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The trace inequality and eigenvalue estimates for Schrödinger operators


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THE TRACE INEQUALITY AND EIGENVALUE ESTIMATES FOR SCHRODINGER OPERATORS

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1. Introduction.

This paper deals with potential operators $T_\Phi$ given at Lebesgue measurable $f$ on $\mathbb{R}^n$ by a convolution integral

$$(T_\Phi f)(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) \, dy,$$

provided this integral exists for almost all $x \in \mathbb{R}^n$. The kernels $\Phi(y)$ are radially decreasing (r.d.) functions; that is, they are nonnegative, locally integrable radial functions on $\mathbb{R}^n$, which are nonincreasing in $|y|$. These $T_\Phi$ include the Riesz potential operator $I_\alpha$ whose kernel $K_\alpha$ is defined directly as

$$K_\alpha(y) = |y|^{a-n}, \quad 0 < \alpha < n$$

and the Bessel potential operator $J_\alpha$ with kernel $G_\alpha$ defined in terms of its Fourier transform $\mathcal{G}_\alpha$ by

$$\mathcal{G}_\alpha(\zeta) = \int_{\mathbb{R}^n} G_\alpha(x) e^{-\kappa \cdot x} \, dx = (1 + |\zeta|^2)^{-\frac{a}{2}}, \quad 0 < \alpha < n.$$ 

Given an r.d. kernel $\Phi$ and $1 < p < \infty$, we wish to characterize the (possibly singular) positive Borel measures $\mu$ on $\mathbb{R}^n$ for which there exists $C > 0$ such that

$$\int_{\mathbb{R}^n} (T_\Phi f)(x)^p \, d\mu(x) \leq C \int_{\mathbb{R}^n} f(x)^p \, dx$$

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for all nonnegative measurable $f$. Clearly this will be true if and only if $T_\Phi$ is a bounded linear operator between the Lebesgue spaces $L^p(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n,\mu)$. An important special case, with $p=2$ and $\Phi=\mathcal{G}_1$, arises in estimating the spectrum of Schrödinger operators and will be considered in detail below. Another special case is treated in Stein [19], where it is shown that (1.1) holds for $J_\alpha$ when $\mu = \mu_k$, $\alpha > \frac{n-k}{p}$, where

$$
\mu_k(E) \equiv m_k(E \cap \mathbb{R}^k),
$$

$m_k$ being $k$-dimensional Lebesgue measure on $\mathbb{R}^k$ considered as a subset of $\mathbb{R}^n$. The inequality of [19] can be stated in the equivalent form

$$
\int_{\mathbb{R}^n} (J_\alpha f)(x_1, \ldots, x_k, 0, \ldots, 0)^p \, dx_1, \ldots, dx_k \leq C \int_{\mathbb{R}^n} f(x_1, \ldots, x_n)^p \, dx_1, \ldots, dx_n.
$$

It is thus a statement about the restriction, or trace, of $J_\alpha f$. For this reason we follow other authors in referring to (1.1) as "the trace inequality". 

Generalizing results of Adams [1] and Maz'ya [14], K. Hansson in [12] has characterized the $\mu$ satisfying (1.1) in terms of capacities (see also B. Dahlberg [8]). He shows the trace inequality holds if and only if $K > 0$ exists for which

$$
(1.2) \quad \mu(E) \leq K \text{cap}(E)
$$

whenever $E$ is a compact subset of $\mathbb{R}^n$. Here $\text{cap}(E)$ denotes the $L^p$ capacity associated with the kernel $\Phi$,

$$
\text{cap}(E) = \inf \left\{ \int_{\mathbb{R}^n} f(x)^p \, dx : f \geq 0 \text{ and } T_\Phi f \geq 1 \text{ on } E \right\}.
$$

A criterion such as (1.2) can be difficult to verify for all compact sets $E$. On the other hand if one only requires (1.2) to hold for a class of simple sets such as all cubes $Q$ with sides parallel to the coordinate axes, the resulting condition is no longer sufficient (D. Adams [2]). For example, when $n=p=2$, $I_{\frac{1}{2}}$ doesn't satisfy (1.1) with $\mu_1$, yet inequality (1.2) for cubes, which amounts to $\mu_1(Q) \leq K|Q|^{\frac{1}{2}}$, holds. In fact, with $f(x) = x_2^{-\frac{1}{2}} |\ln x_2|^{-1} \chi_{\left[0, \frac{1}{2}\right]} \times \chi_{\left[0, \frac{1}{2}\right]}(x_1, x_2)$, $I_{\frac{1}{2}} f$ is infinite on
and thus the left side of (1.1) is infinite while the right side is finite. Examples of this nature were first pointed out in [2].

Theorem 2.3 below gives a necessary and sufficient condition for (1.1) that involves testing an inequality over dyadic cubes Q, namely

\[(1.3) \int_Q (M_{\phi} X_Q \mu)(x) dx \leq K \int_Q d\mu < \infty\]

where \(p' = \frac{p}{p-1}\), the constant \(K > 0\) is independent of \(Q\), and

\[\Phi(x) = \sup_{x \in Q} \left[ \frac{1}{|Q|} \int_{|y| < |Q|^n} \Phi(y) dy \right], \int_Q f(y) d\mu(y).\]

Alternatively, (1.1) is equivalent to

\[(1.4) \int_{\mathbb{R}^n} (T_{\phi} X_Q \mu)(x) dx \leq K \int_Q d\mu < \infty \text{ for all dyadic cubes } Q.\]

To compare (1.2) and (1.4), we note that (1.2) is equivalent by an elementary argument (see Theorem 4 in [2]) to testing the inequality in (1.4) over all compact sets \(Q\). The reduction in (1.4) to testing over dyadic cubes \(Q\) is essential in obtaining sharp estimates for the higher eigenvalues of Schrödinger operators in § 3. For a different characterization involving test functions see Stromberg and Wheeden [21].

In the special case where \(T_{\phi} = I_{\phi}\), the equivalence of (1.1) and (1.3) can be established by dualizing inequality (1.1), using the "good \(\lambda\) inequality" of B. Muckenhoupt and R. L. Wheeden [15] in order to replace \(I_{\phi}\) by its associated maximal operator \(M_{\phi}\), and then using the characterization of the weighted inequality for \(M_{\phi}\) in [18]. The general case of the theorem is proved along similar lines, the crucial new estimate being an extension (Theorem 2.2) of the "good \(\lambda\) inequality" in [15].

As an application of Theorem 2.3 we obtain a sharpened form of recent results of C. L. Fefferman and D. H. Phong on the distribution of eigenvalues of Schrödinger operators, \(H = -\Delta - v\), \(v \geq 0\) ([10]; Theorem 5, 6 and 6' in Chapter II). Roughly speaking, their results show that for many \(v \geq 0\), the negative eigenvalues of \(H = -\Delta - v\) are approximately given by \(-|Q|^{-\frac{2}{n}}\) as \(Q\) varies over the minimal dyadic.
cubes satisfying $|Q|^2^{-1} \int_Q v \geq C$. Theorem 3.3 below shows, as suggested by condition (1.3), that this picture extends to arbitrary $v \geq 0$ if the fractional average, $|Q|^2^{-1} \int_Q v$, is replaced by

$$\frac{1}{|Q|^v} \int_Q [I_1(\chi_{Q^v})(x)]^2 \, dx = \frac{1}{|Q|^v} \int_Q I_2(\chi_{Q^v})(x)v(x) \, dx,$$

the $v$-average over $Q$ of the Newtonian potential of $\chi_{Q^v}$. Certain of the results in [10] have been generalized by S. Y. A. Chang, J. M. Wilson and T. H. Wolff ([5]) and by S. Chanillo and R. L. Wheeden ([6]). This is discussed in more detail in § 3. Further applications of Theorem 2.3 have been announced in [13].

2. The trace inequality.

We begin by deriving the basic properties of r.d. kernels $\Phi$ and Borel measures $\mu$ for which the trace inequality holds. For the sake of completeness, we consider here and in § 3 the more general trace inequality

$$(2.1) \left[ \int_{\mathbb{R}^n} (T_\Phi f)(x)^q \, d\mu(x) \right]^{\frac{1}{q}} \leq C \left[ \int_{\mathbb{R}^n} f(x)^p \, dx \right]^{\frac{1}{p}}$$

for all nonnegative measurable $f$, where $1 < p \leq q < \infty$. For $p < q$ and many r.d. kernels $\Phi$, the trace inequality (2.1) can be characterized in terms of very simple conditions — see e.g. [12]. However, many applications, such as that in the next section, require the case $p = q$.

**Proposition 2.1.** — If (2.1) holds for a non-trivial r.d. kernel $\Phi$ and a non-trivial Borel measure $\mu$, then (i) $\mu$ is locally finite, that is, $\int_Q d\mu < \infty$ for all cubes $Q$, and (ii) $\Phi$ satisfies

$$(2.2) \int_{|y| > r} \Phi(y)^p \, dy < \infty \quad \text{for all} \quad r > 0.$$

**Proof.** — Choose $\varepsilon > 0$ so that $\Phi(2\varepsilon) > 0$. If $B$ is any ball of radius $\varepsilon$, and if $\gamma_n$ denotes the measure of the surface of the unit ball in
\( R^n \), then

\[
\gamma_n \left( \text{Cap}_{2\varepsilon} \right) \left( \int_B d\mu \right)^{\frac{1}{q}} \leq \left[ \int_B (T_\phi \chi_B)^q \ d\mu \right]^{\frac{1}{q}} \\
\leq \left[ \gamma_n \text{Cap}_{2\varepsilon} \right]^{\frac{1}{p}} \| T_\phi \|_{L^p} < \infty.
\]

Hence \( \int_B d\mu < \infty \) and this proves that \( \mu \) is locally finite.

To obtain (2.2), fix \( R > 0 \) so that \( \int_B d\mu > 0 \) where \( B \) is the ball of radius \( R \) centred at the origin. Momentarily fix \( S > 2R \) and let \( f(x) = \Phi(x)^{p' - 1} \chi_{[2R, S]}(x) \). For \( |x| \leq R \), we have

\[
T_\phi f(x) = \int_{2R < |y| \leq S} \Phi(x - y) \Phi(y)^{p' - 1} \ d\mu \geq C \int_{2R < |y| \leq S} \Phi(y)^{p'} \ d\mu.
\]

Indeed, for all \( y \) satisfying \( |x - y| \leq |y| \) and this in turn holds provided \( |x| \leq R, |y| \geq 2R \) and the distance between \( \frac{x}{|x|} \) and \( \frac{y}{|y|} \) is sufficiently small. With this estimate, (2.1) yields

\[
C \int_{2R < |y| \leq S} \Phi(y)^{p'} \ d\mu \left( \int_B d\mu \right)^{\frac{1}{q}} \leq \left[ \int_B (T_\phi f)^q \ d\mu \right]^{\frac{1}{q}} \\
\leq C \left[ \int_{2R < |y| \leq S} \Phi(y)^{p'} \ d\mu \right]^{\frac{1}{p}}.
\]

Letting \( S \to \infty \) yields \( \int_{|y| \geq 2R} \Phi(y)^{p'} \ d\mu < \infty \) and this proves (2.2).

To obtain a criterion for (2.1) to hold, we look at the inequality dual to it. A standard argument shows this dual is, with the same \( C > 0 \),

\[
\left( \int_{R^n} (T_\phi f \mu)(x)^{p'} \ dx \right)^{\frac{1}{p'}} \leq C \left( \int_{R^n} f(x)^{q'} \ d\mu(x) \right)^{\frac{1}{q'}}
\]

where \( p' = \frac{p}{p - 1}, \ q' = \frac{q}{q - 1} \), and

\[
(T_\phi f \mu)(x) = \int_{R^n} \Phi(x - y) f(y) \ d\mu(y).
\]
The behaviour of $T_\Phi$ in (2.3) is determined by that of the maximal operator $M_\Phi$ given at a positive Borel measure $\nu$ by

$$(M_\Phi \nu)(x) = \sup_{x \in Q} \left[ \frac{1}{|Q|} \int_{|y| < |Q|^{\frac{1}{n}}} \Phi(y) \, dy \right] \int_Q dv.$$ 

Note that the first factor on the right side is the average of $\Phi$ over the ball of radius $|Q|^{\frac{1}{n}}$ centred at the origin. In the case when $\Phi$ is the kernel $K_\alpha$ for the Riesz potential operator, then $M_\Phi$ is the usual fractional maximal operator $M_\alpha$ (see e.g. [3] or [15]).

**Theorem 2.2.** — Let $\Phi$ be an r.d. kernel and $\nu$ a positive locally finite Borel measure on $\mathbb{R}^n$. Then

(a) $$(M_\Phi \nu)(x) \leq C_n M(T_\Phi \nu)(x), \quad x \in \mathbb{R}^n$$

where $M$ denotes the usual Hardy-Littlewood maximal operator and the constant $C_n > 0$ depends only on the dimension $n$.

(b) There exists $\gamma > 1$ and a positive constant $C_n$ depending only on $n$ so that for all $\lambda > 0$ and all $\beta \in (0,1)$,

$$\{|\{T_\Phi \nu > \gamma \lambda \text{ and } M_\Phi \nu \leq \beta \gamma\}| \leq C_n \frac{1}{\gamma} |\{M(T_\Phi \nu) > \lambda\}|.$$ 

**Proof.** — To a given cube $Q$ in $\mathbb{R}^n$ associate the cube $Q^*$ having the same centre as $Q$ but edges $7 \sqrt{n}$ times as long as those of $Q$.

To prove (a) fix $x \in \mathbb{R}^n$ and a cube $Q$ containing $x$. Then

$$\int_{Q^*} (T_\Phi \nu)(y) \, dy \geq \int_{Q^*} dy \int_Q \Phi(y-z) \, dv(z)$$

$$\geq \int_Q dv(z) \int_{Q^*} \Phi(y-z) \, dy$$

$$\geq \int_{|y| < |Q|^{\frac{1}{n}}} \Phi(y) \, dy \int_Q dv(y)$$

since $\{y; |y-z| < |Q|^{\frac{1}{n}} \subset Q^*, \text{ whenever } z \in Q \}$. Hence,

$$M(T_\Phi \nu)(x) \geq \frac{7^{-\frac{n}{2}}}{|Q|} \int_{|y| < |Q|^{\frac{1}{n}}} \Phi(y) \, dy \int_Q dv(y)$$
and so

\[ M_\phi(v)(x) \geq 7^n n^2 M(T_\phi v)(x), \ x \in \mathbb{R}^n. \]

We now show (b). Given \( \lambda > 0 \), let

\( \Omega_\lambda = \{ M(T_\phi v) \geq \lambda \}. \)

Decompose \( \Omega_\lambda \) into disjoint Whitney cubes \( Q \) with \( Q^* \cap \Phi_\xi \neq \emptyset \). See De Guzman [11]. Let \( \{ Q_k \} \) be those Whitney cubes for which there is an \( x_k \in Q_k \) satisfying \( (M_\phi v)(x_k) \leq \beta \lambda \). Fixing attention on such a \( Q_k \), which we'll denote simply by \( Q \), we define \( v_1 \) and \( v_2 \) to be restrictions of the measure \( v \); the first to \( Q^* \), the second to \( \mathbb{R}^n - Q^* \). We claim it is enough to obtain a dimensional constant \( C_\alpha > 0 \) such that

\[ T_\phi v_2 \leq C_\alpha \lambda \]

on \( Q \). Suppose for the moment that (2.4) has been proved and take \( \gamma > 2C_\alpha \). Then

\( \{ x \in Q; (T_\phi v)(x) > \gamma \lambda \} \subset \left\{ x \in Q; (T_\phi v_1)(x) > \frac{\gamma \lambda}{2} \right\}. \)

Now,

\[ \int_Q \Phi(x-z) \, dx \leq \int_{|y| < \frac{\sqrt{n}}{2} |Q|^\alpha} \Phi(y) \, dy. \]  

This means

\[
\int_Q (T_\phi v_1)(x) \, dx = \int_Q \Phi(x-y) \, dv(y) \]
\[ = \int_{Q^*} dv(y) \int_Q \Phi(x-y) \, dx \leq \int_{|y| < \frac{\sqrt{n}}{2} |Q|^\alpha} \Phi(y) \, dy \int_{Q^*} dv(y) \]
\[ \leq (7\sqrt{n})^\alpha |Q| (M_\phi v)(x_k) \leq (7\sqrt{n})^\alpha \beta \lambda |Q|. \]

Thus with \( C = 2(7\sqrt{n})^\alpha \),

\[ \left| \left\{ x \in Q; (T_\phi v_1)(x) > \frac{\gamma \lambda}{2} \right\} \right| \leq \frac{2}{\gamma \lambda} \int_Q (T_\phi v_1)(x) \, dx > C \frac{\beta}{\gamma} |Q|. \]

Therefore,

\[ \left| \{ T_\phi v > \gamma \lambda \text{ and } M_\phi v \leq \beta \lambda \} \right| = \sum_k \left| \{ x \in Q_k; (T_\phi v)(x) > \gamma \lambda \} \right| \]
\[ \leq \frac{C \beta}{\gamma} \sum_k |Q_k| \leq C \frac{\beta}{\gamma} \left| \{ M(T_\phi v) > \lambda \} \right|. \]
To prove (2.4) we'll require the fact that \( \epsilon > 0 \) exists with

\[
\Phi(y) \leq \frac{C_n}{r^n} \int_{|y-z| \leq r} \Phi(z) \, dz, \quad 0 < r \leq |y|.
\]

As \( \Phi \) is nonincreasing, this would be true if it were known to hold whenever \( \Phi \) is the characteristic function of a ball centred at the origin. For this it suffices to know that the set of \( z \) in the ball \( |y-z| \leq r \) satisfying \( |z| \leq |y| \) occupies at least a fixed fraction of the ball. The change of variable \( z = |y|v \), followed by the rotation that sends \( e_1 = (1,0,\ldots,0) \), reduces the problem to the relative size of the intersection of the balls \( |v| \leq 1 \) and \( |v-e_1| \leq s, 0 < s < 1 \), to the size of the ball \( |v-e_1| \leq s \) itself. But for these sets the result is clear.

If \( x \in Q \) (where \( Q \) denotes some fixed \( Q_k \)) and \( y \in \mathbb{R}^n - Q^* \), then \( |x-y| \geq |Q|^\frac{1}{n} \). Thus taking \( r = |Q|^\frac{1}{n} \) in (2.6), we get

\[
(Tv_2)(x) = \int_{\mathbb{R}^n - Q^*} \Phi(x-y) \, dv(y)
\leq \frac{C_n}{r^n} \int_{\mathbb{R}^n - Q^*} dv(y) \int_{|z| \leq r} \Phi(x-y-z) \, dz.
\]

Making the substitution \( v = x - z \), the last expression becomes

\[
\frac{C_n}{r^n} \int_{|x-v| \leq r} (T\Phi v_2)(v) \, dv \leq \frac{C_n}{r^n} \int_{Q^*} (T\Phi v)(x) \, dx \leq \frac{C_n}{r^n} \lambda |Q^*| = C_n \lambda.
\]

with \( C_n = (7\sqrt{n})^n C_n \), since \( Q^* \) intersects \( \mathbb{R}^n - Q = \{ M(T\Phi v) \leq \lambda \} \) by the Whitney condition. This completes the proof.

**Theorem 2.3.** — Suppose \( \Phi \) is a nonnegative, locally integrable radially decreasing function satisfying (2.2). Then for \( 1 < p \leq q < \infty \) and \( \mu \) a positive locally finite Borel measure on \( \mathbb{R}^n \), the following statements are equivalent:

1. There exists \( C > 0 \) so that whenever \( f \) is a nonnegative measurable function on \( \mathbb{R}^n \)

\[
\left[ \int_{\mathbb{R}^n} (T\Phi f)(x)^q \, d\mu(x) \right]^\frac{1}{q} \leq C \left[ \int_{\mathbb{R}^n} f(x)^p \, dx \right]^\frac{1}{p}.
\]
2. There exists $C > 0$ so that for all dyadic cubes $Q$
\[
\left( \int_{\mathbb{R}^n} T_\Phi(\chi_Q \mu)(x)^{p'} \, dx \right)^{\frac{1}{p'}} \leq C' \left[ \mu(Q) \right]^{\frac{1}{p'}} < \infty
\]
where $p' = \frac{p}{p-1}$, $q' = \frac{q}{q-1}$.

3. There exists $K > 0$ so that for all dyadic cubes $Q$
\[
\left( \int_{Q} (M_\Phi(\chi_Q \mu)(x))^{p'} \, dx \right)^{\frac{1}{p'}} \leq K \left[ \mu(Q) \right]^{\frac{1}{p'}} < \infty.
\]
Moreover, the least possible $C$, $C'$ and $K$ in the above are all within constant multiples of one another, the constants being independent of $\Phi$ and $\mu$.

Proof. — Let $M_\Phi^{dy}$ denote the dyadic analogue of $M_\Phi$ given by
\[
M_\Phi^{dy}(x) = \sup_{x \in Q \text{ dyadic}} \left[ \frac{1}{|Q|} \int_{|y| < |Q|^{\frac{1}{n}}} \Phi(y) \, dy \right] \int_{Q} \]
for $x \in \mathbb{R}^n$ and $\nu$ a locally finite positive measure. We claim that for all such $\nu$,
\[
(2.7) \quad \int_{\mathbb{R}^n} |M_\Phi^{dy}(\nu)|^{p'} \leq \int_{\mathbb{R}^n} |M_\Phi(\nu)|^{p'} \leq C_1 \int_{\mathbb{R}^n} |T_\Phi(\nu)|^{p'},
\]
\[
(2.8) \quad \int_{\mathbb{R}^n} |T_\Phi(\nu)|^{p'} \leq C_2 \int_{\mathbb{R}^n} |M_\Phi(\nu)|^{p'} \leq C_3 \int_{\mathbb{R}^n} |M_\Phi^{dy}(\nu)|^{p'},
\]
where the constants $C_1$, $C_2$, $C_3$ depend only on $n$ and $p(1 < p < \infty)$. The first inequality in (2.7) is trivial and the second inequality follows from part (a) of Theorem 2.2 and the classical $L^p$ inequality for $M$ ([18]). The first inequality in (2.8) follows from part (b) of Theorem 2.2 as in [6]. Finally, to prove the second inequality in (2.8), we apply a standard covering argument to $\{M_\Phi(\nu) > \lambda\}$ (where $\lambda > 0$) to obtain the existence of cubes $(Q_k)_k$ with disjoint triples satisfying
\[
(i) \quad \left[ \frac{1}{|Q_k|} \int_{|y| < |Q_k|^{\frac{1}{n}}} \Phi(y) \, dy \right] \int_{Q_k} \nu > \lambda \quad \text{for all } k
\]
\[
(ii) \quad |\{M_\Phi(\nu) > \lambda\}| \leq C \sum_k |Q_k|.
\]
Now each $Q_k$ is covered by at most $2^n$ dyadic cubes $(I_k^j)_{1 \leq j \leq 2^n}$ with
2^{-n}|Q_k| \leq |I_k| \leq |Q_k|$. There is at least one of these dyadic cubes, say $I_k = I_k^1$, with \( \int_{I_k} dv \geq 2^{-n} \int_{Q_k} dv \). Then, since $\Phi$ is r.d. and $|I_k| \leq |Q_k|$, 

\[
\left[ \frac{1}{|I_k|} \int_{|y| \leq |I_k|^\frac{1}{n}} \Phi(y) \, dy \right] \int_{I_k} dv > 2^{-n} \lambda \quad \text{for all } k
\]

and so $\bigcup_k I_k \subset \{ M^d \varphi > 2^{-n} \lambda \}$. Since the $I_k$'s are pairwise disjoint, we have

\[
|\{ M^d \varphi > \lambda \}| \leq C \sum_k |Q_k| \leq C \sum_k |I_k| \\
\leq C |\{ M^d \varphi > 2^{-n} \lambda \}|
\]

and (2.8) follows upon multiplying this inequality by $\lambda^{p-1}$ and then integrating over $(0, \infty)$.

From (2.3), (2.7) and (2.8) we obtain that the trace inequality in 1. holds if and only if there is $C > 0$, comparable to the one in (2.1), for which

\[
(2.9) \quad \left[ \int_{\mathbb{R}^n} (M^d(f \varphi^2)(x))^{\frac{1}{p'}} \, dx \right]^{\frac{1}{p}} \leq C \left[ \int_{\mathbb{R}^n} f(x)^{\frac{1}{q}} \, d\mu(x) \right]^{\frac{1}{q}}, \quad \text{for all } f.
\]

Theorem A of [16] (with $M^d_\varphi$ in place of $M^d_\mu$, the proof is unchanged) shows that (2.9) holds if and only if there is $C > 0$, comparable to that in (2.9), for which

\[
(2.10) \quad \left[ \int_{\mathbb{R}^n} [M^d_\varphi(\chi_Q \, d\mu)]^{\frac{1}{p'}} \right]^{\frac{1}{p}} \leq C \mu(Q)^{\frac{1}{q'}} < \infty
\]

for all dyadic cubes $Q$. Theorem 2.3 now follows easily. The trace inequality 1. implies its dual (2.3) which in turn implies 2. upon taking $f = \chi_Q$. Inequality 2. implies 3. by (2.7) and finally, 3. $\Rightarrow$ (2.10) $\Rightarrow$ (2.9) $\Rightarrow$ 1.

### 3. Schrödinger operators.

In this section, Theorem 2.3 is used to refine the estimates for eigenvalues of a Schrödinger operator $H = -\Delta - v$ given in Theorem 5, Chapter II, of [10]. By eigenvalues, we mean the numbers
\( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N \ldots \) where \( \lambda_N \) is the maximum over all \( N - 1 \) tuples \( \Phi_1, \ldots, \Phi_{N-1} \) of the quantity \( \inf \frac{\langle Hu, u \rangle}{\langle u, u \rangle} \), the infimum being over all \( u \in Q(H), \langle u, \Phi_j \rangle = 0, j = 1, \ldots, N - 1 \). Here \( Q(H) \) denotes the form domain of \( H \) (see [16]) and \( \langle Hu, u \rangle = \int_{\mathbb{R}^n} (|\nabla u|^2 - v|u|^2) \) for \( u \in Q(H) \). Recall that \( I_2 f(x) = \int_{\mathbb{R}^n} |x - y|^2 \gamma(y) dy \) denotes the Newtonian potential of \( f \).

**Theorem 3.1.** — Let \( H = -\Delta - v \), where \( v(x) \geq 0 \) is locally integrable on \( \mathbb{R}^n \) and \( n \geq 3 \). Denote the \( v \) measure of \( Q \), \( \int_Q v(x) \, dx \), by \( |Q|_v \). There are positive constants \( C, c \) depending only on the dimension \( n \) such that the least eigenvalue \( \lambda_1 \) of \( H \) satisfies \( E_{sm} \leq -\lambda_1 \leq E_{big} \) where

\[
E_{sm} = \sup \left\{ |Q|^{-2/n}; |Q|_v^{-1} \int_Q I_2(\chi_Q v) v \geq C \right\}
\]

\[
E_{big} = \sup \left\{ |Q|^{-2/n}; |Q|_v^{-1} \int_Q I_2(\chi_Q v) v \geq c \right\}.
\]

**Example 3.2.** — Consider Example V in [10]: a particle in a rectangular box \( B = B_1 \times B_2 \times \cdots B_n \) with side lengths \( \delta_1 \leq \delta_2 \leq \cdots \delta_n \). Let \( v = \chi_B \) and let \( x_B \) denote the centre of \( B \). Since

\[
\sup_{Q} |Q|_v^{-1} \int_Q I_2(\chi_Q v) v \approx I_2 v(x_B) \approx \delta_1^2 + \delta_1 \delta_2 + \delta_1 \delta_2 \log (\delta_3/\delta_2) \\
\approx \delta_1 \delta_2 \log (1 + \delta_3/\delta_2),
\]

Theorem 3.1 yields the correct order of magnitude for the energy \( E_{\text{critical}} \), needed to trap a particle in \( B \), namely

\[
E_{\text{critical}} = \sup \{ 1; -\Delta - Ev \geq 0 \} = 1/\delta_1 \delta_2 \log (1 + \delta_3/\delta_2).
\]

A refinement of Theorems 6 and 6' in Chapter II of [10], similar to the one above, is given in

**Theorem 3.3.** — Let \( H = -\Delta - v \) where \( v(x) \geq 0 \) is locally integrable on \( \mathbb{R}^n \) and \( n \geq 3 \). There are positive constants \( C, c \) depending only on the dimension \( n \) such that:

(A) Suppose \( \lambda \geq 0 \) and let \( Q_1, \ldots, Q_N \) be a collection of cubes of side length at most \( \lambda^{-1/2} \) whose doubles are pairwise disjoint. Suppose further that
\(|Q_j|^{-1} \int_{Q_j} I_2(\chi_Q) v \geq C, \ 1 \leq j \leq N. \) Then \(H\) has at least \(N\) eigenvalues \(\leq -\lambda.\)

(B) Conversely, suppose \(\lambda \geq 0\) and that \(H\) has at least \(CN\) eigenvalues \(\leq -\lambda.\) Then there is a collection of pairwise disjoint (dyadic) cubes \(Q_1, \ldots, Q_N\) of side lengths at most \(\lambda^{-\frac{1}{2}}\) that satisfy
\[|Q_j|^{-1} \int_{Q_j} I_2(\chi_Q) v \geq c, \ 1 \leq j \leq N.\]

Roughly speaking, Theorem 3.3 says that the negative eigenvalues of \(H\) are approximately given by
\[\lambda|Q|^{-2/m} \text{ as } Q \text{ ranges over the minimal dyadic cubes satisfying } |Q|^{-1} \int_{Q} I_2(\chi_Q) v \geq C.\]

In [10], results corresponding to Theorems 3.1 and 3.3 were obtained with the quantity \(|Q|^{-1} \int_{Q} I_2(\chi_Q) v\) replaced by the simpler average \(C|Q|^2 |v| v\) in part (A) of Theorem 3.3 and by \(C_p|Q|^2 |v| v^p \left( \int_{Q} v^p \right)^{\frac{1}{p}}\) in part (B). A comparison of these quantities is made in Remark 3.5 at the end of this section. Chang, Wilson, and Wolff [5] show part (B) of Theorem 3.3 holds for \(v\) if
\[\sup_{Q} |Q|^{-1} \int_{Q} v(\chi_Q) \Phi(|Q|^2 v(x)) \ dx < \infty,\]
where \(\Phi: [0, \infty) \rightarrow [1, \infty)\) is increasing and \(\int_{1}^{\infty} \frac{dx}{x\Phi(x)} < \infty.\) See also Chanillo and Wheeden [6].

Proof of Theorem 3.1. — The Schwartz class \(S\) is dense in \(Q(H)\) and thus we have

\[-\lambda_1 = - \inf_{u \in Q(H)} \frac{\left< Hu, u \right>}{\left< u, u \right>} = \sup_{u \in S} \int |u|^2 v - \int |\nabla u|^2 \]
\[= \inf \{ \alpha > 0 ; \int |u|^2 v \leq \int |\nabla u|^2 + \alpha |u|^2 \}
= \int \{ |\xi|^2 + \alpha \} |\hat{u}(\xi)|^2 d\xi, u \in S\}
= \inf \{ \alpha > 0 ; \int (I^2_x f)^2 v \leq \int f^2, f \geq 0 \} \]
where $I^n_\gamma$ is the operator with r.d. kernel $K_\gamma^n$ defined by

$$(K_\gamma^n)^*(\xi) = (|\xi|^2 + \alpha)^{-\frac{1}{2}}.$$  Thus $K_\gamma^n(x) = G_\gamma(x)$ and

$$K_\gamma^n(x) = \alpha^\frac{n-1}{2} G_\gamma(\alpha^\frac{1}{2}x).$$

If we let $C_a$ denote the least constant such that

$$\int (I^n_\gamma f)^2 \leq C_a \int f^2 \quad \text{for all } f \geq 0,$$

then $-\lambda_1 = \inf \{\alpha; C_a \leq 1\}$. By Theorem 2.3,

$$C_a \approx \sup_{Q} \frac{1}{|Q|^v} \int [I^n_\gamma(\chi_Q)^2]$$

in the sense that the ratio of the left and right sides is bounded between two constants independent of $\alpha$ and $v$. We now show that, in fact, the supremum in (3.1) need only be taken over those cubes $Q$ with $\frac{1}{n} |Q|^\frac{1}{n} \leq \alpha^{-\frac{1}{2}}$. To this end, set $M = \sup_{|Q|^v \leq \alpha^{-\frac{1}{2}}} \frac{1}{|Q|^v} \int [I^n_\gamma(\chi_Q)^2]$ and suppose $Q$ is a cube with $|Q|^\frac{1}{n} > \alpha^{-\frac{1}{2}}$. Express $Q$ as a union of congruent cubes, $Q_i$, having pairwise disjoint interiors and common sidelengths, $|Q_i|^\frac{1}{n}$, satisfying $\frac{1}{2} \alpha^{-\frac{1}{2}} \leq |Q_i|^\frac{1}{n} \leq \alpha^{-\frac{1}{2}}$. Then, we claim

$$\int [I^n_\gamma(\chi_Q)^2] = \sum_{i,j} \int I^n_\gamma(\chi_{Q_i})I^n_\gamma(\chi_{Q_j})$$

$$\leq C \sum_i \int [I^n_\gamma(\chi_{Q_i})]^2$$

$$\leq CM \sum_i |Q_i|^v = CM |Q|^v.$$

The second inequality holds by definition of $M$ and since $|Q_i|^\frac{1}{n} \leq \alpha^{-\frac{1}{2}}$. To prove the first inequality, we consider two cases. First, when $Q_i$ and $Q_j$ are adjacent, we simply use

$$\int I^n_\gamma(\chi_{Q_i})I^n_\gamma(\chi_{Q_j}) \leq \frac{1}{2} \int [I^n_\gamma(\chi_{Q_i})]^2 + \frac{1}{2} \int [I^n_\gamma(\chi_{Q_j})]^2.$$

To treat the case when $Q_i$ and $Q_j$ have a distance of roughly $k$
sidelengths between them, \( k \geq 1 \), we require the facts that
\[ K^2(x) \approx |x|^{-n} \quad \text{if} \quad |x| \leq \alpha^{-\frac{1}{2}} \quad \text{and} \quad K^2(x) \leq C \alpha^{-\frac{n-2}{2}} e^{-\sqrt{|x|}} \quad \text{if} \quad |x| > \alpha^{-\frac{1}{2}}, \]
for which see [4]. We then have
\[
\int_{Q_i} I_1^2(\chi_{Q_i^v}) I_1^2(\chi_{Q_i^v}) = \int_{Q_i} I_2^2(\chi_{Q_i^v})(x) v(x) \, dx \leq C \alpha^{-\frac{n-2}{2}} e^{-k|Q_i|v|Q_j|v}.
\]
However, \( I_1^2(\chi_{Q_i^v})(x) \geq C \alpha^{-\frac{1}{2}} \) for \( x \in Q_i \) and so
\[
|Q_i|v \leq \frac{\alpha^\frac{1}{2}}{C} \int_{Q_i} I_1^2(\chi_{Q_i^v}) v = \frac{\alpha^\frac{1}{2}}{C} \int_{Q_i} I_1^2(\chi_{Q_i^v})(x) \, dx.
\]
Thus
\[
2|Q_i|v|Q_j|v \leq |Q_i|^2 + |Q_j|^2
\]
\[
\leq C \alpha \left( \left[ \int_{Q_i} I_1^2(\chi_{Q_i^v}) \right]^2 + \left[ \int_{Q_j} I_1^2(\chi_{Q_j^v}) \right]^2 \right)
\]
\[
\leq C \alpha^{1 - \frac{n}{2}} \left( \int_{Q_i} [I_1^2(\chi_{Q_i^v})]^2 + \int_{Q_j} [I_1^2(\chi_{Q_j^v})]^2 \right).
\]
Now, for a fixed cube \( Q_i \), there are at most \( C k^{n-1} \) cubes \( Q_j \) at a distance of roughly \( k \) sidelengths from \( Q_i \). Combining all of the above, we obtain
\[
\sum_{i \neq j} \int_{Q_i} I_1^2(\chi_{Q_i^v}) I_1^2(\chi_{Q_j^v}) \leq C \left[ 1 + \sum_{k=1}^{\infty} k^{n-1} e^{-k} \right] \sum_i \int [I_1^2(\chi_{Q_i^v})]^2
\]
which yields the first inequality in (3.2). From (3.1) and (3.2), we have
\[
C_\alpha \approx M \quad \text{and since} \quad \int [I_1^2(\chi_{Q_i^v})]^2 = \int I_2^2(\chi_{Q_i^v}) v \approx \int I_2^2(\chi_{Q_i^v}) v \quad \text{when} \quad |Q_i| \leq \alpha^{-\frac{1}{2}}, \]
we finally have
\[
C_\alpha \approx \sup_{Q} \frac{1}{|Q|^{1/n}} \int_{Q} I_2(\chi_{Q_i^v}) v
\]
and Theorem 3.1 follows readily.

Proof of Theorem 3.3, part (A). — As in [10], it suffices by elementary functional analysis to construct an \( N \)-dimensional subspace \( \Omega \subset Q(H) \) so
that $\langle Hu, u \rangle \leq -\lambda \int |u|^2$ for $u$ in $\Omega$. Our hypothesis implies

$$\frac{1}{|Q_j|} \int_{Q_j} \mathbb{I}^2_{2}(\chi_{Q_j}v) v \geq C \quad \text{for } j = 1, \ldots, N.$$ 

Since $\int_{Q} \mathbb{I}^2_{2}(\chi_{Q}v) v \leq \left( \int_{Q} [\mathbb{I}^2_{2}(\chi_{Q}v)]^2 v \right)^{\frac{1}{2}} |Q|^\frac{1}{2} v$ by H"older's inequality, we actually have

$$\int_{Q_j} [\mathbb{I}^2_{2}(\chi_{Q_j}v)]^2 v \geq C \int_{Q_j} \mathbb{I}^2_{2}(\chi_{Q_j}v) v, \quad 1 \leq j \leq N.$$ 

This suggests we let $\Omega$ be the linear span of $\{f_j\}_{j=1}^N$ where $f_j = \Phi_j \mathbb{I}^2_{2}(\chi_{Q_j}v)$ and $\Phi_j = 1$ on $\frac{3}{2}Q_j$ with supp $\Phi_j$ contained in $2Q_j$. Here the $\Phi_j$ are dilates and translates of a fixed $\Phi \in C_c^\infty(\mathbb{R}^n)$. We have immediately that

$$(3.3) \quad \int f_j^2 v \geq C \int_{Q_j} \mathbb{I}^2_{2}(\chi_{Q_j}v) v \quad \text{for } 1 \leq j \leq N.$$ 

By hypothesis, the supports of the $f_j$ are pairwise disjoint and so we need only establish

$$(3.4) \quad \langle (-\Delta + \lambda)f_j, f_j \rangle \leq \int (f_j)^2 v \quad \text{for } 1 \leq j \leq N$$

in order to conclude $\langle Hu, u \rangle \leq -\lambda \int |u|^2$ for $u$ in $\Omega$, as required. To prove (3.4), we let $G_j = 2Q_j - \frac{3}{2}Q_j$ and compute that

$$(-\Delta + \lambda)f_j = (-\Delta + \lambda)[\Phi_j \mathbb{I}^2_{2}(\chi_{Q_j}v)]$$

$$= \chi_{Q_j}v + \chi_{G_j}(-\Delta + \lambda)[\Phi_j \mathbb{I}^2_{2}(\chi_{Q_j}v)]$$

$$= A_j + B_j$$

since $\mathbb{I}^2_{2} = (-\Delta + \lambda)^{-1}$. Now

$$\langle A_j, f_j \rangle = \int_{Q_j} \mathbb{I}^2_{2}(\chi_{Q_j}v) v \leq \frac{1}{C} \int f_j^2 v \quad \text{(by 4.3))} \leq \frac{1}{2} \int f_j^2 v$$

provided $C$ is chosen $\geq 2$. It remains to verify

$$\langle B_j, f_j \rangle \leq C \int_{Q_j} \mathbb{I}^2_{2}(\chi_{Q_j}v) v$$

for all $j$ since then (3.4) will follow from (3.3)
and the previous estimate provided \( C \geq 2C' \). Now

\[
(3.5) \quad |B_j| \leq k_{ij} |\Phi_j| |\Delta I_2(x_{Q_j}v)| + 2|\nabla \Phi_j||\nabla I_2(x_{Q_j}v)| \\
+ (\lambda + |\Delta \Phi_j|)|I_2(x_{Q_j}v)|
= D_j + E_j + F_j.
\]

Using the estimates \( |D^sK_2(x)| \leq C|x|^{2-s} \), for \( s \geq 0 \) and \( |x| \leq C\lambda^{-\frac{1}{2}} \) (see [4]) we obtain that on \( G_j \),

\[
I_2(x_{Q_j}v)(x) \leq C|Q_j|^{\frac{n-1}{2}} \int_{Q_j} v \\
|\nabla I_2(x_{Q_j}v)(x)| \leq C|Q_j|^{-1} \int_{Q_j} v \\
|\Delta I_2(x_{Q_j}v)(x)| \leq C|Q_j|^{-1} \int_{Q_j} v.
\]

These inequalities, together with \( |\Phi_j| \leq 1 \), \( |\nabla \Phi_j| \leq C|Q_j|^{-\frac{1}{2}} \), \( |\Delta \Phi_j| \leq C|Q_j|^{-\frac{1}{2}} \) and the hypothesis \( \lambda \leq |Q_j|^{-\frac{1}{2}} \), yields

\[
(3.6) \quad D_j, E_j, F_j \leq C|Q_j|^{-1}|Q_j|^v.
\]

Since \( f_j(x) \leq C|Q_j|^{\frac{n-1}{2}} \int_{Q_j} v \) on \( G_j \), (3.5) and (3.6) imply

\[
(3.7) \quad \langle B_j, f_j \rangle \leq C|Q_j|^{\frac{n-1}{2}}|Q_j|^v.
\]

Finally,

\[
|Q_j|^{\frac{n-1}{2}} \left( \int_{Q_j} v \right)^2 \leq C(\min_{x \in Q_j} I_2(x_{Q_j}v)) \left( \int_{Q_j} v \right) \\
\leq C \int_{Q_j} I_2(x_{Q_j}v)v
\]

and this, combined with (3.7), shows that \( \langle B_j, f_j \rangle \leq C' \int_{Q_j} I_2(x_{Q_j}v)v \) and completes the proof of part (A) of Theorem 3.3.

Proof of Theorem 3.3, part (B). — We follow closely the argument of C. L. Fefferman and D. H. Phong in ([10]; proof of Theorem 6 in Chapter II), but with certain modifications designed to avoid the use of a square function. As in [10], it suffices to suppose \( v \) bounded and to show that if \( Q_1, \ldots, Q_N \) are the minimal dyadic cubes satisfying
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\[ \frac{1}{|Q_j|} \int_{Q_j} I_2(\chi_{Q_j} v) v \geq c \quad \text{and} \quad |Q_j|^{\frac{1}{n}} \leq \lambda^{-\frac{1}{2}}, \] then \( H = -\Delta - v \) has at most \( CN \) eigenvalues \( \leq -\lambda \) (where the constant \( C \) is of course independent of the bound on \( v \)). As usual, this will be accomplished by exhibiting a subspace \( \Omega \subset L^2 \) of codimension \( \leq CN \) such that

\begin{equation}
\langle Hu, u \rangle \geq -\lambda \int |u|^2 \quad \text{for all } u \text{ in } \Omega.
\end{equation}

We consider only the case \( \lambda = 0 \), the case \( \lambda > 0 \) requiring easy modifications. We begin by defining additional cubes \( Q_{N+1}, \ldots, Q_M \) as in [10]; i.e. let \( B \) be the collection of all dyadic cubes \( Q \) with

\[ \frac{1}{|Q|} \int_{Q} I_2(\chi_{Q} v) v \geq c \] and define the additional cubes \( Q_{N+1}, \ldots, Q_M \) to consist of (i) the maximal cubes in \( B \), (ii) the branching cubes in \( B \) and (iii) the descendents of branching cubes in \( B \). The descendents of a cube \( Q \) in \( B \) are those \( Q' \in B \) which are maximal with respect to the property of being properly contained in \( Q \). A cube in \( B \) « branches » if it has at least two descendents. As shown in [10], \( M \leq CN \). Still following [10] we define \( E_0 = \mathbb{R}^n - \bigcup_{j=1}^{M} Q_j \) and \( E_j = Q_j \) minus its descendents for \( j \geq 1 \). In analogy with estimates (i) and (ii) of [10], we shall prove that the weights \( v_j = \chi_{E_j} v \) satisfy

\begin{equation}
\frac{1}{|Q_j|} \int_{Q_j} I_2(\chi_{Q} v_j) v_j \leq Cc \quad \text{for all } 0 \leq j \leq M, Q \text{ dyadic cube.}
\end{equation}

In order to make use of (3.9) and the trace inequalities it implies we shall have to define the subspace \( \Omega \) so that

\begin{equation}
|u(x)| \leq C I_1(\chi_{E_j} |\nabla u|)(x) \quad \text{for } x \in E_j, 0 \leq j \leq M, u \in \Omega.
\end{equation}

Indeed, if both (3.9) and (3.10) hold, then for \( u \in \Omega \),

\begin{align*}
\int |u|^2 v &= \sum_{j=0}^{M} \int_{E_j} |u|^2 v_j \\
&\leq C \sum_{j=0}^{M} \int_{E_j} [I_1(\chi_{E_j} |\nabla u|)]^2 v_j \quad \text{by (3.10)} \\
&\leq C c \sum_{j=0}^{M} \int_{E_j} |\nabla u|^2 \quad \text{by (3.9) and Theorem 2.3} \\
&\leq \int |\nabla u|^2 \quad \text{if } c \text{ small enough,}
\end{align*}
and this is (3.8) for $\lambda = 0$. Thus it remains to construct $\Omega$ of codimension $\leq CN$ such that (3.10) holds. In the case $1 \leq j \leq N$, $E_j$ is a cube and (3.10) holds whenever $\int_{E_j} u = 0$ by the following inequality of E. Fabes, C. Kenig and R. Serapioni ([9]; Lemma 1.4)

$$
(3.11) \quad \left| u(x) - \frac{1}{|Q|} \int_Q u \right| \leq C L_1(\chi_Q |\nabla u|)(x) \quad \text{for } x \in Q, Q \text{ a cube.}
$$

For the case when $E_j$ is not a cube we will need the following lemma.

**Lemma 3.4.** — Suppose $Q_1, \ldots, Q_k$ are pairwise disjoint dyadic subcubes of a dyadic cube $Q$ in $\mathbb{R}^n$. Then there are (not necessarily dyadic or disjoint) cubes $I_1, \ldots, I_m$ such that $Q - \bigcup_{j=1}^k Q_j = \bigcup_{i=1}^m I_i$ and $m \leq C k$

where $C$ is a constant depending only on the dimension $n$. The above holds also for $Q = \mathbb{R}^n$ if we allow the cubes $I_i$ to be infinite, i.e. of the form $J_1 \times J_2 \times \cdots J_n$ where each $J_i$ is a semi-infinite interval.

This lemma has been obtained independently by S. Chanillo and R. L. Wheeden [6], with a proof much simpler than that appearing in a previous version of this paper. As a result, we refer the reader to [6] for a proof of the lemma.

We can now define the subspace $\Omega$. For each $j$ with $j = 0$ or $N + 1 \leq j \leq M$, apply Lemma 3.4 with $Q = Q_j$ and $Q_1, \ldots, Q_k$ the descendents of $Q_j$ (for $j = 0$, take $Q = \mathbb{R}^n$ and $Q_1, \ldots, Q_k$ to be the maximal cubes in $B$), to obtain cubes $I_1^{(j)}, \ldots, I_m^{(j)}$ with $E_j = \bigcup_{i=1}^m I_i^{(j)}$ and $m_j \leq C$ (# of descendents of $Q_j$). Note that $E_j = Q_j$ for $1 \leq j \leq N$. Now define

$$
\Omega = \{ u; \int_{Q_j} u = 0 \text{ for } 1 \leq j \leq N \text{ and } \int_{I_i^{(j)}} u = 0 \text{ for } N + 1 \leq j \leq M, j = 0 \text{ and } 1 \leq i \leq m_j \}. 
$$

If $x \in E_j$, $N + 1 \leq j \leq M$ or $j = 0$, then $x \in$ some $I_i^{(j)}$ and thus for $u \in \Omega$, $|u(x)| \leq C L_1(\chi_{E_j^{(j)}} |\nabla u|)(x) \leq C L_1(\chi_{E_j} |\nabla u|)(x)$ by (3.11). Thus (3.10) holds. Finally, the codimension of $\Omega$ is at most

$$
N + \sum_{j=0}^{N+1} m_j \leq N + C \sum_{j=0}^{N+1} m_j \leq N + C(M+1) \leq CM.
$$
It remains now to establish (3.9). We begin with the case \( j \neq 0 \) of (3.9), and follow the corresponding argument in [10]. Since \( \text{supp} \ v_j \subset Q_j \), we need only check (3.9) for dyadic cubes \( Q \in B \) with \( Q \subset Q_j \) and in fact, only for proper dyadic subcubes of \( Q_j \) (since if \( Q = \bigcup_{i=1}^{2^n} Q_i \), then

\[
\int_Q I_2(\chi_{Q^0}) = \int [I_1(\chi_{Q^0})]^2
\]

\[
= \sum_{i,j} I_1(\chi_{Q_{ij}^0}) I_1(\chi_{Q_{ij}}) \leq \frac{1}{2} \sum_{i,j} [I_1(\chi_{Q^0})]^2
\]

\[
\leq C_n \sum_{i=1}^{2^n} [I_1(\chi_{Q_i^0})]^2
\]

\[
= C_n \sum_{i=1}^{2^n} I_2(\chi_{Q_{ij}^0} v).
\]

As in [10], the only «non-trivial» case occurs when \( Q_j \in B \) is neither minimal nor branching and \( Q \) contains \( Q^* \), the unique maximal \( Q_i, 1 \leq i \leq M \), that is properly contained in \( Q_j \) (see the argument on p. 157-158 of [10]). To obtain (3.9) in this case we use a Whitney decomposition in place of the Calderon-Zygmund decomposition used in [10]. There is a dimensional constant \( C \) so large that we can choose pairwise disjoint dyadic subcubes \( \hat{Q}_a \) of \( Q - Q^* (= E_j \cap Q) \) such that each \( \hat{Q}_a \) satisfies

(3.12) either \( |\hat{Q}_a| = |Q^*| \) and \( \text{dist} (\hat{Q}_a, \hat{Q}_a^*) \leq C \)

or \( 2 \leq \frac{\text{dist} (\hat{Q}_a, \hat{Q}_a^*)}{\text{diam } \hat{Q}_a} \leq 2C. \)

Then

\[
\int_Q I_2(\chi_{Q^0} v_j) = \sum_{\alpha, \beta} \int_{Q_{\alpha}} I_2(\chi_{Q_{\alpha}^0} v)
\]

\[
\leq C \sum_{[\alpha, \beta; Q_\alpha \text{ touches } Q_\beta]} \int I_1(\chi_{Q_{\alpha}^0}) I_1(\chi_{Q_{\beta}^0} v)
\]

\[
+ C \sum_{[\alpha, \beta; Q_\alpha \text{ touches } Q_\beta]} \int_{Q_{\alpha}} I_2(\chi_{Q_{\beta}^0} v) = D + E.
\]

Now (3.12) shows that the number of \( \hat{Q}_{\beta} \) touching a given \( \hat{Q}_{\alpha} \) doesn’t
exceed a dimensional constant and so
\[ D \leq C \sum_{\alpha} \int_{Q_\alpha} |I_1(\chi_{Q_\alpha})|^2 = C \sum_{\alpha} \int_{Q_\alpha} I_2(\chi_{Q_\alpha})v \leq Cc \sum_{\alpha} \int_{Q_\alpha} v_j = Cc \int_{Q} v_j \]

since the \( Q_\alpha \) are not in \( B \). Condition (3.12) also shows that if \( |\hat{Q}_\beta| \leq |\hat{Q}_\alpha| \) and \( \hat{Q}_\beta, \hat{Q}_\alpha \) do not touch, then \( \text{dist}(\hat{Q}_\beta, \hat{Q}_\alpha) \geq c |\hat{Q}_\beta|^{1/n} \). Thus
\[ E \leq C \sum_{\alpha} \left( \int_{Q_\alpha} v \right) |\hat{Q}_\alpha|^2 \sum_{\beta: |\beta| < |\alpha|} \left[ \int_{Q_\beta} v \right]. \]

But \( |\hat{Q}_\beta|^2 \sum_{\alpha} \left( \int_{Q_\alpha} v \right) \leq \frac{1}{|\beta|^2} \int_{Q_\beta} I_2(\chi_{\hat{Q}_\beta})v \leq c \) since \( \hat{Q}_\beta \notin B \) and, by (3.12), the number of \( \hat{Q}_\beta \) of a given size does not exceed a dimensional constant. Thus
\[ E \leq Cc \sum_{\alpha} \left( \int_{Q_\alpha} v \right) |\hat{Q}_\alpha|^2 \sum_{\beta: |\beta| = 2^n |\alpha|} \left[ \int_{Q_\beta} v \right] \]
\[ \leq Cc \sum_{\alpha} \int_{Q_\alpha} v = Cc \int_{Q} v_j \quad \text{(since } n \geq 3) \]

and this completes the verification of (3.9) for \( j \neq 0 \). For \( j = 0 \), we again suppose \( Q \) dyadic in \( B \). If \( Q \subset \text{some } Q_1, \ldots, Q_M \), then \( \text{supp } v_\alpha \cap Q = \emptyset \) and (3.9) holds trivially. Otherwise, \( Q \) contains a unique maximal \( Q_i \) (\( 1 \leq i \leq M \)), say \( Q^* \), and we may argue as above to obtain (3.9). This completes the proof of Theorem 3.3.

Remark 3.5. — In [10] it is shown that \( \sup_Q |Q|^\frac{2}{n} \int_Q v \leq C \) is necessary and \( \sup_Q |Q|^\frac{2}{n} \left( \int_Q v^p \right)^{1/p} \leq C_p, p > 1 \), sufficient for the \( L^2 \) trace inequality (1.1) with \( T_\Phi = I_1 \). We give here a direct proof that
\[ (3.20) \sup_Q |Q|^\frac{2}{n} \int_Q v \leq C \sup_Q |Q|^{-1} \int_Q I_2(\chi_Qv)v \]
\[ \leq C_p \sup_Q |Q|^\frac{2}{n} \left( \int_Q v^p \right)^{1/p}, \quad p > 1. \]

The first inequality in (3.20) follows from the observation that \( I_2(\chi_Qv)(x) \geq C |Q|^\frac{2}{n} \int_Q v \) for \( x \) in a cube \( Q \).
Let $B_p = \sup_Q |Q|^{2-1/p} \left( \int_Q v^p \right)^{1/p}$. Suppose first that $v$ satisfies the $A_\infty$ condition of B. Muckenhoupt. Choose $p$ so close to 1 that the reverse Hölder condition $\left( |Q|^{-1} \int_Q v^p \right)^{1/p} \leq C_p |Q|^{-1} \int_Q v$ holds for all cubes $Q$. Let $M_2 f(x) = \sup_{x \in Q} |Q|^{2-1} \int_Q |f|$. Since $M_2(\chi_Q v) \leq B_p$ on $Q$,

$$\begin{align*}
(3.21) \quad \int_Q I_2(\chi_Q v) u & \leq \left( \int_Q I_2(\chi_Q v)^{p'} \right)^{1/p'} \left( \int_Q v^p \right)^{1/p} \\
& \leq C_p \left( \int_Q M_2(\chi_Q v)^{p'} \right)^{1/p'} \left( \int_Q v^p \right)^{1/p} \quad \text{(see [15])}
& \leq C_p B_p |Q|^{1/p'} \left( \int_Q v^p \right)^{1/p} \leq C_p B_p \int_Q v.
\end{align*}$$

For the general case, we use the observations in [10] that $v^+(x) = \sup_{x \in Q} \left( |Q|^{-1} \int_Q v^p \right)^{1/p}$ satisfies the $A_\infty$ condition and $M_2 v^+ \leq C_p B_p ([10]; p. 153)$. The above argument then yields (3.21) with $v^+$ in place $v$. Since $v \leq v^+$, (3.20) follows. This is of course obvious from Theorem 2.3, but can also be proved directly. Finally, we point out that the condition $M_2 p(v^p) \leq C_p$ is equivalent to the boundedness of $M_p$ from $L^2$ to $L^2(v^p)$ ([17]). Together with the inequality $|I_1 f(x)| \leq C_p M_p f(x)^{1/p} Mf(x)^{1/p'}$ of D. R. Adams, this yields another proof that $M_2 p(v^p) \leq C_p$ is sufficient for the $L^2$ trace inequality (1.1) with $T_\phi = I_1$. J. M. Wilson has recently communicated to us yet another proof.

**BIBLIOGRAPHY**


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