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<http://www.numdam.org/item?id=AIF_1986__36_4_207_0>
THE TRACE INEQUALITY AND EIGENVALUE ESTIMATES FOR SCHRODINGER OPERATORS

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1. Introduction.

This paper deals with potential operators $T_\Phi$ given at Lebesgue measurable $f$ on $\mathbb{R}^n$ by a convolution integral

$$(T_\Phi f)(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)\,dy,$$

provided this integral exists for almost all $x \in \mathbb{R}^n$. The kernels $\Phi(y)$ are radially decreasing (r.d.) functions; that is, they are nonnegative, locally integrable radial functions on $\mathbb{R}^n$, which are nonincreasing in $|y|$. These $T_\Phi$ include the Riesz potential operator $I_\alpha$ whose kernel $K_\alpha$ is defined directly as

$$K_\alpha(y) = |y|^{n-\alpha}, \quad 0 < \alpha < n$$

and the Bessel potential operator $J_\alpha$ with kernel $G_\alpha$ defined in terms of its Fourier transform $\hat{G}_\alpha$ by

$$\hat{G}_\alpha(\xi) = \int_{\mathbb{R}^n} G_\alpha(x)e^{-\xi\cdot x}\,dx = (1 + |\xi|^2)^{-\frac{\alpha}{2}}, \quad 0 < \alpha < n.$$ 

Given an r.d. kernel $\Phi$ and $1 < p < \infty$, we wish to characterize the (possibly singular) positive Borel measures $\mu$ on $\mathbb{R}^n$ for which there exists $C > 0$ such that

$$(1.1) \quad \int_{\mathbb{R}^n} (T_\Phi f)(x)\,d\mu(x) \leq C \int_{\mathbb{R}^n} f(x)^p\,dx$$

Key-words: Schrödinger operators - Eigenvalues - Weighted inequalities.

(1) Research supported in part by N.S.E.R.C. grant A 4021.
(2) Research supported in part by N.S.E.R.C. grant A 5149.
for all nonnegative measurable $f$. Clearly this will be true if and only if $T_\phi$ is a bounded linear operator between the Lebesgue spaces $L^p(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n,\mu)$. An important special case, with $p=2$ and $\Phi=\Delta_1$, arises in estimating the spectrum of Schrödinger operators and will be considered in detail below. Another special case is treated in Stein [19], where it is shown that (1.1) holds for $J_\mu$ when $\mu = \mu_k, \alpha > \frac{n-k}{p}$, where

$$\mu_k(E) \equiv m_k(E \cap \mathbb{R}^k),$$

$m_k$ being $k$-dimensional Lebesgue measure on $\mathbb{R}^k$ considered as a subset of $\mathbb{R}^n$. The inequality of [19] can be stated in the equivalent form

$$\left( \int_{\mathbb{R}^n} (J_\phi f)(x_1, \ldots, x_k, 0, \ldots, 0)^p \, dx_1, \ldots, dx_k \right)^{\frac{1}{p}} \leq C \int_{\mathbb{R}^n} f(x_1, \ldots, x_n)^p \, dx_1, \ldots, dx_n.

It is thus a statement about the restriction, or trace, of $J_\phi f$. For this reason we follow other authors in referring to (1.1) as «the trace inequality».

Generalizing results of Adams [1] and Maz'ya [14], K. Hansson in [12] has characterized the $\mu$ satisfying (1.1) in terms of capacities (see also B. Dahlberg [8]). He shows the trace inequality holds if and only if $K > 0$ exists for which

$$\mu(E) \leq K \text{cap}(E)$$

whenever $E$ is a compact subset of $\mathbb{R}^n$. Here $\text{cap}(E)$ denotes the $L^p$ capacity associated with the kernel $\Phi$,

$$\text{cap}(E) = \inf \left\{ \int_{\mathbb{R}^n} f(x)^p \, dx : f \geq 0 \text{ and } T_\phi f \geq 1 \text{ on } E \right\}.

A criterion such as (1.2) can be difficult to verify for all compact sets $E$. On the other hand if one only requires (1.2) to hold for a class of simple sets such as all cubes $Q$ with sides parallel to the coordinate axes, the resulting condition is no longer sufficient (D. Adams [2]). For example, when $n=p=2$, $I_{\frac{1}{2}}$ doesn’t satisfy (1.1) with $\mu_1$, yet inequality (1.2) for cubes, which amounts to $\mu_1(Q) \leq K |Q|^\frac{1}{2}$, holds. In fact, with $f(x) = x_2^{-\frac{1}{2}} \ln x_2^{-1} \chi_{[0,1]}(x_1) \chi_{[0,1]}(x_2), I_{\frac{1}{2}} f$ is infinite on
\[(x_1, 0) : 0 \leq x_1 \leq \frac{1}{2}\] and thus the left side of (1.1) is infinite while the right side is finite. Examples of this nature were first pointed out in [2].

Theorem 2.3 below gives a necessary and sufficient condition for (1.1) that involves testing an inequality over dyadic cubes $Q$, namely

\[
(1.3) \int_Q (M_{\varphi} X_Q \mu)(x) dx \leq K \int_Q d\mu < \infty
\]

where $p' = \frac{p}{p - 1}$, the constant $K > 0$ is independent of $Q$, and

\[
(M_{\varphi} f \mu)(x) = \sup_{x \in Q} \left\{ \frac{1}{|Q|} \int_{|y| < |Q|^n} \Phi(y) dy \right\} \int_Q f(y) d\mu(y).
\]

Alternatively, (1.1) is equivalent to

\[
(1.4) \int_{\mathbb{R}^n} (T_{\varphi} X_Q \mu)(x)^{p'} dx \leq K \int_Q d\mu < \infty \text{ for all dyadic cubes } Q.
\]

To compare (1.2) and (1.4), we note that (1.2) is equivalent by an elementary argument (see Theorem 4 in [2]) to testing the inequality in (1.4) over all compact sets $Q$. The reduction in (1.4) to testing over dyadic cubes $Q$ is essential in obtaining sharp estimates for the higher eigenvalues of Schrödinger operators in § 3. For a different characterization involving test functions see Stromberg and Wheeden [21].

In the special case where $T_{\varphi} = I_\varphi$, the equivalence of (1.1) and (1.3) can be established by dualizing inequality (1.1), using the «good $\lambda$ inequality» of B. Muckenhoupt and R. L. Wheeden [15] in order to replace $I_\varphi$ by its associated maximal operator $M_\varphi$, and then using the characterization of the weighted inequality for $M_\varphi$ in [18]. The general case of the theorem is proved along similar lines, the crucial new estimate being an extension (Theorem 2.2) of the «good $\lambda$ inequality» in [15].

As an application of Theorem 2.3 we obtain a sharpened form of recent results of C. L. Fefferman and D. H. Phong on the distribution of eigenvalues of Schrödinger operators, $H = -\Delta - \mu$, $\mu \geq 0$ ([10]; Theorem 5, 6 and 6' in Chapter II). Roughly speaking, their results show that for many $\mu \geq 0$, the negative eigenvalues of $H = -\Delta - \mu$ are approximately given by $-|Q|^{-\frac{n}{2}}$ as $Q$ varies over the minimal dyadic...
cubes satisfying $|Q|^{2^{-1}} \int_Q v \geq C$. Theorem 3.3 below shows, as suggested by condition (1.3), that this picture extends to arbitrary $v \geq 0$ if the fractional average, $|Q|^{2^{-1}} \int_Q v$, is replaced by

$$\frac{1}{|Q|} \int_Q [I_1(\chi_Q v)(x)]^2 \, dx = \frac{1}{|Q|} \int_Q I_2(\chi_Q v)(x)v(x) \, dx,$$

the $v$-average over $Q$ of the Newtonian potential of $\chi_Q v$. Certain of the results in [10] have been generalized by S. Y. A. Chang, J. M. Wilson and T. H. Wolff ([5]) and by S. Chanillo and R. L. Wheeden ([6]). This is discussed in more detail in § 3. Further applications of Theorem 2.3 have been announced in [13].

2. The trace inequality.

We begin by deriving the basic properties of r.d. kernels $\Phi$ and Borel measures $\mu$ for which the trace inequality holds. For the sake of completeness, we consider here and in § 3 the more general trace inequality

$$(2.1) \quad \left[ \int_{\mathbb{R}^n} (T_\Phi f)(x)^q \, d\mu(x) \right]^{1/q} \leq C \left[ \int_{\mathbb{R}^n} f(x)^p \, dx \right]^{1/p}$$

for all nonnegative measurable $f$, where $1 < p < q < \infty$. For $p < q$ and many r.d. kernels $\Phi$, the trace inequality (2.1) can be characterized in terms of very simple conditions — see e.g. [12]. However, many applications, such as that in the next section, require the case $p = q$.

**Proposition 2.1.** — If (2.1) holds for a non-trivial r.d. kernel $\Phi$ and a non-trivial Borel measure $\mu$, then (i) $\mu$ is locally finite, that is, $\int_Q d\mu < \infty$ for all cubes $Q$, and (ii) $\Phi$ satisfies

$$(2.2) \quad \int_{|y| > r} \Phi(y)^p \, dy < \infty \quad \text{for all } r > 0.$$

**Proof.** — Choose $\varepsilon > 0$ so that $\Phi(2\varepsilon) > 0$. If $B$ is any ball of radius $\varepsilon$, and if $\gamma_n$ denotes the measure of the surface of the unit ball in
Then
\[
\gamma_n e^\alpha \Phi(2\varepsilon) \left( \int_B d\mu \right)^{\frac{1}{q}} \leq \left[ \int_B (T_\phi \chi_B)^q d\mu \right]^{\frac{1}{q}} \leq [\gamma_n e^\alpha]^{\frac{1}{q}} \|T_\phi\|_{L^p} < \infty .
\]

Hence \( \int_B d\mu < \infty \) and this proves that \( \mu \) is locally finite.

To obtain (2.2), fix \( R > 0 \) so that \( \int_B d\mu > 0 \) where \( B \) is the ball of radius \( R \) centred at the origin. Momentarily fix \( S > 2R \) and let \( f(x) = \Phi(x)^{p'-1} \chi_{\{2R \leq |y| \leq S\}}(x) \). For \( |x| \leq R \), we have
\[
T_\phi f(x) = \int_{2R \leq |y| \leq S} \Phi(x-y) \Phi(y)^{p'-1} dy \geq C \int_{2R \leq |y| \leq S} \Phi(y)^{p'} dy .
\]
Indeed, \( \Phi(x-y) \geq \Phi(y) \) for all \( y \) satisfying \( |x-y| \leq |y| \) and this in turn holds provided \( |x| \leq R, |y| \geq 2R \) and the distance between \( \frac{x}{|x|} \) and \( \frac{y}{|y|} \) is sufficiently small. With this estimate, (2.1) yields
\[
C \int_{2R \leq |y| \leq S} \Phi(y)^{p'} dy \left( \int_B d\mu \right)^{\frac{1}{q}} \leq \left[ \int_B (T_\phi f)^q d\mu \right]^{\frac{1}{q}} \leq C \left[ \int_{2R \leq |y| \leq S} \Phi(y)^{p'} dy \right]^{\frac{1}{p}}.
\]

Letting \( S \to \infty \) yields \( \int_{|y| \geq 2R} \Phi(y)^{p'} dy < \infty \) and this proves (2.2).

To obtain a criterion for (2.1) to hold, we look at the inequality dual to it. A standard argument shows this dual is, with the same \( C > 0 \),
\[
(2.3) \quad \left( \int_{\mathbb{R}^n} (T_\phi f \mu)(x)^{p'} dx \right)^{\frac{1}{p'}} \leq C \left( \int_{\mathbb{R}^n} f(x)^{q'} d\mu(x) \right)^{\frac{1}{q'}} ,
\]
where \( p' = \frac{p}{p-1} \), \( q' = \frac{q}{q-1} \), and
\[
(T_\phi f \mu)(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y) d\mu(y) .
\]
The behaviour of $T_\Phi$ in (2.3) is determined by that of the maximal operator $M_\Phi$ given at a positive Borel measure $\nu$ by

$$(M_\Phi \nu)(x) = \sup_{x \in Q} \left[ \frac{1}{|Q|} \int_{|y| \leq |Q|^\gamma} \Phi(y) \, dy \right] \int_Q \, d\nu.$$

Note that the first factor on the right side is the average of $\Phi$ over the ball of radius $|Q|^\frac{1}{\gamma}$ centred at the origin. In the case when $\Phi$ is the kernel $K_\alpha$ for the Riesz potential operator, then $M_\Phi$ is the usual fractional maximal operator $M_\alpha$ (see e.g. [3] or [15]).

**Theorem 2.2.** Let $\Phi$ be an r.d. kernel and $\nu$ a positive locally finite Borel measure on $\mathbb{R}^n$. Then

(a) $$(M_\Phi \nu)(x) \leq C_n M(T_\Phi \nu)(x), \quad x \in \mathbb{R}^n$$

where $M$ denotes the usual Hardy-Littlewood maximal operator and the constant $C_n > 0$ depends only on the dimension $n$.

(b) There exists $\gamma > 1$ and a positive constant $C_n$ depending only on $n$ so that for all $\lambda > 0$ and all $\beta \in (0,1]$,

$$|\{T_\Phi \nu > \gamma \lambda \text{ and } M_\Phi \nu \leq \beta \gamma\}| \leq C_n \frac{\beta}{\gamma} |\{M(T_\Phi \nu) > \lambda\}|.$$

**Proof.** To a given cube $Q$ in $\mathbb{R}^n$ associate the cube $Q^*$ having the same centre as $Q$ but edges $7\sqrt{n}$ times as long as those of $Q$.

To prove (a) fix $x \in \mathbb{R}^n$ and a cube $Q$ containing $x$. Then

$$\int_{Q^*} (T_\Phi \nu)(y) \, dy \geq \int_Q \Phi(y-z) \, dv(z)$$

$$\geq \int_Q dv(z) \int_{Q^*} \Phi(y-z) \, dy$$

$$\geq \int_{|y| \leq |Q|^\gamma} \Phi(y) \, dy \int_Q dv(y)$$

since $\{y; |y-z| \leq |Q|^\frac{1}{\gamma} \subset Q^*\}$, whenever $z \in Q$. Hence,

$$M(T_\Phi \nu)(x) \geq \frac{7^{-n} n^{-\frac{n}{2}}}{|Q|} \int_{|y| \leq |Q|^\gamma} \Phi(y) \, dy \int_Q dv(y)$$
and so
\[ M_\phi \nu(x) \geq \frac{n}{2} M(T_\phi \nu)(x), \quad x \in \mathbb{R}^n. \]

We now show (b). Given \( \lambda > 0 \), let
\[ \Omega_\lambda = \{ M(T_\phi \nu) > \lambda \}. \]

Decompose \( \Omega_\lambda \) into disjoint Whitney cubes \( Q \) with \( Q^* \cap \Phi_\xi \neq \emptyset \). See De Guzman [11]. Let \( \{Q_k\} \) be those Whitney cubes for which there is an \( x_k \in Q_k \) satisfying \( (M_\phi \nu)(x_k) \leq \beta \lambda \). Fixing attention on such a \( Q_k \), which we'll denote simply by \( Q \), we define \( \nu_1 \) and \( \nu_2 \) to be restrictions of the measure \( \nu \); the first to \( Q^* \), the second to \( \mathbb{R}^n - Q^* \). We claim it is enough to obtain a dimensional constant \( C_n > 0 \) such that
\begin{align*}
(2.4) \quad T_\phi \nu_2 & \leq C_n \lambda
\end{align*}
on \( Q \). Suppose for the moment that (2.4) has been proved and take \( \gamma > 2C_n \). Then
\[ \{ x \in Q; (T_\phi \nu)(x) > \gamma \lambda \} \subseteq \left\{ x \in Q; (T_\phi \nu_1)(x) > \frac{\gamma \lambda}{2} \right\}. \]
Now,
\begin{align*}
(2.5) \quad \int_Q \Phi(x-z) \, dx & \leq \int_{|y| < \sqrt{n}|Q|^n} \Phi(y) \, dy.
\end{align*}
This means
\begin{align*}
\int_Q (T_\phi \nu_1)(x) \, dx & = \int_Q dx \int_{Q^*} \Phi(x-y) \, dv(y) \\
& = \int_{Q^*} dv(y) \int_Q \Phi(x-y) \, dx \leq \int_{|y| < \sqrt{n}|Q|^n} \Phi(y) \, dy \int_{Q^*} dv(y) \\
& \leq (7\sqrt{n})^n |Q| (M_\phi \nu)(x_k) \leq (7\sqrt{n})^n \beta \lambda |Q|.
\end{align*}
Thus with \( C = 2(7\sqrt{n})^n \),
\[ \left\| \left\{ x \in Q; (T_\phi \nu_1)(x) > \frac{\gamma \lambda}{2} \right\} \right\| \leq \frac{2}{\gamma \lambda} \int_Q (T_\phi \nu_1)(x) \, dx > \frac{C \beta}{\gamma} |Q|. \]
Therefore,
\[ |\{ T_\phi \nu > \gamma \lambda \text{ and } M_\phi \nu \leq \beta \lambda \}| = \sum_k |\{ x \in Q_k; (T_\phi \nu)(x) > \gamma \lambda \}| \leq \frac{C \beta}{\gamma} \sum_k |Q_k| \leq C \frac{\beta}{\gamma} |\{ M(T_\phi \nu) > \lambda \}|. \]
To prove (2.4) we'll require the fact that $C'_n > 0$ exists with

$$(2.6) \quad \Phi(y) \leq \frac{C'_n}{r^n} \int_{|y-z| \leq r} \Phi(z) \, dz, \; 0 < r \leq |y|.$$ 

As $\Phi$ is nonincreasing, this would be true if it were known to hold whenever $\Phi$ is the characteristic function of a ball centred at the origin. For this it suffices to know that the set of $z$ in the ball $|y-z| \leq r$ satisfying $|z| \leq |y|$ occupies at least a fixed fraction of the ball. The change of variable $z = |y|v$, followed by the rotation that sends $-y$ to $e_1 = (1,0,\ldots,0)$, reduces the problem to the relative size of the intersection of the balls $|v| \leq 1$ and $|v-e_1| \leq s$, $0 < s < 1$, to the size of the ball $|v-e_1| \leq s$ itself. But for these sets the result is clear.

If $x \in Q$ (where $Q$ denotes some fixed $Q_0$) and $y \in \mathbb{R}^n - Q^*$, then $|x-y| \geq |Q|^1$. Thus taking $r = |Q|^1$ in (2.6), we get

$$(Tv_2)(x) = \int_{\mathbb{R}^n - Q^*} \Phi(x-y) \, dv(y) \leq \frac{C'_n}{r^n} \int_{\mathbb{R}^n - Q^*} dv(y) \int_{|z| \leq r} \Phi(x-y-z) \, dz.$$ 

Making the substitution $v = x - z$, the last expression becomes

$$\frac{C'_n}{r^n} \int_{|x-r| \leq r} (T\Phi v_2)(v) \, dv \leq \frac{C'_n}{r^n} \int_Q (T\Phi v)(x) \, dx \leq \frac{C'_n}{r^n} \lambda |Q^*| = C_n \lambda,$$

with $C_n = (7\sqrt{n})^n C'_n$, since $Q^*$ intersects $\mathbb{R}^n - \Omega_1 = \{M(T\Phi v) \leq \lambda\}$ by the Whitney condition. This completes the proof.

**Theorem 2.3.** — Suppose $\Phi$ is a nonnegative, locally integrable radially decreasing function satisfying (2.2). Then for $1 < p \leq q < \infty$ and $\mu$ a positive locally finite Borel measure on $\mathbb{R}^n$, the following statements are equivalent:

1. There exists $C > 0$ so that whenever $f$ is a nonnegative measurable function on $\mathbb{R}^n$

$$\left[ \int_{\mathbb{R}^n} (T\Phi f)(x)^q \, d\mu(x) \right]^{\frac{1}{q}} \leq C \left[ \int_{\mathbb{R}^n} f(x)^p \, dx \right]^{\frac{1}{p}}.$$
2. There exists $C > 0$ so that for all dyadic cubes $Q$

\[ \left[ \int_{\mathbb{R}^n} T_{\Phi}(\chi_Q \mu)(x)^{p'} \, dx \right]^{\frac{1}{p'}} \leq C \left[ \mu(Q) \right]^{\frac{1}{q'}} < \infty \]

where $p' = \frac{p}{p - 1}$, $q' = \frac{q}{q - 1}$.

3. There exists $K > 0$ so that for all dyadic cubes $Q$

\[ \left[ \int_Q (M_{\Phi} \chi_Q \mu)(x)^{p'} \, dx \right]^{\frac{1}{p'}} \leq K \left[ \mu(Q) \right]^{\frac{1}{q'}} < \infty . \]

Moreover, the least possible $C$, $C'$ and $K$ in the above are all within constant multiples of one another, the constants being independent of $\Phi$ and $\mu$.

Proof. — Let $M_{\Phi}^{dy}$ denote the dyadic analogue of $M_{\Phi}$ given by

\[
M_{\Phi}^{dy}(x) = \sup_{x \in Q \text{ dyadic}} \left[ \frac{1}{|Q|} \int_{|y| \leq |Q|^n} \Phi(y) \, dy \right] \int_Q dv
\]

for $x \in \mathbb{R}^n$ and $\nu$ a locally finite positive measure. We claim that for all such $\nu$,

\begin{align*}
(2.7) & \quad \int_{\mathbb{R}^n} |M_{\Phi}^{dy}(x)|^{p'} \, dx \leq \int_{\mathbb{R}^n} |M_{\Phi}(x)|^{p'} \, dx \leq C_1 \int_{\mathbb{R}^n} |T_{\Phi}(x)|^{p'}, \\
(2.8) & \quad \int_{\mathbb{R}^n} |T_{\Phi}(x)|^{p'} \, dx \leq C_2 \int_{\mathbb{R}^n} |M_{\Phi}(x)|^{p'} \, dx \leq C_3 \int_{\mathbb{R}^n} |M_{\Phi}^{dy}(x)|^{p'},
\end{align*}

where the constants $C_1$, $C_2$, $C_3$ depend only on $n$ and $p(1 < p < \infty)$. The first inequality in (2.7) is trivial and the second inequality follows from part (a) of Theorem 2.2 and the classical $L^p$ inequality for $M$ ([18]). The first inequality in (2.8) follows from part (b) of Theorem 2.2 as in [6]. Finally, to prove the second inequality in (2.8), we apply a standard covering argument to \{ $M_{\Phi} \nu \geq \lambda$ \} (where $\lambda > 0$) to obtain the existence of cubes $(Q_k)_k$ with disjoint triples satisfying

\begin{align*}
(i) & \quad \left[ \frac{1}{|Q_k|} \int_{|y| \leq |Q|^n} \Phi(y) \, dy \right] \int_{Q_k} dv > \lambda \quad \text{for all } k, \\
(ii) & \quad |\{ M_{\Phi} \nu > \lambda \}| \leq C \sum_k |Q_k|.
\end{align*}

Now each $Q_k$ is covered by at most $2^n$ dyadic cubes $(I_{ij})_{1 \leq i \leq 2^n}$ with
There is at least one of these dyadic cubes, say $I_k = I_k^*$, with $\int_{I_k} dv \geq 2^{-n} \int_{Q_k} dv$. Then, since $\Phi$ is r.d. and $|I_k| \leq |Q_k|$, 

$$\left[ \frac{1}{|I_k|} \int_{|y| \leq |I_k|^{1/2}} \Phi(y) \, dy \right] \int_{I_k} dv > 2^{-n} \lambda \quad \text{for all } k$$

and so $\bigcup_k I_k \subseteq \{ M_{\phi}^d v > 2^{-n} \lambda \}$. Since the $I_k$'s are pairwise disjoint, we have

$$\left| \{ M_{\phi}^d v \geq \lambda \} \right| \leq C \sum_k |Q_k| \leq C \sum_k |I_k| \leq C |\{ M_{\phi}^d v > 2^{-n} \lambda \}|$$

and (2.8) follows upon multiplying this inequality by $\lambda^{p' - 1}$ and then integrating over $(0, \infty)$.

From (2.3), (2.7) and (2.8) we obtain that the trace inequality in 1. holds if and only if there is $C > 0$, comparable to the one in (2.1), for which

$$\left[ \int_{\mathbb{R}^n} (M_{\phi}^d f \mu)(x) v^p \, dx \right]^{\frac{1}{p'}} \leq C \left[ \int_{\mathbb{R}^n} f(x) v^q \, d\mu(x) \right]^{\frac{1}{q'}} \quad \text{for all } f.$$  

Theorem A of [16] (with $M_{\phi}^d$ in place of $M_{\mu, \sigma}$, the proof is unchanged) shows that (2.9) holds if and only if there is $C > 0$, comparable to that in (2.9), for which

$$\left[ \int_{\mathbb{R}^n} [M_{\phi}^d (\chi_Q d\mu)] v^p \right]^{\frac{1}{p'}} \leq C \mu(Q)^\frac{1}{q'} < \infty$$

for all dyadic cubes $Q$. Theorem 2.3 now follows easily. The trace inequality 1. implies its dual (2.3) which in turn implies 2. upon taking $f = \chi_Q$. Inequality 2. implies 3. by (2.7) and finally, 3. $\Rightarrow$ (2.10) $\Rightarrow$ (2.9) $\Rightarrow$ 1.

3. Schrödinger operators.

In this section, Theorem 2.3 is used to refine the estimates for eigenvalues of a Schrödinger operator $H = -\Delta - v$ given in Theorem 5, Chapter II, of [10]. By eigenvalues, we mean the numbers
\[ \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N \] \quad \text{where } \lambda_N \text{ is the maximum over all } N-1 \text{ tuples } \Phi_j, \ldots, \Phi_{N-1} \text{ of the quantity } \inf \frac{\langle Hu, u \rangle}{\langle u, u \rangle}, \text{ the infimum being over all } u \in Q(H), \langle u, \Phi_j \rangle = 0, j = 1, \ldots, N-1. \text{ Here } Q(H) \text{ denotes the form domain of } H \text{ (see [16]) and } \langle Hu, u \rangle = \int_{\mathbb{R}^n} (|\nabla u|^2 - v|u|^2) \text{ for } u \in Q(H). \text{ Recall that } I_2 f(x) = \int_{\mathbb{R}^n} |x-y|^2 f(y) \, dy \text{ denotes the Newtonian potential of } f.

**Theorem 3.1.** Let \( H = -\Delta - v \), where \( v(x) \geq 0 \) is locally integrable on \( \mathbb{R}^n \) and \( n \geq 3 \). Denote the \( v \) measure of \( Q \), \( \int_Q v(x) \, dx \), by \( |Q|_v \). There are positive constants \( C, c \) depending only on the dimension \( n \) such that the least eigenvalue \( \lambda_1 \) of \( H \) satisfies
\[ E_{\text{sm}} \leq -\lambda_1 \leq E_{\text{big}} \]
where
\[ E_{\text{sm}} = \sup \left\{ |Q|^{-2/n}; |Q|_v^{-1} \int_Q I_2(\chi_Q v) \right\} \]
\[ E_{\text{big}} = \sup \left\{ |Q|^{-2/n}; |Q|_v^{-1} \int_Q I_2(\chi_Q v) \right\}. \]

**Example 3.2.** Consider Example V in [10]: a particle in a rectangular box \( B = B_1 \times B_2 \times \cdots B_n \) with side lengths \( \delta_1 \leq \delta_2 \leq \cdots \delta_n \). Let \( v = \chi_B \) and let \( x_B \) denote the centre of \( B \). Since
\[ \sup |Q|^{-1} \int_Q I_2(\chi_Q v) \approx I_2 v(x_B) \approx \delta_1^2 + \delta_1 \delta_2 + \delta_1 \delta_2 \log(\delta_3/\delta_2), \]
Theorem 3.1 yields the correct order of magnitude for the energy, \( E_{\text{critical}} \), needed to trap a particle in \( B \), namely
\[ E_{\text{critical}} = \sup \{ 1 - \delta_1; -\Delta - Ev \geq 0 \} = 1/\delta_1 \delta_2 \log(1 + \delta_3/\delta_2). \]

A refinement of Theorems 6 and 6' in Chapter II of [10], similar to the one above, is given in

**Theorem 3.3.** Let \( H = -\Delta - v \) where \( v(x) \geq 0 \) is locally integrable on \( \mathbb{R}^n \) and \( n \geq 3 \). There are positive constants \( C, c \) depending only on the dimension \( n \) such that:

(A) Suppose \( \lambda \geq 0 \) and let \( Q_1, \ldots, Q_N \) be a collection of cubes of side length at most \( \lambda^{-1/2} \) whose doubles are pairwise disjoint. Suppose further that
\[ |Q_j|^{-1} \int_{Q_j} I_2(\chi_{Q_j} v) v \geq C, \quad 1 \leq j \leq N. \text{ Then } H \text{ has at least } N \text{ eigenvalues} \leq -\lambda. \]

(B) Conversely, suppose \( \lambda \geq 0 \) and that \( H \) has at least \( CN \) eigenvalues \( \leq -\lambda \). Then there is a collection of pairwise disjoint (dyadic) cubes \( Q_1, \ldots, Q_N \) of side lengths at most \( \lambda^{-\frac{1}{2}} \) that satisfy

\[ |Q_j|^{-1} \int_{Q_j} I_2(\chi_{Q_j} v) v \geq c, \quad 1 \leq j \leq N. \]

Roughly speaking, Theorem 3.3 says that the negative eigenvalues of \( H \) are approximately given by \(- |Q|^{-2/n} \) as \( Q \) ranges over the minimal dyadic cubes satisfying \( |Q|^{-1} \int_{Q} I_2(\chi_{Q} v) v \geq C \).

In [10], results corresponding to Theorems 3.1 and 3.3 were obtained with the quantity \( |Q|^{-1} \int_{Q} I_2(\chi_{Q} v) v \) replaced by the simpler average \( C|Q|^2|\int_{Q} v|^{2/p} \int_{Q} v^{2/p} \) in part (A) of Theorem 3.3 and by \( C_p |Q|^{2-1/p} \left( \int_{Q} v^{2/p} \right)^{1/p} \) in part (B). A comparison of these quantities is made in Remark 3.5 at the end of this section. Chang, Wilson, and Wolff [5] show part (B) of Theorem 3.3 holds for \( v \) if \( \sup_{Q} |Q|^{-1} \int_{Q} v(x) \Phi(|Q|^{2/p}(x)) \, dx < \infty \), where \( \Phi: [0,\infty] \to [1,\infty] \) is increasing and \( \int_{1}^{\infty} \frac{dx}{x\Phi(x)} < \infty \). See also Chanillo and Wheeden [6].

**Proof of Theorem 3.1.** — The Schwartz class \( S \) is dense in \( Q(H) \) and thus we have

\[
\lambda_1 = \inf_{u \in Q(H)} \frac{\langle Hu, u \rangle}{\langle u, u \rangle} = \sup_{u \in S} \frac{\int |u|^2 v - \int |\nabla u|^2}{\int |u|^2} = \inf \{ \alpha > 0; \int |u|^2 v \leq \int |\nabla u|^2 + \alpha |u|^2 \} \leq \int (|\xi|^2 + \alpha) |\hat{u}(\xi)|^2 \, d\xi, \quad u \in S \}
\[
= \inf \{ \alpha > 0; \int (I^2 f)^2 v \leq \int f^2, f \geq 0 \}
\]
where $I_1^v$ is the operator with r.d. kernel $K_1^v$ defined by

$$(K_1^v)^{\alpha} (\xi) = (|\xi|^2 + \alpha)^{-\frac{1}{2}}.$$  Thus $K_1^v(x) = G_1(\alpha^{\frac{1}{2}}x)$ and

$$K_1^v(x) = \alpha^{\frac{1}{2}} G_1(\alpha^{\frac{1}{2}}x).$$

If we let $C_a$ denote the least constant such that

$$\int (I_1^v f)^2 v \leq C_a \int f^2 \quad \text{for all } f \geq 0,$$

then $-\lambda_1 = \inf \{\alpha; C_a \leq 1\}$. By Theorem 2.3,

$$(3.1) \quad C_a \approx \sup_Q \frac{1}{|Q|^v} \int |I_1^v(\chi_Q^v)|^2$$

in the sense that the ratio of the left and right sides is bounded between two constants independent of $\alpha$ and $v$. We now show that, in fact, the supremum in (3.1) need only be taken over those cubes $Q$ with

$|Q|^\frac{1}{n} \leq \alpha^{-\frac{1}{2}}$. To this end, set

$$M = \sup_{|Q|^\frac{1}{n} \leq \alpha^{-\frac{1}{2}}} \frac{1}{|Q|^v} \int |I_1^v(\chi_Q^v)|^2$$

and suppose $Q$ is a cube with $|Q|^\frac{1}{n} > \alpha^{-\frac{1}{2}}$. Express $Q$ as a union of congruent cubes, $Q_i$, having pairwise disjoint interiors and common sidelengths, $|Q_i|^\frac{1}{n}$, satisfying $\frac{1}{2} \alpha^{-\frac{1}{2}} \leq |Q_i|^\frac{1}{n} \leq \alpha^{-\frac{1}{2}}$. Then, we claim

$$(3.2) \quad \int |I_1^v(\chi_Q^v)|^2 = \sum_{i,j} \int I_1^v(\chi_{Q_i}^v) I_1^v(\chi_{Q_j}^v)$$

$$\leq C \sum_i \int |I_1^v(\chi_{Q_i}^v)|^2$$

$$\leq CM \sum_i |Q_i|^v = CM |Q|^v.$$

The second inequality holds by definition of $M$ and since $|Q_i|^\frac{1}{n} \leq \alpha^{-\frac{1}{2}}$. To prove the first inequality, we consider two cases. First, when $Q_i$ and $Q_j$ are adjacent, we simply use

$$\int I_1^v(\chi_{Q_i}^v) I_1^v(\chi_{Q_j}^v) \leq \frac{1}{2} \int |I_1^v(\chi_{Q_i}^v)|^2 + \frac{1}{2} \int |I_1^v(\chi_{Q_j}^v)|^2.$$

To treat the case when $Q_i$ and $Q_j$ have a distance of roughly $k$
sidelengths between them, $k \geq 1$, we require the facts that $K_2^*(x) \approx |x|^{2-n}$ if $|x| \leq \alpha^{-1/2}$ and $K_2^*(x) \leq C \alpha^{-2} e^{-\sqrt{a|x|}}$ if $|x| > \alpha^{-1/2}$, for which see [4]. We then have

$$\int I_1^*(\chi_{Q_i}^v)I_1^*(\chi_{Q_j}^v) = \int_{Q_i} I_2^*(\chi_{Q_j}^v)(x)v(x) \, dx \leq C \alpha^{-2} e^{-k|Q_i|v|Q_j|v}.$$  

However, $I_1^*(\chi_{Q_i}^v)(x) \geq C \alpha^{-1/2}$ for $x \in Q_i$ and so

$$|Q_i|v \leq \frac{\alpha^{1/2}}{C} \int_{Q_i} I_1^*(\chi_{Q_i}^v) = \frac{\alpha^{1/2}}{C} \int_{Q_i} I_1^*(\chi_{Q_i}^v)(x) \, dx.$$  

Thus

$$2|Q_i|v|Q_j|v \leq |Q_i|^2 + |Q_j|^2 \leq C \alpha \left( \left[ \int_{Q_i} I_1^*(\chi_{Q_i}^v) \right]^2 + \left[ \int_{Q_j} I_1^*(\chi_{Q_j}^v) \right]^2 \right) \leq C \alpha^{1-\frac{n}{2}} \left( \int_{Q_i} [I_1^*(\chi_{Q_i}^v)]^2 + \int_{Q_j} [I_1^*(\chi_{Q_j}^v)]^2 \right).$$  

Now, for a fixed cube $Q_i$, there are at most $Ck^{n-1}$ cubes $Q_j$ at a distance of roughly $k$ sidelengths from $Q_i$. Combining all of the above, we obtain

$$\sum_{i \neq j} I_1^*(\chi_{Q_i}^v)I_1^*(\chi_{Q_j}^v) \leq C \left[ 1 + \sum_{k=1}^{\infty} k^{n-1} e^{-k} \right] \sum_{i} \left[ I_1^*(\chi_{Q_i}^v) \right]^2$$

which yields the first inequality in (3.2). From (3.1) and (3.2), we have $C_\alpha \approx M$ and since $\int [I_1^*(\chi_{Q}^v)]^2 = \int I_2^*(\chi_{Q}^v) v \approx \int I_2(\chi_{Q}^v) v$ when $|Q|^n \leq \alpha^{-1/2}$, we finally have

$$C_\alpha \approx \sup_{|Q|^{1/n} \leq \alpha^{-1/2}} \frac{1}{|Q|v} \int_{Q} I_2(\chi_{Q}^v)v$$

and Theorem 3.1 follows readily.

Proof of Theorem 3.3, part (A). — As in [10], it suffices by elementary functional analysis to construct an N-dimensional subspace $\Omega \subset Q(H)$ so
that \( \langle Hu, u \rangle \leq -\lambda \int |u|^2 \) for \( u \) in \( \Omega \). Our hypothesis implies

\[
\frac{1}{|Q_j|} \int_{Q_j} I_2(\chi_{Q_j}v)v \geq C \quad \text{for } j = 1, \ldots, N.
\]

Since \( \int_{Q} I_2(\chi_{Q_j}v)v \leq \left( \int_{Q} [I_2(\chi_{Q_j}v)]^2 v \right)^{\frac{1}{2}} |Q|^{\frac{1}{2}} \) by Holder's inequality, we actually have

\[
\int_{Q_j} [I_2(\chi_{Q_j}v)]^2 v \geq C \int_{Q_j} I_2(\chi_{Q_j}v)v, \quad 1 \leq j \leq N.
\]

This suggests we let \( \Omega \) be the linear span of \( \{f_j\}_{j=1}^N \) where \( f_j = \Phi_j I_2(\chi_{Q_j}v) \) and \( \Phi_j = 1 \) on \( \frac{3}{2} Q_j \) with supp \( \Phi_j \) contained in \( 2Q_j \). Here the \( \Phi_j \) are dilates and translates of a fixed \( \Phi \in C_c^\infty(\mathbb{R}^n) \). We have immediately that

\[
(3.3) \quad \int_{Q_j} f_j^2 v \geq C \int_{Q_j} I_2(\chi_{Q_j}v)v \quad \text{for } 1 \leq j \leq N.
\]

By hypothesis, the supports of the \( f_j \) are pairwise disjoint and so we need only establish

\[
(3.4) \quad \langle (-\Delta + \lambda) f_j, f_j \rangle \leq \int (f_j)^2 v \quad \text{for } 1 \leq j \leq N
\]

in order to conclude \( \langle Hu, u \rangle \leq -\lambda \int |u|^2 \) for \( u \) in \( \Omega \), as required. To prove (3.4), we let \( G_j = 2Q_j - \frac{3}{2} Q_j \) and compute that

\[
(-\Delta + \lambda)f_j = (-\Delta + \lambda)[\Phi_j I_2(\chi_{Q_j}v)]
\]

\[
= \chi_{Q_j}v + \chi_{G_j}(-\Delta + \lambda)[\Phi_j I_2(\chi_{Q_j}v)]
\]

\[
= A_j + B_j
\]

since \( I_2 = (-\Delta + \lambda)^{-1} \). Now

\[
\langle A_j, f_j \rangle = \int_{Q_j} I_2(\chi_{Q_j}v)v \leq \frac{1}{C} \int f_j^2 v \quad (\text{by } 4.3) \leq \frac{1}{2} \int f_j^2 v
\]

provided \( C \) is chosen \( \geq 2 \). It remains to verify

\[
\langle B_j, f_j \rangle \leq C \int_{Q_j} I_2(\chi_{Q_j}v)v \quad \text{for all } j \quad \text{since then } (3.4) \text{ will follow from } (3.3)
\]
and the previous estimate provided $C \geq 2C'$. Now

$$
|B_j| \leq x_{Q_j} |\Phi_j| \Delta I^2_2(x_{Q_j}v) + 2|\nabla \Phi_j| |\nabla I^2_2(x_{Q_j}v)|
$$

$$
+ (\lambda + |\Delta \Phi_j|)[I^2_2(x_{Q_j}v)]
$$

$$
= D_j + E_j + F_j.
$$

Using the estimates $|D^sK^2(x)| \leq C|x|^{2-s}$, for $s \geq 0$ and $|x| \leq C \lambda^{-\frac{1}{2}}$ (see [4]) we obtain that on $G_j$,

$$
I^2_2(x_{Q_j}v)(x) \leq C|Q_j|^{\frac{2}{n} - 1} \int_{Q_j} v
$$

$$
|\nabla I^2_2(x_{Q_j}v)(x)| \leq C|Q_j|^{\frac{1}{n} - 1} \int_{Q_j} v
$$

$$
|\Delta I^2_2(x_{Q_j}v)(x)| \leq C|Q_j|^{-1} \int_{Q_j} v.
$$

These inequalities, together with $|\Phi_j| \leq 1$, $|\nabla \Phi_j| \leq C|Q_j|^{-\frac{1}{n}}$, $|\Delta \Phi_j| \leq C|Q_j|^{-\frac{2}{n}}$ and the hypothesis $\lambda \leq |Q_j|^{-\frac{2}{n}}$, yields

$$
D_j, E_j, F_j \leq C|Q_j|^{-1}|Q_j|^v.
$$

Since $f_j(x) \leq C|Q_j|^{\frac{2}{n} - 1} \int_{Q_j} v$ on $G_j$, (3.5) and (3.6) imply

$$
\langle B_j, f_j \rangle \leq C|Q_j|^{\frac{2}{n} - 1}|Q_j|^v.
$$

Finally,

$$
|Q_j|^{\frac{2}{n} - 1}\left(\int_{Q_j} v\right)^2 \leq C(\min_{x \in Q_j} I^2_2(x_{Q_j}v))\left(\int_{Q_j} v\right)
$$

$$
\leq C\int_{Q_j} I^2_2(x_{Q_j}v)v
$$

and this, combined with (3.7), shows that $\langle B_j, f_j \rangle \leq C' \int_{Q_j} I^2_2(x_{Q_j}v)v$ and completes the proof of part (A) of Theorem 3.3.

Proof of Theorem 3.3, part (B). We follow closely the argument of C. L. Fefferman and D. H. Phong in ([10]; proof of Theorem 6 in Chapter II), but with certain modifications designed to avoid the use of a square function. As in [10], it suffices to suppose $v$ bounded and to show that if $Q_1, \ldots, Q_N$ are the minimal dyadic cubes satisfying
\[
\frac{1}{|Q_j|} \int_{Q_j} I_2(\chi_{Q_j}v)v \geq c \quad \text{and} \quad |Q_j|^{\frac{1}{n}} \leq \lambda^{-\frac{1}{2}}, \quad \text{then} \quad H = -\Delta - v \quad \text{has at most} \quad CN \quad \text{eigenvalues} \quad \leq -\lambda \quad \text{(where the constant} \quad C \quad \text{is of course independent of the bound on} \quad v). \quad \text{As usual, this will be accomplished by exhibiting a subspace} \quad \Omega \subset L^2 \quad \text{of codimension} \quad \leq CN \quad \text{such that}
\]

\[(3.8) \quad \langle Hu, u \rangle \geq -\lambda \int |u|^2 \quad \text{for all} \quad u \in \Omega.
\]

We consider only the case \( \lambda = 0 \), the case \( \lambda > 0 \) requiring easy modifications. We begin by defining additional cubes \( Q_{N+1}, \ldots, Q_M \) as in [10]; i.e. let \( B \) be the collection of all dyadic cubes \( Q \) with
\[
\frac{1}{|Q|} \int_{Q} I_2(\chi_{Q}v)v \geq c \quad \text{and define the additional cubes} \quad Q_{N+1}, \ldots, Q_M \quad \text{to consist of (i) the maximal cubes in} \quad B, \quad \text{(ii) the branching cubes in} \quad B \quad \text{and (iii) the descendents of branching cubes in} \quad B. \quad \text{The descendents of a cube} \quad Q \quad \text{in} \quad B \quad \text{are those} \quad Q' \in B \quad \text{which are maximal with respect to the property of being properly contained in} \quad Q. \quad \text{A cube in} \quad B \quad \text{« branches » if it has at least two descendents. As shown in [10],} \quad M \leq CN. \quad \text{Still following [10] we define} \quad E_0 = R^n - \bigcup_{j=1}^{M} Q_j \quad \text{and} \quad E_j = Q_j \quad \text{minus its descendents for} \quad j \geq 1. \quad \text{In analogy with estimates (i) and (ii) of [10], we shall prove that the weights} \quad v_j = \chi_{E_j}v \quad \text{satisfy}
\]

\[(3.9) \quad \frac{1}{|Q_j|} \int_{Q_j} I_2(\chi_{Q}v_j)v_j \leq Cc \quad \text{for all} \quad 0 \leq j \leq M, Q \quad \text{dyadic cube.}
\]

In order to make use of (3.9) and the trace inequalities it implies we shall have to define the subspace \( \Omega \) so that

\[(3.10) \quad |u(x)| \leq C I_1(\chi_{E_j}|Vu|)(x) \quad \text{for} \quad x \in E_j, 0 \leq j \leq M, u \in \Omega.
\]

Indeed, if both (3.9) and (3.10) hold, then for \( u \in \Omega,
\]

\[
\int |u|^2v = \sum_{j=0}^{M} \int_{E_j} |u|^2v_j \\
\leq C \sum_{j=0}^{M} \int_{E_j} [I_1(\chi_{E_j}|Vu|)]^2v_j \quad \text{by (3.10)} \\
\leq Cc \sum_{j=0}^{M} \int_{E_j} |Vu|^2 \quad \text{by (3.9) and Theorem 2.3} \\
\leq \int |Vu|^2 \quad \text{if} \quad c \quad \text{small enough},
\]

THE TRACE INEQUALITY 223
and this is (3.8) for \( \lambda = 0 \). Thus it remains to construct \( \Omega \) of codimension \( \leq CN \) such that (3.10) holds. In the case \( 1 \leq j \leq N \), \( E_j \) is a cube and (3.10) holds whenever \( \int_{E_j} u = 0 \) by the following inequality of E. Fabes, C. Kenig and R. Serapioni ([9]; Lemma 1.4)

\[
\left| u(x) - \frac{1}{|Q|} \int_Q u \right| \leq CI_1(\chi_Q|\nabla u|)(x) \quad \text{for } x \in Q, \ Q \text{ a cube.}
\]

For the case when \( E_j \) is not a cube we will need the following lemma.

**Lemma 3.4.** Suppose \( Q_1, \ldots, Q_k \) are pairwise disjoint dyadic subcubes of a dyadic cube \( Q \) in \( \mathbb{R}^n \). Then there are (not necessarily dyadic or disjoint) cubes \( I_1, \ldots, I_m \) such that \( Q - \bigcup_{j=1}^k Q_j = \bigcup_{i=1}^m I_i \) and \( m \leq Ck \) where \( C \) is a constant depending only on the dimension \( n \). The above holds also for \( Q = \mathbb{R}^n \) if we allow the cubes \( I_i \) to be infinite, i.e. of the form \( J_1 \times J_2 \times \cdots \times J_n \) where each \( J_i \) is a semi-infinite interval.

This lemma has been obtained independently by S. Chanillo and R. L. Wheeden [6], with a proof much simpler than that appearing in a previous version of this paper. As a result, we refer the reader to [6] for a proof of the lemma.

We can now define the subspace \( \Omega \). For each \( j \) with \( j = 0 \) or \( N + 1 \leq j \leq M \), apply Lemma 3.4 with \( Q = Q_j \) and \( Q_1, \ldots, Q_k \) the descendents of \( Q_j \) (for \( j = 0 \), take \( Q = \mathbb{R}^n \) and \( Q_1, \ldots, Q_k \) to be the maximal cubes in \( B \)), to obtain cubes \( I_1^{(0)}, \ldots, I_m^{(0)} \) with \( E_j = \bigcup_{i=1}^m I_i^{(0)} \) and \( m \leq C \) (\# of descendents of \( Q_j \)). Note that \( E_j = Q_j \) for \( 1 \leq j \leq N \).

Now define

\[
\Omega = \{ u; \int_{Q_j} u = 0 \text{ for } 1 \leq j \leq N \text{ and } \int_{I_i^{(0)}} u = 0 \text{ for } N+1 \leq j \leq M, j = 0 \text{ and } 1 \leq i \leq m_j \}.
\]

If \( x \in E_j, N + 1 \leq j \leq M \) or \( j = 0 \), then \( x \in \text{some } I_i^{(0)} \) and thus for \( u \in \Omega \), \( |u(x)| \leq CI_1(\chi_{E_j^{(0)}}|\nabla u|)(x) \leq CI_1(\chi_{E_j}|\nabla u|)(x) \) by (3.11). Thus (3.10) holds. Finally, the codimension of \( \Omega \) is at most

\[
N + \sum_{j=0}^{N+1} m_j \leq N + C \sum_{j=0}^{N+1} \sum_{N+1 \leq j \leq M} \text{(# of descendents of } Q_j) \leq N + C(M+1) \leq CM.
\]
It remains now to establish (3.9). We begin with the case \( j \neq 0 \) of (3.9), and follow the corresponding argument in [10]. Since \( \text{supp} \ v_j \subset Q_j \), we need only check (3.9) for dyadic cubes \( Q \in B \) with \( Q \subset Q_j \) and in fact, only for proper dyadic subcubes of \( Q_j \) (since if \( Q = \bigcup_{i=1}^{2^n} Q_i \), then

\[
\int_Q I_2(\chi_Q v) = \int [I_1(\chi_Q v)]^2 \\
= \sum_{i,j} \int I_1(\chi_{Q_i} v)I_1(\chi_{Q_j} v) \leq \frac{1}{2} \sum_{i,j} [I_1(\chi_Q v)]^2 \\
\leq C_n \sum_{i=1}^{2^n} [I_1(\chi_{Q_i} v)]^2 \\
= C_n \sum_{i=1}^{2^n} \int_{Q_i} I_2(\chi_{Q_i} v) v.
\]

As in [10], the only "non-trivial" case occurs when \( Q \in B \) is neither minimal nor branching and \( Q \) contains \( Q^* \), the unique maximal \( Q_i, 1 \leq i \leq M, \) that is properly contained in \( Q_j \) (see the argument on p. 157-158 of [10]). To obtain (3.9) in this case we use a Whitney decomposition in place of the Calderon-Zygmund decomposition used in [10]. There is a dimensional constant \( C \) so large that we can choose pairwise disjoint dyadic subcubes \( \hat{Q}_a \) of \( Q - Q^* (= \bigcap_{i=1}^{2^n} Q) \) such that each \( \hat{Q}_a \) satisfies

\[(3.12) \quad \text{either } |\hat{Q}_a| = |Q_j^*| \text{ and } \text{dist}(\hat{Q}_a, Q_j^*) \leq C \]

\[\text{or } 2 \leq \frac{\text{dist}(\hat{Q}_a, Q_j^*)}{\text{diam } \hat{Q}_a} \leq 2C.\]

Then

\[
\int_Q I_2(\chi_Q v_j)v_j = \sum_{\alpha, \beta} \int_{Q_a} I_2(\chi_{Q_{\beta}} v) v \\
\leq C \sum_{|\alpha:Q_{\alpha} \text{ touches } Q_{\beta}|} \int I_1(\chi_{Q_{\alpha}} v) I_1(\chi_{Q_{\beta}} v) \\
+ C \sum_{|\alpha:Q_{\alpha} \text{ touches } Q_{\beta}|} \int_{Q_{\alpha}} I_2(\chi_{Q_{\alpha}} v) v = D + E.
\]

Now (3.12) shows that the number of \( \hat{Q}_\beta \) touching a given \( \hat{Q}_a \) doesn't
exceed a dimensional constant and so

\[ D \leq C \sum_{a} \int_{Q_{a}} |I_{2}(\chi_{Q_{a}}v)|^{2} = C \sum_{a} \int_{Q_{a}} I_{2}(\chi_{Q_{a}}v)v \leq Cc \sum_{a} \int_{Q_{a}} v_{j} = Cc \int_{Q} v_{j} \]

since the \( Q_{a} \) are not in \( B \). Condition (3.12) also shows that if \( |Q_{p}| \leq |Q_{a}| \) and \( Q_{p}, Q_{a} \) do not touch, then \( \text{dist}(Q_{p},Q_{a}) \geq c |Q_{a}|^{\frac{1}{n}} \). Thus

\[ E \leq C \sum_{a} \left( \int_{Q_{a}} v \right)^{\frac{n}{2}} \sum_{|Q_{p}| \leq |Q_{a}|} \left[ \int_{Q_{p}} v \right]. \]

But \( |Q_{p}|^{\frac{n}{2}} \int_{Q_{p}} v \leq \frac{1}{|Q_{p}|^{\frac{n}{2}}} \int_{Q_{p}} I_{2}(\chi_{Q_{p}}v)v \leq c \) since \( Q_{p} \notin B \) and, by (3.12), the number of \( Q_{p} \) of a given size does not exceed a dimensional constant. Thus

\[ E \leq Cc \sum_{a} \left( \int_{Q_{a}} v \right)^{\frac{n}{2}} \sum_{|Q_{p}| \leq |Q_{a}|} \left[ \int_{Q_{p}} v \right]. \]

and this completes the verification of (3.9) for \( j \neq 0 \). For \( j = 0 \), we again suppose \( Q \) dyadic in \( B \). If \( Q \subset \) some \( Q_{1}, \ldots Q_{M} \), then \( \text{supp } v_{0} \cap Q = \emptyset \) and (3.9) holds trivially. Otherwise, \( Q \) contains a unique maximal \( Q_{i}(1 \leq i \leq M) \), say \( Q^{*} \), and we may argue as above to obtain (3.9). This completes the proof of Theorem 3.3.

**Remark 3.5.** — In [10] it is shown that \( \sup_{Q} |Q|^{\frac{n}{p} - 1} \int_{Q} v \leq C \) is necessary and \( \sup_{Q} |Q|^{\frac{n}{p} - 1} \left( \int_{Q} v^{p} \right)^{1/p} \leq C_{p}, p > 1 \), sufficient for the \( L^{2} \) trace inequality (1.1) with \( T_{\varphi} = I_{1} \). We give here a direct proof that

\[ \sup_{Q} |Q|^{\frac{n}{p} - 1} \int_{Q} v \leq C \sup_{Q} |Q|^{\frac{n}{p} - 1} \int_{Q} I_{2}(\chi_{Q}v)v \]

\[ \leq C_{p} \sup_{Q} |Q|^{\frac{n}{p} - 1} \left( \int_{Q} v^{p} \right)^{1/p}, \quad p > 1. \]

The first inequality in (3.20) follows from the observation that \( I_{2}(\chi_{Q}v)(x) \geq C |Q|^{\frac{n}{p} - 1} \int_{Q} v \) for \( x \) in a cube \( Q \).
Let $B_p = \sup_{Q} |Q|^{\frac{2}{p} - 1} \left( \int_Q |v|^p \right)^{1/p}$. Suppose first that $v$ satisfies the $A_\infty$ condition of B. Muckenhoupt. Choose $p$ so close to 1 that the reverse Hölder condition $\left( \frac{1}{|Q|} \int_Q |v|^p \right)^{1/p} \leq C_p |v|^{-1} \int_Q v$ holds for all cubes $Q$. Let $M^v f(x) = \sup_{x \in Q} |Q|^{\frac{2}{p} - 1} \int_Q |f|$. Since $M^v \leq B_p$ on $Q$,

$$\int_Q I_2(\chi_Q v) \leq \left( \int_Q I_2(\chi_Q v)^{p'} \right)^{1/p'} \left( \int_Q v^p \right)^{1/p} \leq C_p \left( \int_Q M^v(\chi_Q v)^{p'} \right)^{1/p'} \left( \int_Q v^p \right)^{1/p} \leq C_p B_p |Q|^{1/p'} \int_Q v^p \leq C_p B_p \int_Q v.$$ (3.21)

For the general case, we use the observations in [10] that $v^+(x) = \sup_{x \in Q} \left( \frac{1}{|Q|} \int_Q |v|^p \right)^{1/p}$ satisfies the $A_\infty$ condition and $M^v \leq C_p B_p ([10]; p. 153)$. The above argument then yields (3.21) with $v^+$ in place $v$. Since $v \leq v^+$, (3.20) follows. This is of course obvious from Theorem 2.3, but can also be proved directly. Finally, we point out that the condition $M^v \leq C_p$ is equivalent to the boundedness of $M_p$ from $L^2$ to $L^2(v^p)$ ([17]). Together with the inequality $|I_1(f(x))| \leq C_p M_p |f(x)|^{1/p} Mf(x)^{1/p'}$ of D. R. Adams, this yields another proof that $M^v \leq C_p$ is sufficient for the $L^2$ trace inequality (1.1) with $T_\phi = I_1$. J. M. Wilson has recently communicated to us yet another proof.

**BIBLIOGRAPHY**


