_{K,K'}$ is an open subset of $G_K$.

(ii) Let $\mathcal{R}$ be the pull back of $R_K$ by the inclusion $G_{K,K'} \hookrightarrow G_K$. Then the map from $G_{K,K'}$ to $G_K$, which maps every $B(K)$-module $E$ element of $G_{K,K'}$ onto the $B(K')$-module $E \otimes_{B(K)} B(K')$ is given by a unique morphism

$$\rho_{K,K'} : G_{K,K'} \to G_K,$$

satisfying

$$\rho_{K,K'}^* R_K = \mathcal{R}.$$

**Proof.** — Proposition 2 follows from ([4] 14 prop. 4).

Let $\rho_1 : B(K,\Theta_X) \times B(K,\Theta_X) \to \Theta_X(\hat{K}) \times \Theta_X(\hat{K})$ and

$$\rho_2 : \Theta_X(\hat{K}) \to B(K',\Theta_X)$$

be the restriction homomorphisms and

$m : \Theta_X(\hat{K}) \times \Theta_X(\hat{K}) \to \Theta_X(\hat{K})$

the bracket map.

Let

$$m_{K,K'} : B(K,\Theta_X) \times B(K,\Theta_X) \to B(K',\Theta_X)$$

be the continuous $C$-bilinear map defined by

$$m_{K,K'} = \rho_2 \circ m \circ \rho_1.$$

**Definition 2.** — A $B(K)$-submodule $Y$ of $B(K,\Theta_X)$ is said to be $m_{K,K'}$-stable if it verifies:

(i) $Y$ is an element of $G_{K,K'}$,

(ii) for every $f$ and $g$ in $Y$ one has

$$m_{K,K'}(f,g) \in \rho_{K,K'}(Y).$$

Then, if $\mathfrak{S}$ is a $m$-stable $\Omega_X$-submodule of $\Theta_X$ such that $K$ and $K'$ are $\mathfrak{S}$-privileged, $B(K,\mathfrak{S})$ is $m_{K,K'}$-stable; the converse is not necessarily true; however we have the following proposition:
PROPOSITION 3. — Let $Y$ be a $m_{K,K}$-stable $B(K)$-submodule of $B(K,\Theta_X)$; then $Y$ defines in a natural way a coherent $O_X$-submodule of $\Theta_X$ on $\hat{K}$, the restriction to $\hat{K}'$ of which is $m$-stable (i.e. stable by the bracket-map).

Proof. — Let $B_Y$ be the privileged $B_K$-module given by $Y$ ([6]); the restriction to $\hat{K}$ of $B_Y$ is a coherent sheaf; therefore one has ([6] th. 2.3 (ii) and prop. 2.11)

$$Y = \hat{H}(K,B_Y)$$

and the restriction homomorphism

$$i: Y = H^0(K,B_Y) \to H^0(\hat{K},B_Y)$$

is injective and has dense image; therefore the restriction $B_Y|_K$ is a submodule of $\Theta_X$ ([4] § 8 lemme 1 (b)), hence $H^0(\hat{K}',B_Y)$ is a closed subspace of the Frechet space $H^0(\hat{K}',\Theta_X)$.

Let us show that $m_{K,K}$ induces a $C$-bilinear continuous map

$$\hat{m}: H^0(\hat{K},B_Y) \times H^0(\hat{K},B_Y) \to H^0(\hat{K}',B_Y).$$

Take $t_1, t_2$ two elements of $H^0(\hat{K},B_Y)$ and $(t^n_1)$ and $(t^n_2)$ two sequences of elements of $Y$ with

$$\lim_{n \to \infty} t^n_i = t_i, \quad i = 1, 2.$$ 

Because the bracket-map $m: H^0(\hat{K},\Theta_X) \times H^0(\hat{K},\Theta_X) \to H^0(\hat{K},\Theta_X)$ is continuous one has

$$\lim_{n \to \infty} m(t^n_{1|K}, t^n_{2|K}) = m(t_1, t_2) \in H^0(\hat{K},\Theta_X).$$

Therefore the $m_{K,K}$-stability of $Y$ implies for every $m$

$$m_{K,K}(t^n_1, t^n_2) \in B(K',B_Y) \subset H^0(\hat{K}',B_Y)$$

then $m(t_1, t_2)|_K \in H^0(\hat{K}',B_Y)$ follows.

In order to prove the proposition it is sufficient to remark that, for every polycylinder $K'' \subset \hat{K}'$, the restriction homomorphism

$$H^0(\hat{K}',B_Y) \to H^0(\hat{K}'',B_Y)$$

has a dense image. Q.E.D.
Recall some properties of infinite dimensional spaces: let $V$ be an open subset of a Banach $\mathbb{C}$-vector space; let $F$ be a Banach vector space and $f: V \to F$ an analytic map. Let $\mathcal{X}$ the Banach analytic space defined by the equation $f = 0$; $\mathcal{X}$ is a local model of general Banach analytic space; the morphisms from $\mathcal{X}$ into a Banach vector space $G$ extend locally in analytic maps on open subsets of $V$; for such a morphism $\varphi: \mathcal{X} \to G$ the equation $\varphi = 0$ defines in a natural way a Banach analytic subspace of $\mathcal{X}$; the morphisms from a Banach analytic space $\mathcal{Y}$ into $\mathcal{X}$ are exactly the morphisms $\psi: \mathcal{Y} \to V$ such that $f \circ \psi = 0$.

**Proposition 4.** — Let $S_{K,K'}$ be the subset of elements of $G_{K,K'}$ which are $m_{K,K'}$-stable. Then $S_{K,K'}$ is a Banach analytic subspace of $G_{K,K'}$.

**Proof.** — Let $Y_0 \in S_{K,K'}$ and $Y_0' = \rho_{K,K}(Y_0)$; let $G_0$ (resp. $G_0'$) a closed $\mathbb{C}$-vector subspace of $B(K,\Theta_X)$ (resp. $B(K',\Theta_X)$) which is a topological supplementary of $Y_0$ (resp. $Y_0'$). Let $U_0$ (resp. $U_0'$) the set of closed $\mathbb{C}$-vector subspaces of $B(K,\Theta_X)$ (resp. $B(K',\Theta_X)$) which are topological supplementaries of $G_0$ (resp. $G_0'$); we identify $U_0$ and $L(Y_0, G_0)$, hence $U_0 \cap G_K$ is a Banach analytic subspace of $U_0([4] \S 4)$.

For every $Y$ in $U_0$ one denotes $p_Y: B(K,\Theta_Y) = Y \oplus G_0 \to G_0$ the projection and $j_Y: Y_0 \to Y \subset B(K,\Theta_Y)$ the reciprocal map of the restriction to $Y$ of the projection $B(K,\Theta_X) = Y_0 \oplus G_0 \to Y_0$.

Then the two maps

\[
p^K: G_K \to L(B(K,\Theta_X), G_0)
\]
\[
j^K: G_K \to L(Y_0, B(K,\Theta_X))
\]

defined by $p^K(Y) = p_Y$ and $j^K(Y) = j_Y$ are induced by morphisms ([4] § 4, n° 1); associated to the polycylinder $K'$ we have in the same way morphisms $p^{K'}$ and $j^{K'}$. Put $W_0 = G_{K,K'} \cap U_0 \cap \rho_{K,K}^{-1}(U_0)$; $W_0$ is an open subset of $G_{K,K'}$. Let be

\[
\varphi_1 = p^{K'} \circ \rho_{K,K}: W_0 \to L(B(K',\Theta_X), G_0)
\]

and $\Delta: G_K \to L(Y_0 \otimes Y_{0}, B(K',\Theta_X))$ the morphism defined by

\[
\Delta(Y) = m_{K,K'} \circ (j_Y \times j_Y).
\]

Let be $\varphi_2 = \Delta \circ j^K: W_0 \to L(Y_0 \otimes Y_{0}, B(K',\Theta_X))$; $\varphi_1$ and $\varphi_2$ are
morphisms; let

\[ \varphi : W_0 \rightarrow L(Y_0 \otimes Y_0, G_0) \]

be the morphism defined by

\[ \varphi(Y) = \varphi_2(Y) \circ \varphi_1(Y). \]

We have \( W_0 \cap S_{K,K'} = \varphi^{-1}(0) \), hence \( S_{K,K'} \cap W_0 \) is a Banach analytic subspace of \( W_0 \); following ([4] § 4, n° 1 (i) and (ii)) one easily proves that the analytic structures obtained in the different charts of \( G_K \) and \( G_{K'} \) patch together in an analytic structure on \( S_{K,K'} \); that proves proposition 4.

Remark 1. — With the previous notations the morphisms of Banach analytic spaces \( g : Z \rightarrow S_{K,K'} \cap W_0 \) are the morphisms \( g : Z \rightarrow W_0 \) satisfying \( \varphi \circ g = 0 \).

Let \( i : S_{K,K'} \rightarrow G_K \) be the inclusion and \( R_{K,K'} \) the pullback of \( R_K \) by \( i \); \( R_{K,K'} \) is \( S_{K,K'} \)-anaflat; by construction \( R_{K,K'} \) is a quotient of \( p^*_{S_{K,K'}} \Theta_X \), then put

\[ R_{K,K'} = p^*_{S_{K,K'}} \Theta_X / T_{K,K'}. \]

By anaflatness one obtains for every \( s \in S_{K,K'} \) an exact sequence of coherent sheaves on \( K' \):

\[ 0 \rightarrow T_{K,K'}(s) \rightarrow \Theta_X \rightarrow R_{K,K'}(s) \rightarrow 0. \]

From the definition of the analytic structure of \( S_{K,K'} \) and from proposition 3 one deduces the following theorem:

THEOREM 3. — (i) For every \( s \in S_{K,K'} \) the restriction to \( K' \) of the coherent subsheaf \( T_{K,K'}(s) \) of \( \Theta_X \) is stable by the bracket-map.

(ii) For every Banach analytic space \( Z \) and every quotient \( \mathcal{F} = p^*_Z \Theta_X / T \) of \( p^*_Z \Theta_X \) by a \( O_{Z \times X} \)-submodule \( T \) such that

- \( \mathcal{F} \) is \( Z \)-anaflat.
- \( T \) is \( m_Z \)-stable and for any \( z \in Z \) the poly cylinders \( K \) et \( K' \) are \( \mathcal{F}(z) \)-privileged;

then the unique morphism \( g : Z \rightarrow G_K \) satisfying

\[ (g \times 1_K)^* R_K = \mathcal{F} \]

factorizes through \( S_{K,K'} \) (i.e. it exists a unique morphism \( f : Z \rightarrow S_{K,K'} \) with \( r \circ f = g \)).
Remark 2. — We don’t know if the restriction of $R^*,K$ to $S^*,K \times \hat{K}'$ is $m_{S,K,K}$-stable; but if $S$ is a finite dimensional analytic space then the pull back of $R^*,K$ by any morphism $S \to S^*,K$ is $m_S$-stable.


In this section $X$ denotes a compact reduced normal space and $\Theta_X$ its tangent sheaf. Let $H$ be the universal space of quotient $O_X$-modules of $\Theta_X$ and $\mathcal{R}$ the $H$-flat universal sheaf on $H \times X$ ([4]). Put $\mathcal{R} = p_H^*\Theta_X/\mathcal{E}$, $\mathcal{E}$ being a coherent submodule of $p_H^*\Theta_X$; for any $h \in H$, $\mathcal{E}(h)$ is a coherent submodule of $\Theta_X$. We shall construct the space $\mathcal{H}$ as an analytic subspace of an open subset of $H$.

1. Refining of a privileged « cuirasse ».

Let $M$ be a $\Theta_X$-privileged « cuirasse » ([4] § 9, no 2); $M$ is given by,

(i) a finite family $(\varphi_i)_{i \in I}$ of charts of $X$, i.e. for every $i \in I$, $\varphi_i$ is an isomorphism from an open set $X_i \subset X$ onto a closed analytic subspace of an open set $U_i$ in $\mathbb{C}^n$,

(ii) for every $i \in I$ a $\Theta_X$-privileged polycylinder $K_i \subset U_i$ (i.e. a $\varphi_{i*}\Theta_X$-privileged polycylinder) and an open set $V_i \subset X_i$ satisfying

\[ V_i = \varphi_i^{-1}(\hat{K}_i) \subset X_i \]

\[ X = \bigcup_{i \in I} V_i \]

(iii) for every $(i,j) \in I \times J$ a chart $\varphi_{ij}$ defined on $X_i \cap X_j$ with values in an open $U_{ij} \subset \mathbb{C}^n$ and a finite family $(K_{ij})$ of $\Theta_X$-privileged polycylinders in $U_{ij}$ such that conditions

\[ V_i \cap V_j \subset \bigcup_{a} \psi_{ij}^{-1}(K_{ij}) \]

\[ \varphi_{ij}^{-1}(K_{ij}) \subset \varphi_i^{-1}(\hat{K}_i) \cap \varphi_j^{-1}(\hat{K}_j) \]

are fulfilled.

As in ([4]) let us denote $H_M$ the open subset of the elements $F$ of $H$ for which $M$ is $F$-privileged (i.e. all the polycylinders $K_i, K_{ij}$ are $F$-privileged); we shall construct $\mathcal{H}$ as union of open subsets $\mathcal{H} \cap H_M$.

For any $\Theta_X$-privileged polycylinder $K$ let us denote $G_K$ (§ 2) the Banach analytic space of quotients of $B(K, \Theta_X)$ with finite direct resolution.
For every $i \in I$ let $G_i$ be the open subset of $G_{K_i}$ on which, for any $\alpha$, the restriction homomorphisms $B(K_i) \to B(K_{ij})$ induce morphisms $G_i \to G_{K_{ij}}$. The Douady construction of $H_M$ gives a natural injective morphism

$$i : H_M \to \prod_{i \in I} G_i.$$

**Definition 5.** — A refining of the « cuirasse » $M$ is given by a family $(K_i)_{i \in I}$ of polycylinders satisfying:

(i) for every $i$, $\varphi_i(V_i) \subset \hat{K}_i \subset K'_i \subset \hat{K}_i$,

(ii) for every $i, j, \alpha$, $\varphi_{ij}^{-1}(K_{ij}) \subset \varphi_i^{-1}(\hat{K}_i) \cap \varphi_j^{-1}(\hat{K}_j)$,

(iii) for every $i$, $K_i$ is $\Theta_X$-privileged.

We denote by $M((K_i))$ such a refining; for any coherent sheaf $\mathcal{F}$ on $X$ we shall say that $M((K_i))$ is $\mathcal{F}$-privileged if $M$ is $\mathcal{F}$-privileged and if, for every $i$, $K_i$ is $\mathcal{F}$-privileged.

**Lemma 1.** — (i) Let $\mathcal{F}$ be a coherent sheaf such that $M$ is $\mathcal{F}$-privileged; then it exists a $\mathcal{F}$-privileged refining of $M$.

(ii) Let $M((K_i))$ a refining of $M$; then the set of quotient $\mathcal{F}$ of $\Theta_X$ such that $M((K_i))$ is $\mathcal{F}$-privileged is open in $H_M$.

**Proof.** — (i) follows from ([4] § 7, n° 3 corollary of prop. 6) and (ii) is an immediate consequence of flatness and privilege.

2. Now we fix a $\Theta_X$-privileged « cuirasse » $M = M(I,(K_i),(V_i),(K_{ij}))$ and a $\Theta_X$-privileged refining $M((K_i))$ of $M$.

**Lemma 2.** — Let $H'_M$ be the subset of $H_M$ the points of which are quotients $\Theta_X/T$ satisfying:

(i) $M((K_i))$ is $\Theta_K/T$-privileged,

(ii) $T$ is a subsheaf of $\Theta_X$ stable by the bracket-map.

Then $H'_M$ is an analytic subspace of an open subset of $H_M$.

**Proof.** — Using notations of section 2 one puts for every $i \in I$

$$G'_i = G_{K_i,K_i} \cap G_i$$

$G'_i$ is an open subset of $G_i$ and $G_{K_i}$; put $S_i = S_{K_i,K_i} \cap G'_i$. 
One knows that the category of Banach analytic spaces has finite products, kernel of double arrows and hence fiber products (for all this notions the reader can refer to ([4] § 3, n° 3). Then \( \prod_{i \in I} S_i \) is a Banach analytic subspace of \( \prod_{i \in I} G_i \); since \( \prod_{i \in I} G_i \) is an open subset of \( \prod_{i \in I} G_i \) it follows from (§ II Theorem 3)

\[
H_M = H_M \times \prod_{i \in I} G_i \times \prod_{i \in I} S_i
\]

and the lemma is proved.

Now let \( R_M' \) (resp. \( T_M' \)) be the pull back of \( R \) (resp. \( T \)) by the inclusion morphism \( H'_M \times X \to H \times X \); \( R_M \) is the quotient of \( p_{H_M}^* \Theta_X \) by \( T_M \) (the sheaves \( T_M \) and \( \ker [p_{H_M}^* \Theta_X \to R_M] \) are \( H_M \)-flat and equal on the fibers \( \{ h \} \times X \).

**Lemma 3.** — \( T_M' \) is a \( m_{H_M}' \)-stable submodule of \( p_{H_M}^* \Theta_X \).

The proof follows immediately of the remark 2 of paragraph 2 and of

\[
X = \bigcup_{i \in I} V_i = \bigcup_{i \in I} \varphi_i^{-1}(K_i).
\]

Using the universal property of \( H_M \), Theorem 3 § 2 and the commutative diagram

\[
\begin{array}{ccc}
H'_M \times X & \to & H_M \times X \\
\downarrow & & \downarrow \\
\left( \prod_{i \in I} G_i \right) \times X & \to & \left( \prod_{i \in I} G_i \right) \times X
\end{array}
\]

we obtain the following proposition:

**Proposition 1.** — Let \( Z \) be an analytic space and \( T_Z \) a coherent subsheaf of \( p_Z^* \Theta_X \) satisfying :

(i) \( p_Z^* \Theta_X / T_Z \) is \( Z \)-flat.

(ii) For every \( z \in Z \) the cuirasse \( M((K_i)) \) is \( \Theta_X / T_Z(z) \)-privileged.

(iii) \( T_Z \) is a \( m_Z \)-stable submodule of \( p_Z^* \Theta_X \).
Then the unique morphism $g : Z \to H$ such that
\[(g \times I_X)^* \mathcal{R} = p^*_Z \Theta_X/T_Z\]
factorizes through $H^*_M$ and verifies
\[(g \times I_X)^* T^*_M = T_Z.\]

3. End of the proof of Theorem 1.

Notations are those of the previous proposition; the unicity of $g$ implies
the unicity of its factorization through the subspace $H^*_M$ of $H$. Hence, when
the refinings of a given $M$ are varying, one obtains analytic spaces $H^*_M$
which patch together in an analytic subspace of an open subset of $H_M$.

When the « cuirasse » $M$ varies in the family of all the $\Theta_X$-privileged
« cuirasse » the spaces $H_M$ form an open covering of $H$; then the universal
property of the $H_M$'s implies that $\tilde{H} = \bigcup_M H^*_M$ is an analytic subspace of an
open subset of $H$. Theorem 4 is proved.

Remark. — More generally if $X$ is not compact, let $\Theta$ be a coherent
sheaf on $X$ and $m : \Theta \times \Theta \to \Theta$ a sheaf morphism inducing for each open
set $U$ a continuous $C$-bilinear map $m_U : \Theta(U) \times \Theta(U) \to \Theta(U)$; let $H$
be the Douady space of the coherent quotients of $\Theta$ with compact support
([4]). We get a universal analytic structure on the subset of those quotients
which are $m$-stable.

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