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MULTIPLE SINGULAR INTEGRALS AND MAXIMAL FUNCTIONS ALONG HYPERSURFACES

by **Javier DUOANDIKOETXEA**

0. Introduction.

Homogeneous singular integrals on the product space $\mathbf{R}^{n+m}$, $Tf = p.v.\ K*f$, with $K(x,y) = \Omega\left(\frac{x}{|x|}, \frac{y}{|y|}\right) |x|^{-n} |y|^{-m}$ and $\int_{S^{n-1}} \Omega(u,v)\, du = \int_{S^{m-1}} \Omega(u,v)\, dv = 0$, are bounded in $L^p(\mathbf{R}^{n+m})$, $1 < p < \infty$, when some regularity conditions are assumed on Ω, as can be seen in [5]. In [6] weighted inequalities are obtained for these operators, always assuming some regularity on the kernel. In this paper we get both L^p-boundedness and weighted inequalities for T with size conditions on Ω instead of regularity, namely: T is bounded in $L^p(\mathbf{R}^{n+m})$, $1 < p < \infty$, if $\Omega \in L^q(S^{n-1} \times S^{m-1})$, $q > 1$, and in $L^p(w)$, for the natural class of weights w (described below in § 2), if $\Omega \in L^\infty$. Our study of T is based on its decomposition as

$$Tf = \sum_{k,j} \sigma_{k,j} * f$$

where $\sigma_{k,j}$ are Borel measures given by

$$\sigma_{k,j}(g) = \iint_{\substack{2^k \leqslant |x| < 2^{k+1} \\ 2^j \leqslant |y| < 2^{j+1}}} K(x,y) g(x,y)\, dx\, dy .$$

Also Hilbert transforms along surfaces can be decomposed in an

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analogous way. For a surface S in $\mathbf{R}^3$ parametrized as $(s,t,\varphi(s,t))$ we define

$$\mathrm{H}f(x) = \mathrm{p.v.}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1-s,x_2-t,x_3-\varphi(s,t))\,\frac{ds\,dt}{st}$$

and we can write

$$\mathrm{H}f = \sum_{k,j=-\infty}^{\infty} \sigma_{k,j} * f$$

where the Borel measures $\sigma_{k,j}$ are now defined by

$$\sigma_{k,j}(g) = \iint_{\substack{2^k\leqslant|s|<2^{k+1}\\ 2^j\leqslant|t|<2^{j+1}}} g(s,t,\varphi(s,t))\,\frac{ds\,dt}{st}\cdot$$

When $\varphi(s,t) = |s|^\alpha|t|^\beta$, $\alpha,\beta > 0$, H is known to be bounded in $\mathrm{L}^p(\mathbf{R}^3)$, $|1/p-1/2| < \varepsilon$ for some $\varepsilon > 0$ (see [8], [12] and [13]). Associated to the surface S we have the maximal function

$$\mathrm{M}f(x) = \sup_{h_1,h_2>0}\frac{1}{h_1h_2}\left|\int_0^{h_1}\int_0^{h_2} f(x_1-s,x_2-t,x_3-\varphi(s,t))\,ds\,dt\right|$$

which is controlled by

$$\mathcal{N}f(x) = \sup_{k,j}|\mu_{k,j} * f(x)|$$

where the $\mu_{k,j}$ are positive Borel measures given by

$$\mu_{k,j}(g) = 2^{-k-j}\int_{2^k}^{2^{k+1}}\int_{2^j}^{2^{j+1}} g(s,t,\varphi(s,t))\,ds\,dt.$$

M is bounded in $\mathrm{L}^p(\mathbf{R}^3)$, $1 < p < \infty$, if $\varphi(s,t) = |s|^\alpha|t|^\beta$ (see [3]) and if $\varphi(0,0) = \nabla\varphi(0,0) = 0$ and φ has nonvanishing second order derivatives at the origin (see [2]). We get the boundedness of M and H in the whole range $1 < p < \infty$ for these surfaces and also for some others having a contact of infinite order at the origin with the $\mathrm{OX_1X_2}$ plane. These problems on surfaces appear as a natural generalization of their analogues on curves, and are posed in [11].

All the results are obtained from the two general theorems stated in § 1. Two families of measures $\{\mu_{k,j}\}$ and $\{\sigma_{k,j}\}$ being given, we study the boundedness in L^p of the operators

$$\mathcal{M}f(x) = \sup_{k,j} |\mu_{k,j} * f(x)|$$

$$Tf(x) = \sum_{k,j} \sigma_{k,j} * f(x).$$

Here $\mu_{k,j}$ will be positive Borel measures and $\sigma_{k,j}$ will have zero integral. The technique is based on the cutting of the multipliers (the Fourier transforms of $\mu_{k,j}$ and $\sigma_{k,j}$) according to a certain Littlewood-Paley decomposition which allows us to obtain L^p-norm inequalities. The method works when some decay and regularity conditions are supposed on the multipliers. The extension of the results to more than two parameters is straightforward but it complicates the notations considerably. In § 2 we give the applications to multiple singular integrals and in § 3 we deal with maximal functions and Hilbert transforms along hypersurfaces.

This paper extends to the multiparametric case the results in [4] and the proofs of § 1 and § 2 follow the same pattern as for the one-parameter case of [4]. Nevertheless, for the sake of completeness, we state them here. In § 3 the estimates on the Fourier transforms and the boundedness of maximal functions required by the conditions of theorems 1 and 2 must be proved (lemmas 2 and 3). As expected, Van der Corput's lemma is an important tool in obtaining the estimates; the boundedness of the maximal functions is a consequence of some inequalities involving lower dimensional maximal functions which in some cases are bounded by an induction hypothesis.

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1. General Results.

Let us introduce some notation. We write $\mathbf{R}^n = \mathbf{R}^{n_1} \times \mathbf{R}^{n_2} \times \mathbf{R}^{n_3}$ and $x \in \mathbf{R}^n$ as $x = (x_1, x_2, x_3)$ with $x_i \in \mathbf{R}^{n_i}$, $i = 1, 2, 3$.

If f is a function defined on $\mathbf{R}^n$ and $h_i \in \mathbf{R}^{n_i} (i=1,2)$ we define

$$\Delta^1_{h_1} f(x_1, x_2, x_3) = f(x_1 + h_1, x_2, x_3) - f(x_1, x_2, x_3)$$

$$\Delta^2_{h_2} f(x_1, x_2, x_3) = f(x_1, x_2 + h_2, x_3) - f(x_1, x_2, x_3)$$

$$\Delta^{1,2}_{h_1, h_2} f(x) = \Delta^1_{h_1}(\Delta^2_{h_2} f(x)).$$

Given a measure μ in $\mathbf{R}^n$ we define $\mu^{(1)}$ in $\mathbf{R}^{n_2+n_3}$, $\mu^{(2)}$ in $\mathbf{R}^{n_1+n_3}$, $\mu^{(1,2)}$ in $\mathbf{R}^{n_3}$, by $\mu^{(1)}(E) = \mu(\mathbf{R}^{n_1}\times E)$, $\mu^{(2)}(F) = \mu(\mathbf{R}^{n_2}\times F)$, $\mu^{(1,2)}(G) = \mu(\mathbf{R}^{n_1}\times\mathbf{R}^{n_2}\times G)$ where E, F, G are Borel sets in $\mathbf{R}^{n_2+n_3}$, $\mathbf{R}^{n_1+n_3}$, $\mathbf{R}^{n_3}$ respectively.

Finally, we write $|\sigma|$ for the total variation of the measure σ and $t^{\pm\alpha} = \inf(t^\alpha, t^{-\alpha})$ for $t > 0$.

The main theorems of this paper are the following:

THEOREM 1. — *Let $\mu_{k,j}$ be uniformly bounded positive measures in $\mathbf{R}^n$. Suppose that for some $a, b > 1$, $\alpha, \beta > 0$, and for all $k, j \in \mathbf{Z}$*

$$|\hat{\mu}_{k,j}(\xi)| \leqslant C|a^k\xi_1|^{-\alpha}|b^j\xi_2|^{-\beta}$$

$$|\Delta^1_{\xi_1}\hat{\mu}_{k,j}(0,\xi_2,\xi_3)| \leqslant C|a^k\xi_1|^{\alpha}|b^j\xi_2|^{-\beta}$$

$$|\Delta^2_{\xi_2}\hat{\mu}_{k,j}(\xi_1,0,\xi_3)| \leqslant C|a^k\xi_1|^{-\alpha}|b^j\xi_2|^{\beta}$$

$$|\Delta^{1,2}_{\xi_1,\xi_2}\hat{\mu}_{k,j}(0,0,\xi_3)| \leqslant C|a^k\xi_1|^{\alpha}|b^j\xi_2|^{\beta}$$

with C independent of ξ. Suppose also that the maximal functions

$$\tilde{M}^{(i)}g_i = \sup_{k,j}|\mu^{(i)}_{k,j} * g_i|, \qquad i = 1,2$$

$$\tilde{M}^{(1,2)}g = \sup_{k,j}|\mu^{(1,2)}_{k,j} * g|$$

are bounded in L^p for every $p > 1$. Then,

$$\mathcal{M}f(x) = \sup_{k,j}|\mu_{k,j} * f(x)|$$

is bounded in $L^p(\mathbf{R}^n)$ for all $p > 1$.

THEOREM 2. — *Let $\sigma_{k,j}$ be Borel measures in $\mathbf{R}^n$ such that $\|\sigma_{k,j}\| \leqslant 1$ and*

$$|\hat{\sigma}_{k,j}(\xi)| \leqslant C|a^k\xi_1|^{\pm\alpha}|b^j\xi_2|^{\pm\beta}$$

for some $a, b > 1$, $\alpha, \beta > 0$ and for all $k, j \in \mathbf{Z}$. If $\sigma^*(f) = \sup_{k,j}\||\sigma_{k,j}| * f|$ is bounded in $L^q(\mathbf{R}^n)$ for some $q > 1$, then,

$$Tf(x) = \sum_{k,j}\sigma_{k,j} * f(x)$$

and $g(f)(x) = \left(\sum_{k,j} |\sigma_{k,j} * f(x)|^2\right)^{\frac{1}{2}}$ *are bounded in* $L^p(\mathbf{R}^n)$ *for* $\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{2q}$.

We shall prove first theorem 2 and use it to prove theorem 1. We begin with a lemma needed in the proof of theorem 2 :

LEMMA 1. — *Let* $\sigma_{k,j}$ *be Borel measures in* $\mathbf{R}^n$ *such that* $\|\sigma_{k,j}\| \leqslant 1$. *If* $\sigma^*(f) = \sup_{k,j} \||\sigma_{k,j}| * f|$ *is bounded in* $L^q(\mathbf{R}^n)$ *for some* $q > 1$, *the following vector valued inequality holds*

$$\left\|\left(\sum_{k,j} |\sigma_{k,j} * g_{k,j}|^2\right)^{\frac{1}{2}}\right\|_{p_0} \leqslant C \left\|\left(\sum_{k,j} |g_{k,j}|^2\right)^{\frac{1}{2}}\right\|_{p_0}$$

for $\left|\frac{1}{p_0} - \frac{1}{2}\right| = \frac{1}{2q}$.

Proof. — We present here a proof different from that of [4]. The inequality

$$\left\|\sum_{k,j} |\sigma_{k,j} * g_{k,j}|\right\|_1 \leqslant \left\|\sum_{k,j} |g_{k,j}|\right\|_1$$

is obvious because $\|\sigma_{k,j}\| \leqslant 1$. On the other hand, the hypothesis on σ^* gives

$$\|\sup_{k,j} |\sigma_{k,j} * g_{k,j}|\,\|_q \leqslant \|\sigma^*(\sup_{k,j} |g_{k,j}|)\|_q \leqslant C\|\sup_{k,j} |g_{k,j}|\,\|_q .$$

Interpolation between these inequalities provides the lemma when $\frac{1}{p_0} = \frac{1}{2}\left(1 + \frac{1}{q}\right)$. The case $p_0 > 2$ is then obtained by duality. □

Proof of theorem 2. — Take two Schwartz functions, $\psi^1 \in \mathcal{S}(\mathbf{R}^{n_1})$ $\psi^2 \in \mathcal{S}(\mathbf{R}^{n_2})$ such that

$$\operatorname{supp} (\psi^i)^\wedge \subset \left\{\frac{1}{2} < |\xi_i| < 2\right\}, \qquad i = 1,2$$

$$0 \leqslant (\psi^i)^\wedge \leqslant 1, \qquad i = 1,2$$

$$\sum_{k=-\infty}^{+\infty} |(\psi^1)^\wedge(a^k\xi_1)|^2 = \sum_{j=-\infty}^{+\infty} |(\psi^2)^\wedge(b^j\xi_2)|^2 = 1 .$$

If ψ_k^1 and ψ_j^2 are defined by $(\psi_k^1)\hat{}(\xi_1) = (\psi^1)\hat{}(a^k\xi_1)$ and $(\psi_j^2)\hat{}(\xi_2) = (\psi^2)\hat{}(b^j\xi_2)$, we can write

$$Tf = \sum_{k,j}\sum_{l,m} \sigma_{k,j} * (\psi_{k+l}^1 \otimes \psi_{j+m}^2) * (\psi_{k+l}^1 \otimes \psi_{j+m}^2) * f = \sum_{l,m} T_{l,m}f.$$

Then, $\|Tf\|_p \leqslant \sum_{l,m} \|T_{l,m}f\|_p$ and we estimate each of the terms of this sum by interpolation between the L^2-norm and the L^{p_0}-norm (p_0 as in lemma 1).

$$\begin{aligned}\|T_{l,m}f\|_{p_0} &\leqslant C\left\|\left(\sum_{k,j}|\sigma_{k,j} * (\psi_{k+l}^1 \otimes \psi_{j+m}^2) * f|^2\right)^{\frac{1}{2}}\right\|_{p_0}\\ &\leqslant C'\left\|\left(\sum_{k,j}|(\psi_{k+l}^1 \otimes \psi_{j+m}^2) * f|^2\right)^{\frac{1}{2}}\right\|_{p_0} \leqslant C''\|f\|_{p_0}\end{aligned}$$

where the first and last inequalities are given by Littlewood-Paley theory (see [10]) and the second one follows from lemma 1.

The L^2 estimate is provided, as usual, by Plancherel's theorem

$$\|T_{l,m}f\|_2^2 \leqslant \sum_{k,j} \iint_{\Delta_{k,j}^{l,m}} |\hat{\sigma}_{k,j}(\xi)|^2 |\hat{f}(\xi)|^2 \, d\xi$$

where

$$\Delta_{k,j}^{l,m} = \{\xi \in \mathbf{R}^n / a^{-k-l-1} < |\xi_1| < a^{-k-l+1}, b^{-j-m-1} < |\xi_2| < b^{-j-m+1}\}.$$

The hypotheses on $\hat{\sigma}_{k,j}(\xi)$ imply

$$\|T_{l,m}f\|_2 \leqslant C a^{-\alpha|l|} b^{-\beta|m|} \|f\|_2 .$$

Now, when $\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{2q}$, we have $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{p_0}$ for some θ with $0 < \theta \leqslant 1$ and then

$$\|Tf\|_p \leqslant \sum_{l,m} \|T_{l,m}f\|_p \leqslant C \sum_{l,m} a^{-\theta\alpha|l|} b^{-\theta\beta|m|} \|f\|_p \leqslant C_p \|f\|_p .$$

The proof of $\|g(f)\|_p \leqslant C\|f\|_p$ is similar. It can also be deduced from the preceding result, by observing that for every sequence $\{\varepsilon_{k,j}\}$, $\varepsilon_{k,j} = \pm 1$, the operator $T_\varepsilon f = \sum_{k,j} \varepsilon_{k,j}\sigma_{k,j} * f$ has a bound in L^p independent of the sequence of signs; then, the inequality for $g(f)$ is obtained by randomization. $\square$

Proof of theorem 1. — Let Φ^1, Φ^2 be positive Schwartz functions in $\mathbf{R}^{n_1}$ and $\mathbf{R}^{n_2}$ respectively, such that $(\Phi^1)\hat{}(0) = (\Phi^2)\hat{}(0) = 1$ and define

$$(\Phi_k^1)\hat{}(\xi_1) = (\Phi^1)\hat{}(a^k\xi_1)$$
$$(\Phi_j^2)\hat{}(\xi_2) = (\Phi^2)\hat{}(b^j\xi_2).$$

Then, we define the measures $\sigma_{k,j}$ by

$$\begin{aligned}\hat{\sigma}_{k,j}(\xi) = \hat{\mu}_{k,j}(\xi) - (\Phi_k^1)\hat{}(\xi_1)\hat{\mu}_{k,j}(0,\xi_2,\xi_3) &\\ - (\Phi_j^2)\hat{}(\xi_2)\hat{\mu}_{k,j}(\xi_1,0,\xi_3) &\\ + (\Phi_k^1)\hat{}(\xi_1)(\Phi_j^2)\hat{}(\xi_2)\hat{\mu}_{k,j}(0,0,\xi_3) &\end{aligned}$$

which satisfy the size hypotheses of theorem 2. Moreover, for every $f \geqslant 0$,

$$\begin{aligned}\mathcal{M}f(x) \leqslant \sup_{k,j} (\Phi_k^1 \otimes \mu_{k,j}^{(1)}) * f(x) + \sup_{k,j} (\Phi_j^2 \otimes \mu_{k,j}^{(2)}) * f(x) &\\ + \sup (\Phi_k^1 \otimes \Phi_j^2 \otimes \mu_{k,j}^{(1,2)}) * f(x) + g(f)(x) &\end{aligned}$$

where $g(f)(x)$ is the quadratic mean of $\{\sigma_{k,j} * f(x)\}_{k,j}$ as in theorem 2. If M_i is the Hardy-Littlewood maximal function acting on the x_i-variable, we have

$$\sup_{k,j} (\Phi_k^1 \otimes \mu_{k,j}^{(1)}) * f(x) \leqslant C\mathrm{M}_1\tilde{\mathrm{M}}^{(1)}f(x)$$

$$\sup_{k,j} (\Phi_j^2 \otimes \mu_{k,j}^{(2)}) * f(x) \leqslant C\mathrm{M}_2\tilde{\mathrm{M}}^{(2)}f(x)$$

$$\sup_{k,j} (\Phi_k^1 \otimes \Phi_j^2 \otimes \mu_{k,j}^{(1,2)}) * f(x) \leqslant C\mathrm{M}_1\mathrm{M}_2\tilde{\mathrm{M}}^{(1,2)}f(x)$$

where $\tilde{\mathrm{M}}^{(1)}$, $\tilde{\mathrm{M}}^{(2)}$ and $\tilde{\mathrm{M}}^{(1,2)}$ act on the corresponding variables. M_1 and M_2 are known to be bounded in L^p, $p > 1$; $\tilde{\mathrm{M}}^{(1)}$, $\tilde{\mathrm{M}}^{(2)}$ and $\tilde{\mathrm{M}}^{(1,2)}$ are also bounded in L^p, $p > 1$, by hypothesis. Since g is bounded in L^2, so is $\mathcal{M}$. From the definition of $\sigma_{k,j}$ we deduce the boundedness of σ^* in L^2; then, theorem 2 applies and yields the boundedness of g (and, therefore, of $\mathcal{M}$) for $\frac{4}{3} < p < 4$; but then σ^* is also bounded in such range of $p's$ and a new application of theorem 2 gives the boundedness of g and $\mathcal{M}$ when $\frac{8}{7} < p < 8$. Successive applications of theorem 2 allow us to obtain the whole range $1 < p < \infty$. For $p = \infty$ the boundedness is trivial. □

2. Multiple Singular Integrals.

The homogeneous singular integral operators in $\mathbf{R}^n$ defined by

$$Tf(x) = \text{p.v.} \int_{\mathbf{R}^n} \frac{\Omega(y)}{|y|^n} f(x-y)\, dy$$

with Ω homogeneous of degree 0 and mean value zero over the unit sphere are known to be bounded in $L^p(\mathbf{R}^n)$, $1 < p < \infty$, under the hypothesis $\Omega \in L^q(S^{n-1})$, $q > 1$. This result is usually obtained by the method of rotations of Calderón and Zygmund [1]. It is elementary when Ω is odd (even with $\Omega \in L^1(S^{n-1})$) and when Ω is even, the operator is written as $T = -\sum_{i=1}^{n} R_i^2 T$, where $\{R_i\}_{i=1}^n$ are the Riesz transforms, which are bounded in $L^p(\mathbf{R}^n)$, $1 < p < \infty$. Then, the result for odd kernels is used to handle $R_i T$ (see [1] for the details). A general Ω can now be decomposed in its odd and even parts.

We can generalize these operators to $\mathbf{R}^{n+m}$ in the following way: Let

$$Tf(x_1,x_2) = \text{p.v.} \iint_{\mathbf{R}^n \times \mathbf{R}^m} \frac{\Omega(y_1',y_2')}{|y_1|^n |y_2|^m} f(x_1-y_1,x_2-y_2)\, dy_1\, dy_2$$

where $y_1 \in \mathbf{R}^n$, $y_2 \in \mathbf{R}^m$ and $y_i' = y_i/|y_i|$ $i = 1,2$. The cancellation hypothesis on Ω becomes

$$\int_{S^{n-1}} \Omega(y_1',y_2')\, dy_1' = \int_{S^{m-1}} \Omega(y_1',y_2')\, dy_2' = 0\,.$$

If we intend to apply the method of rotations to this case we find that for the easy part, the oddness of Ω in each one of the variables y_1' and y_2' is needed, i.e.,

$$\Omega(y_1',y_2') = -\Omega(-y_1',y_2') = -\Omega(y_1',-y_2') = \Omega(-y_1',-y_2')$$

for every $y_1 \in \mathbf{R}^n$, $y_2 \in \mathbf{R}^m$. Therefore, the rest of the method seems too difficult to be adapted here.

Anyway, we can show that, also in this case, $\Omega \in L^q(S^{n-1} \times S^{m-1})$, $q > 1$, is enough to obtain the boundedness of T in $L^p(\mathbf{R}^{n+m})$, $1 < p < \infty$. We do this by decomposing T as

$$\begin{aligned} Tf(x) &= \sum_{k,j=-\infty}^{\infty} \int_{2^k}^{2^{k+1}} \int_{2^j}^{2^{j+1}} \iint_{S^{n-1}\times S^{m-1}} \Omega(u,v) f(x_1-ru,x_2-sv)\, du\, dv \frac{dr\, ds}{rs} \\ &= \sum_{k,j} \sigma_{k,j} * f(x) \end{aligned}$$

and applying theorem 2. Actually, we shall prove a somewhat more general result because an extra factor of the type $h(|y_1|,|y_2|)$ is allowed in the kernel.

COROLLARY 1. – *Let* $K(x_1,x_2)$ *be a kernel of the form*

$$K(ru,sv) = r^{-n}s^{-m}h(r,s)\Omega(u,v)$$

for $r,s > 0$, $(u,v) \in S^{n-1} \times S^{m-1}$. *Suppose that*

$$a)\ \int_{S^{n-1}} \Omega(u,v)\,du = 0, \quad \forall v \in S^{m-1}$$

$$\int_{S^{m-1}} \Omega(u,v)\,dv = 0, \quad \forall u \in S^{n-1}$$

$$\Omega \in L^q(S^{n-1} \times S^{m-1}), \quad q > 1.$$

$b)$ $\displaystyle\int_0^{R_1}\int_0^{R_2} |h(r,s)|^2\,dr\,ds \leqslant CR_1R_2$ *for every* $R_1, R_2 > 0$. *Then,* $Tf = \text{p.v. } K * f$ *is bounded in* $L^p(\mathbf{R}^n)$, $1 < p < \infty$.

Proof. – Write $Tf = \sum_{k,j} \sigma_{k,j} * f$ with

$$\hat{\sigma}_{k,j}(\xi) = \int_{2^k}^{2^{k+1}}\int_{2^j}^{2^{j+1}}\iint_{S^{n-1}\times S^{m-1}} \Omega(u,v)h(r,s) \cdot \exp\left(-2\pi i(\xi_1\cdot ru+\xi_2\cdot sv)\right)du\,dv\,\frac{dr\,ds}{rs}\cdot$$

We apply theorem 2 (without the third variable ξ_3); let us verify the conditions of the theorem :

$$|\hat{\sigma}_{k,j}(\xi)|^2 \leqslant C\int_{2^k}^{2^{k+1}}\int_{2^j}^{2^{j+1}}\left|\iint_{S^{n-1}\times S^{m-1}} \Omega(u,v) \exp\left(-2\pi i(\xi_1\cdot ru+\xi_2\cdot sv)\right)du\,dv\right|^2 \cdot\frac{dr\,ds}{rs}$$

$$= C\int_{2^k}^{2^{k+1}}\int_{2^j}^{2^{j+1}}\left[\iint_{(S^{n-1}\times S^{m-1})^2} \Omega(u,v)\overline{\Omega(u',v')} \exp\left(-2\pi i(\xi_1\cdot r(u-u')+\xi_2\cdot s(v-v'))\right)\cdot du\,dv\,du'\,dv'\right]\frac{dr\,ds}{rs}\cdot$$

Now

$$\left|\int_{2^k}^{2^{k+1}}\int_{2^j}^{2^{j+1}} \exp\left(-2\pi i(\xi_1\cdot r(u-u')+\xi_2\cdot s(v-v'))\right)\frac{dr\,ds}{rs}\right| \leqslant \frac{C}{|2^k\xi_1\cdot(u-u')||2^j\xi_2\cdot(v-v')|}.$$

Moreover, this integral is also bounded by 1; thus, the bound can be taken as $C|2^k\xi_1\cdot(u-u')|^{-\varepsilon}|2^j\xi_2\cdot(v-v')|^{-\varepsilon}$ for arbitrary ε, $0<\varepsilon<1$. Hölder's inequality gives

$$|\hat{\sigma}_{k,j}(\xi)| \leqslant C\|\Omega\|_q\left(\int_{S^{n-1}\times S^{n-1}}\frac{du\,du'}{|2^k\xi_1\cdot(u-u')|^{\varepsilon q'}}\right)^{1/q'} \cdot\left(\int_{S^{m-1}\times S^{m-1}}\frac{dv\,dv'}{|2^j\xi_2\cdot(v-v')|^{\varepsilon q'}}\right)^{1/q'}$$

and choosing ε such that $\varepsilon q'<1$ we get

$$|\hat{\sigma}_{k,j}(\xi)| \leqslant C|2^k\xi_1|^{-\varepsilon}|2^j\xi_2|^{-\varepsilon}.$$

For the other estimates the cancellation properties are needed:

$$|\Delta^1_{\xi_1}\hat{\sigma}_{k,j}(0,\xi_2)| \leqslant \int_{2^k}^{2^{k+1}}\int_{S^{n-1}}\left|\int_{2^j}^{2^{j+1}}\int_{S^{m-1}}\Omega(u,v)h(r,s) \exp(-2\pi i\xi_2\cdot sv)\,dv\,\frac{ds}{s}\right| |\exp(-2\pi i\xi_1\cdot ru)-1|\,du\,\frac{dr}{r}.$$

The inner integral can be estimated as before and we obtain the bound

$$C\int_{2^k}^{2^{k+1}}\int_{S^{n-1}}\left(\int_{2^j}^{2^{j+1}}|h(r,s)|^2\frac{ds}{s}\right)^{\frac{1}{2}}\|\Omega(u,\cdot)\|_{L^q(S^{m-1})} du\,\frac{dr}{r}\,|2^k\xi_1||2^j\xi_2|^{-\varepsilon} \leqslant C\|\Omega\|_q|2^k\xi_1||2^j\xi_2|^{-\varepsilon}.$$

The symmetric condition, interchanging the roles of ξ_1 and ξ_2, is similar. The fourth condition

$$|\Delta^{1,2}_{\xi_1,\xi_2}\hat{\sigma}_{k,j}(0,0)| \leqslant C|2^k\xi_1||2^j\xi_2|$$

is a simple consequence of the cancellation properties.

Finally, we must obtain the boundedness of σ^* and this will follow from theorem 1. In fact, since

$$(|\sigma_{k,j}|)\hat{}(\xi) = \int_{2^k}^{2^{k+1}} \int_{2^j}^{2^{j+1}} \iint_{S^{n-1}\times S^{m-1}} |\Omega(u,v)|\,|h(r,s)| \cdot \exp(-2\pi i(\xi_1\cdot ru+\xi_2\cdot sv))\,du\,dv\,\frac{dr\,ds}{rs}$$

the above estimates are also available here and

$$(|\sigma_{k,j}|)\hat{}(0) \leqslant \left(\int_{2^k}^{2^{k+1}} \int_{2^j}^{2^{j+1}} |h(r,s)|^2\,\frac{dr\,ds}{rs}\right)^{\frac{1}{2}} \|\Omega\|_q$$

is a uniformly bounded sequence. On the other hand, the partial maximal functions of theorem 1 are controlled (if $g \geqslant 0$) by

$$|\sigma_{k,j}|^{(1)} * g(x_1) \leqslant \int_{2^k}^{2^{k+1}} \int_{S^{n-1}} \|\Omega(u,\cdot)\|_{L^q(S^{m-1})}\tilde{h}(r) g(x_1 - ru)\,du\,\frac{dr}{r}$$

with $\tilde{h}(r) = \displaystyle\int_{2^j}^{2^{j+1}} |h(r,s)|^2\,\frac{ds}{s}$ and its analogue for x_2. Then, the boundedness of the partial maximal functions is a consequence of the one-parameter result ([4], corollary 4.1). Thus, σ^* is bounded in L^p, $p > 1$, and then theorem 2 implies that T is bounded in L^p, $1 < p < \infty$. □

After this corollary where L^p-estimates have been obtained with size conditions on Ω instead of the regularity assumed in [5], we give another corollary where we get weighted norm inequalities with somewhat more restrictive conditions on the size of Ω (now, $\Omega \in L^\infty$), but without the regularity conditions used in [6].

The class A_p^* of weights w for which we obtain estimates in $L^p(w)$ can be described as those nonnegative locally integrable functions such that

$$\sup_R \left(\frac{1}{|R|}\int_R w(x)\,dx\right)\left(\frac{1}{|R|}\int_R w(x)^{-1/(p-1)}\,dx\right)^{p-1} < +\infty$$

where R is the product of two arbitrary cubes of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively. Alternatively, we can say that $w \in A_p^*$ if for each $x_1 \in \mathbf{R}^n$, $w(x_1,\cdot)$ is in $A_p(\mathbf{R}^m)$ with constant independent of x_1 and a similar condition holds for

$w(\cdot,x_2)$. These are the weights for which the strong maximal function

$$f_s^*(x_1,x_2) = \sup_{h_1,h_2>0} \frac{1}{|B(0,h_1)\times B(0,h_2)|} \cdot \iint_{B(0,h_1)\times B(0,h_2)} |f(x_1-y_1,x_2-y_2)|\,dy_1\,dy_2$$

is bounded in $L^p(w)$ [7].

COROLLARY 2. — *Let*

$$K(x_1,x_2) = \Omega(x_1',x_2')h(|x_1|,|x_2|)|x_1|^{-n}|x_2|^{-m}$$

be a kernel such that $h \in L^\infty(\mathbf{R}^2)$, $\Omega \in L^\infty(S^{n-1}\times S^{m-1})$ *and*

$$\int_{S^{n-1}} \Omega(u,v)\,du = \int_{S^{m-1}} \Omega(u,v)\,dv = 0$$

then $Tf(x) = \text{p.v. } K * f(x)$ *is a bounded operator in* $L^p(w)$, $w \in A_p^*$, $1 < p < \infty$.

The proof of this corollary follows step by step that of corollary 4.2 of [4]. We only need to be sure that all of them generalize to the product setting :

i) The extrapolation theorem of Rubio de Francia [9] allows us to restrict the problem to $p = 2$;

ii) if $w \in A_2^*$ the Littlewood-Paley inequalities associated to a product decomposition hold in $L^2(w)$ [7];

iii) $|\sigma_{k,j}| * f(x) \leqslant f_s^*(x)$, $\forall k,j$; therefore σ^* is bounded in $L^2(w)$, $w \in A_2^*$;

iv) if $w \in A_2^*$, then for some $\varepsilon > 0$, $w^{1+\varepsilon}$ is also in A_2^*.

From (ii) and (iii), the operators $T_{l,m}$ in the proof of theorem 2 are uniformly bounded in $L^2(w)$. By (iv), the same is true in $L^2(w^{1+\varepsilon})$. Interpolating this with the unweighted inequality $\|T_{l,m}f\|_2 \leqslant Ca^{-\alpha|l|}b^{-\beta|m|}\|f\|_2$, we obtain an exponential decay in the $L^2(w)$-norm of $T_{l,m}$, finishing the proof.

3. Maximal functions and Hilbert transforms along surfaces.

Consider a surface S in $\mathbf{R}^3$, parametrized as $(s,t,\varphi(s,t))$ with $\varphi(0,0) = 0$ and even in each one of the variables, i.e.,

$$\varphi(-s,t) = \varphi(s,-t) = \varphi(-s,-t) = \varphi(s,t), \qquad s,t > 0.$$

The maximal function along S is defined by

$$\mathcal{M}f(x) = \sup_{\substack{0 \leqslant h_i \leqslant C_i \\ i=1,2}} \frac{1}{h_1 h_2} \left| \int_0^{h_1} \int_0^{h_2} f(x_1 - s, x_2 - t, x_3 - \varphi(s,t))\, ds\, dt \right|$$

and the Hilbert transform along S is

$$\mathrm{H}f(x) = \mathrm{p.v.} \iint_{\substack{|s| < C_1 \\ |t| < C_2}} f(x_1 - s, x_2 - t, x_3 - \varphi(s,t)) \frac{ds\, dt}{st}.$$

Both integrals will be a priori defined only for Schwartz functions. The numbers C_1 and C_2 appearing in such definitions depend on S and can be $+\infty$. When they are finite we can suppose, without loss of generality, that they are exact powers of 2.

Let us define a new maximal function

$$\begin{aligned} \mathcal{N}f(x) &= \sup_{\substack{k \leqslant N_1 \\ j \leqslant N_2}} \frac{1}{2^{k+j}} \int_{2^k}^{2^{k+1}} \int_{2^j}^{2^{j+1}} f(x_1 - s, x_2 - t, x_3 - \varphi(s,t))\, ds\, dt \\ &= \sup_{\substack{k \leqslant N_1 \\ j \leqslant N_2}} \mu_{k,j} * f(x) \end{aligned}$$

where $2^{N_1+1} = C_1$, $2^{N_2+1} = C_2$. It is easy to see that

$$\mathcal{M}f(x) \leqslant 4\mathcal{N}(|f|)(x)$$

and the estimates for $\mathcal{N}$ in $L^p(\mathbf{R}^3)$ will provide those of $\mathcal{M}$.

On the other hand, write

$$\begin{aligned} \mathrm{H}f(x) &= \sum_{k=-\infty}^{N_1} \sum_{j=-\infty}^{N_2} \int_{2^k}^{2^{k+1}} \int_{2^j}^{2^{j+1}} f(x_1 - s, x_2 - t, x_3 - \varphi(s,t)) \frac{ds\, dt}{st} \\ &= \sum_{k=-\infty}^{N_1} \sum_{j=-\infty}^{N_2} \sigma_{k,j} * f(x). \end{aligned}$$

$\mathcal{N}$ and H are now written in such a way that we can apply theorems 1 and 2. Observe that the boundedness of σ^* required in theorem 2 will here be a consequence of that of $\mathcal{N}$.

In this paragraph $D_i\varphi(s,t)$ stands for the derivative of φ with respect to the i-th variable $(i=1,2)$; $D_i^2\varphi(s,t) = D_i(D_i\varphi(s,t))$ and $D_{12}\varphi(s,t) = D_1(D_2\varphi(s,t))$.

Three types of surface will be considered here:

Type 1. — $\varphi(s,t) = |s|^{\alpha}|t|^{\beta}$, $\alpha, \beta > 0$. See [3] for the maximal function and [8], [12] and [13] for the Hilbert transform. Here $C_1 = C_2 = +\infty$.

Type 2. — $\varphi(s,t)$ is an even function of class C^2 in a neighbourhood of the origin with $D_1^2\varphi(0,0)$ and $D_2^2\varphi(0,0) \neq 0$, $D_{12}\varphi(s,0)$ nonnegative if $D_1^2\varphi(0,0) > 0$ (resp. nonpositive if $D_1^2\varphi(0,0) < 0$) and a similar condition over $D_{12}\varphi(0,t)$. See [2] where the result for maximal functions is obtained without conditions on D_{12}. C_1 and C_2 must be chosen such that $|D_i^2\varphi(s,t)| \geqslant A$ $(i=1,2)$ for some $A > 0$ in $0 < s \leqslant C_1$, $0 < t \leqslant C_2$.

Type 3. — $\varphi(s,t)$ is an even function of class C^2 such that $D_1^2\varphi(s,t)$ and $D_{12}\varphi(s,0)$ (resp. $D_2^2\varphi(s,t)$ and $D_{12}\varphi(0,t)$) are nonnegative and nondecreasing in $s > 0$ (resp. in $t > 0$). In this case C_1 and C_2 must be chosen such that these conditions hold in $0 < s \leqslant C_1$, $0 < t \leqslant C_2$. Observe that surfaces with a contact of infinite order at the origin with the coordinate plane OX_1X_2 are allowed; for example, $\varphi(s,t) = s^2t^2(e^{-1/|s|}+e^{-1/|t|})$ for which $C_1 = C_2 = +\infty$.

We can then state the following result:

COROLLARY 3. — *If* φ *is a function of one of the preceding types, then*

$$\|\mathcal{M}f\|_p \leqslant C_p\|f\|_p, \qquad 1 < p \leqslant +\infty$$
$$\|Hf\|_p \leqslant C_p\|f\|_p, \qquad 1 < p < +\infty.$$

As a consequence of the boundedness of $\mathcal{M}$ and following a well-known method we can deduce that

$$\lim_{\substack{h_1\to 0\\ h_2\to 0}} \frac{1}{h_1h_2}\int_0^{h_1}\int_0^{h_2} f(x_1-s,x_2-t,x_3-\varphi(s,t))\,ds\,dt = f(x) \quad \text{a.e.}$$

for every f which is locally in $L^p(\mathbf{R}^3)$, $p > 1$.

The proof of this corollary is nothing but an application of theorems 1 and 2 to the measures $\mu_{k,j}$ and $\sigma_{k,j}$ defined above. As indicated the boundedness of σ^* is a consequence of that of $\mathcal{N}$ and the remaining conditions of the theorems are contained in the following two lemmas.

LEMMA 2. — *If* φ *is of one of the preceding types*

$$\hat{\mu}_{k,j}(\xi) = \frac{1}{2^{k+j}} \int_{2^k}^{2^{k+1}} \int_{2^j}^{2^{j+1}} \exp\left(-2\pi i(\xi_1 s + \xi_2 t + \xi_3 \varphi(s,t))\right) ds\, dt$$

and

$$\hat{\sigma}_{k,j}(\xi) = \iint_{\substack{2^k < |s| \leqslant 2^{k+1} \\ 2^j < |t| \leqslant 2^{j+1}}} \exp\left(-2\pi i(\xi_1 s + \xi_2 t + \xi_3 \varphi(s,t))\right) \frac{ds\, dt}{st}$$

verify the size properties of theorems 1 *and* 2 *with* $a = b = 2$.

In the following lemma M_i represents the Hardy-Littlewood maximal function acting on the i-th variable $(i = 1,2,3)$.

LEMMA 3. — *a) If* φ *is of type* 1, *then*

$$\tilde{M}^{(1)}f(x) \leqslant CM_2M_3f(x)$$
$$\tilde{M}^{(2)}f(x) \leqslant CM_1M_3f(x)$$
$$\tilde{M}^{(1,2)}f(x) \leqslant CM_3f(x)$$

b) If φ *is of type* 2 *or* 3, *then*

$$\tilde{M}^{(1)}f(x) \leqslant CM_{\gamma_2}M_3f(x)$$
$$\tilde{M}^{(2)}f(x) \leqslant CM_{\gamma_1}M_3f(x)$$
$$\tilde{M}^{(1,2)}f(x) \leqslant CM_3M_3f(x)$$

where γ_1 *and* γ_2 *are the plane curves* $(s,\varphi(s,0))$ *and* $(t,\varphi(0,t))$ *and* M_{γ_1} *and* M_{γ_2} *denote their associated maximal functions.*

From the results of maximal functions along curves we know that M_{γ_1} and M_{γ_2} are bounded in $L^p(\mathbf{R}^2)$, $p > 1$ (see [4]); the Hardy-Littlewood maximal function is also bounded in such L^p's therefore, the inequalities in this lemma give the boundedness of the « partial » maximal functions required in theorem 1.

Proof of lemma 2. — It is enough to prove the lemma for $\hat{\mu}_{k,j}$ because the estimates for $\hat{\sigma}_{k,j}$ are similar after application of the second mean value theorem for integrals and the evenness of φ.

After a change of variables the integrals are taken over $1 < s \leqslant 2$, $1 < t \leqslant 2$ and the exponential function in the integrand becomes $\exp\left[-2\pi i(2^k\xi_1 s + 2^j\xi_2 t + \xi_3\varphi(2^k s, 2^j t))\right]$. The estimates near zero are now trivial, they come from the factors $[\exp(-2\pi i 2^k \xi_1 s) - 1]$ and

$[\exp(-2\pi i 2^j\xi_2 t)-1]$ present in the integrand. Then, we need only to obtain the part of the estimates near infinity. Both are symmetric, therefore, we shall show that

$$|\hat{\mu}_{k,j}(\xi)| \leqslant C|2^k\xi_1|^{-\alpha}$$

for the three types of surface.

Type 1. – Since $\hat{\mu}_{k,j}(\xi) = \hat{\mu}(2^k\xi_1, 2^j\xi_2, 2^{k\alpha+j\beta}\xi_3)$ with $\mu = \mu_{0,0}$, the only thing to verify is $|\hat{\mu}(\xi)| \leqslant C|\xi_1|^{-\frac{1}{2}}$.

If $\alpha \neq 1$ we know (see [11]) that

$$\left|\int_1^2 \exp(-2\pi i(\eta_1 s + \eta_2 s^\alpha))\, ds\right| \leqslant C(\eta_1^2+\eta_2^2)^{-1/4}$$

then,

$$\begin{aligned}|\hat{\mu}(\xi)| &\leqslant \int_1^2 \left|\int_1^2 \exp(-2\pi i(\xi_1 s + \xi_3 s^\alpha t^\beta))\, ds\right| dt \\ &\leqslant C\int_1^2 |\xi_1^2+\xi_3^2 t^{2\beta}|^{-1/4}\, dt \leqslant C|\xi_1|^{-1/2}.\end{aligned}$$

When $\alpha = 1$ slight changes are needed:

$$\begin{aligned}|\hat{\mu}(\xi)| &\leqslant \int_1^2 \left|\int_1^2 \exp(-2\pi i(\xi_1 + \xi_3 t^\beta)s)\, ds\right| dt \\ &\leqslant \int_1^2 \left|\frac{\sin \pi(\xi_1+\xi_3 t^\beta)}{\pi(\xi_1+\xi_3 t^\beta)}\right| dt \leqslant C|\xi_1|^{-1/2}\end{aligned}$$

because $I(A,B) = \displaystyle\int_1^2 \left|\frac{\sin(A+Bt^\beta)}{A+Bt^\beta}\right| dt \leqslant C|A|^{-1/2}$.

In fact, if $|A| \geqslant 2^{\beta+1}|B|$ it is trivial. If $|A| \leqslant 2^{\beta+1}|B|$ we make the change of variable $\tau = A + Bt$; then $d\tau = B\beta t^{\beta-1}\, dt$ and $dt \leqslant C|B|^{-1}\, d\tau$. We get

$$\begin{aligned}I(A,B) &\leqslant C|B|^{-1}\int_{A+B}^{A+2^\beta B} \left|\frac{\sin\tau}{\tau}\right| d\tau \leqslant C|B|^{-1}\log|B| \leqslant C|B|^{-1/2} \\ &\leqslant C|A|^{-1/2}.\end{aligned}$$

Type 2. – Take the domain $\{0<s\leqslant C_1, 0<t\leqslant C_2\}$ such that $|D_1^2\varphi(s,t)|$, $|D_2^2\varphi(s,t)| \geqslant A > 0$. In such a domain $D_{12}\varphi(s,t)$ is bounded (because $\varphi \in C^2$), then $|D_1\varphi(2^k s, 2^j t)|$, $|D_2\varphi(2^k s, 2^j t)| \leqslant B \sup(2^k, 2^j)$.

Write

$$g_1(s) = 2^k\xi_1 s + \xi_3\varphi(2^k s, 2^j t)$$
$$g_2(t) = 2^j\xi_2 t + \xi_3\varphi(2^k s, 2^j t)$$

and $N = \sup(k,j)$. Then,

$$|\hat{\mu}_{k,j}(\xi)| \leqslant \min\left(\int_1^2 \left|\int_1^2 \exp(-2\pi i g_1(s))\, ds\right| dt, \int_1^2 \left|\int_1^2 \exp(-2\pi i g_2(t))\, dt\right| ds\right).$$

From

$$g_1''(s) = 2^{2k}\xi_3 D_1^2\varphi(2^k s, 2^j t)$$
$$g_2''(t) = 2^{2j}\xi_3 D_2^2\varphi(2^k s, 2^j t)$$

and the condition for the second derivatives we have $|g_1''(s)| \geqslant 2^{2k}|\xi_3|A$, $|g_2''(t)| \geqslant 2^{2j}|\xi_3|A$; then, Van der Corput's lemma implies

$$(*) \qquad |\hat{\mu}_{k,j}(\xi)| \leqslant C|2^{2N}\xi_3|^{-1/2}.$$

On the other hand, $g_1'(s) = 2^k\xi_1 + 2^k\xi_3 D_1\varphi(2^k s, 2^j t)$ is monotone and when $|\xi_1| \geqslant B2^{N+1}|\xi_3|$, $|g_1'(s)| \geqslant \frac{1}{2}|2^k\xi_1|$. Then, in such region, Van de Corput's lemma gives

$$|\hat{\mu}_{k,j}(\xi)| \leqslant C|2^k\xi_1|^{-1}.$$

In the complementary region the estimate $(*)$ yields

$$|\hat{\mu}_{k,j}(\xi)| \leqslant C|2^N\xi_1|^{-1/2}B^{-1} \leqslant C|2^k\xi_1|^{-1/2}B^{-1}.$$

Type 3.

$$|\hat{\mu}_{k,j}(\xi)| \leqslant \int_1^2 \left|\int_1^2 \exp(-2\pi i(2^k\xi_1 s + \xi_3\varphi(2^k s, 2^j t)))\, ds\right| dt.$$

Write $g(s) = 2^k\xi_1 s + \xi_3\varphi(2^k s, 2^j t)$; then,

$$g'(s) = 2^k\xi_1 + 2^k\xi_3 D_1\varphi(2^k s, 2^j t)$$

and

$$g''(s) = 2^{2k}\xi_3 D_1^2\varphi(2^k s, 2^j t).$$

For each t, we split the interval $1 < s \leqslant 2$ into two parts, I_1 and I_2, as follows: on I_1,

$$(D_1\varphi(2^k s, 2^j t) - D_1\varphi(0, 2^j t))|\xi_3| \leqslant \frac{|\xi_1 + D_1\varphi(0, 2^j t)\xi_3|}{2}$$

and on I_2 the opposite inequality holds. I_1 and I_2 are intervals because $D_1\varphi(\cdot,2^j t)$ is increasing. Moreover, on I_1, g' is monotone and $|g'(s)| \geqslant 2^{k-1}|\xi_1 + D_1\varphi(0,2^j t)\xi_3|$. On the other hand, since $D_1^2\varphi$ is nondecreasing in s

$$D_1^2\varphi(2^k s, 2^j t) \geqslant \frac{D_1\varphi(2^k s,2^j t) - D_1\varphi(0,2^j t)}{2^k s}$$

and we have $|g''(s)| \geqslant 2^{k-2}|\xi_1 + D_1\varphi(0,2^j t)\xi_3|$ on I_2. From Van der Corput's lemma we get for the inner integral the bound $C(2^k|\xi_1 + D_1\varphi(0,2^j t)\xi_3|)^{-1/2}$. Now,

$$|\hat{\mu}_{k,j}(\xi)| \leqslant C|2^k\xi_1|^{-1/2}\int_1^2 \left|1 + D_1\varphi(0,2^j t)\frac{\xi_3}{\xi_1}\right|^{-1/2} dt .$$

It is clear that the part of this last integral where $D_1\varphi(0,2^j t)\xi_3/\xi_1 \geqslant -1/2$ or $\leqslant -2$ is bounded; then, it remains to see that the integral over the interval (a,b) where $-2 \leqslant D_1\varphi(0,2^j t)\xi_3/\xi_1 \leqslant -1/2$ is also bounded. After a change of variable $u = D_1\varphi(0,2^j t)\xi_3/\xi_1$ we have the bound

$$\int_{-2}^{-1/2} |1+u|^{-1/2}|2^j D_{12}\varphi(0,2^j t)\xi_3/\xi_1|^{-1}\, du \leqslant 4\int_{-2}^{-1/2} |1+u|^{-1/2}\, du$$

because from the hypothesis on $D_{12}\varphi(0,t)$ we have

$$D_{12}\varphi(0,2^j t) \geqslant \frac{D_1\varphi(0,2^j t)}{2^j t} \geqslant \frac{|\xi_1|}{2^{j+2}|\xi_3|}. \qquad \square$$

Proof of lemma 3. — *a*) The first two inequalities are similar and we prove only the first. In fact, we shall prove the following

$$\frac{1}{h_1 h_2}\int_0^{h_1}\int_0^{h_2} f(x_1 - s, x_2, x_3 - s^\alpha t^\beta)\, ds\, dt \leqslant CM_1M_3 f(x).$$

After a change of variables the left-hand side becomes

$$\frac{1}{h_1 h_2}\int_0^{h_1}\int_0^{s^\alpha h_2^\beta} f(x_1 - s, x_2, x_3 - t)t^{(1/\beta)-1}s^{-(\alpha/\beta)}\beta^{-1}\, dt\, ds .$$

If $\beta > 1$, $t^{(1/\beta)-1}$ is decreasing and

$$\int_0^{s^\alpha h_2^\beta} t^{(1/\beta)-1}\, dt = \beta s^{\alpha/\beta} h_2 ;$$

then, the integral is bounded by

$$\frac{2}{h_1}\int_0^{h_1} M_3 f(x_1-s,x_2,x_3)\,ds \leqslant 2M_1M_3f(x).$$

If $\beta \leqslant 1$, $t^{(1/\beta)-1} \leqslant s^{(\alpha/\beta)-\alpha}h_2^{1-\beta}$ and we obtain the bound

$$\frac{1}{\beta h_1}\int_0^{h_1} M_3 f(x_1-s,x_2,x_3)\,ds \leqslant \frac{2}{\beta} M_1M_3f(x).$$

The third inequality is easier,

$$\begin{aligned}\sup_{k,j}&\int_0^1\int_0^1 f(x_1,x_2,x_3-2^{k\alpha+j\beta}s^\alpha t^\beta)\,ds\,dt\\ &\leqslant \sup_{h>0}\int_0^1\int_0^1 f(x_1,x_2,x_3-hs^\alpha t^\beta)\,ds\,dt\\ &\leqslant \sup_{h>0}\int_0^1 f(x_1,x_2,x_3-hs^\alpha)\,ds \leqslant 2\max(1,\alpha^{-1})M_3f(x).\end{aligned}$$

b) Suppose that the second derivatives are positive. In the inner integral of

$$\frac{1}{h_1h_2}\int_0^{h_1}\left[\int_0^{h_2} f(x_1-s,x_2,x_3-\varphi(s,t))\,dt\right]ds$$

we make the change of variable $\tau = \varphi(s,t) - \varphi(s,0)$; we get

$$\frac{1}{h_1h_2}\int_0^{h_1}\left[\int_0^{\varphi(s,h_2)-\varphi(s,0)} f(x_1-s,x_2,x_3-\varphi(s,0)-\tau)(\varphi_s^{-1})'(\tau+\varphi(s,0))\,dt\right]ds$$

where $\varphi_s(t) = \varphi(s,t)$ and φ_s^{-1} is the inverse function. But $(\varphi_s^{-1})'(\tau+\varphi(s,0)) = (D_2\varphi(s,t))^{-1}$ is positive and decreases when t (and therefore τ) increases; its integral over the interval $(0,\varphi(s,h_2)-\varphi(s,0))$ is h_2. Then the double integral above is bounded by

$$\frac{1}{h_1}\int_0^{h_1} M_3f(x_1-s,x_2,x_3-\varphi(s,0))\,ds \leqslant M_{\gamma_1}M_3f(x).$$

Finally, for the third inequality we apply to

$$\frac{1}{h_1h_2}\int_0^{h_1}\int_0^{h_2} f(x_1,x_2,x_3-\varphi(s,t))\,dt\,ds$$

the same change of variable. This gives

$$\frac{1}{h_1}\int_0^{h_1} M_3 f(x_1,x_2,x_3-\varphi(s,0))\,ds\,.$$

If $\varphi(s,0)=0$ the proof is finished; if not, a new change of variables $\tau=\varphi(s,0)$ yields the desired bound. □

If φ verifies $D_1^2\varphi(0,0)$, $D_2^2\varphi(0,0)\neq 0$ as in type 2, but without conditions on D_{12}, lemma 2 still holds. Then, we can use the estimates for the associated maximal function proved in [2] and deduce the boundedness of the Hilbert transform for a class of surfaces larger than those of type 2 :

COROLLARY 4. — *Let φ be an even function of class C^2 in a neighbourhood of the origin with $D_1^2\varphi(0,0)$ and $D_2^2\varphi(0,0)\neq 0$. Then,*

$$\|Hf\|_p \leqslant C_p\|f\|_p, \qquad 1<p<+\infty\,.$$

The extension of these results to hypersurfaces in $\mathbf{R}^{n+1}$ $(n>2)$ is straightforward. The three types of hypersurface corresponding to the preceding ones are :

1) $\varphi(t_1,\ldots,t_n)=\prod_{i=1}^{n}|t_i|^{\alpha_i},\quad \alpha_i>0$

2) $D_j^2\varphi(0,\ldots,0)\neq 0$, $j=1,\ldots,n$ and $D_{ij}\varphi(t_1,\ldots,t_n)|_{t_j=0}$ is zero or has the same sign as $D_j^2\varphi(0,\ldots,0)$, $i,j=1,\ldots,n$.

3) $D_j^2\varphi(t_1,\cdots,t_n)$ is nonnegative and nondecreasing in t_j' and $D_{ij}\varphi(t_1,\ldots,t_n)|_{t_j=0}$ is nonnegative and nondecreasing in t_i.

In all cases, $\varphi(0,\cdots,0)=0$, φ is even in each one of the variables and $\varphi\in C^2$ in a neighbourhood of the origin in the last two cases.

The proof of the analogue of corollary 3 is by induction. Lemma 2 and its proof are similar in this context. The inequalities in Lemma 3 involve now maximal functions along lower dimensional manifolds which are the sections of the hypersurface with the coordinate hyperplanes, i.e.,

$$(t_1,\cdots,t_{n-1},0,\varphi(t_1,\cdots,t_{n-1},0))$$

and its analogues. The boundedness of these maximal functions needed in the application of theorem 2 is a consequence of the induction hypothesis.

Finally we apply the boundedness of the Hilbert transform along surfaces of type 1 to get estimates for convolution operators homogeneous with respect to a multiparameter group. We state the result in $\mathbf{R}^3$ where the kernel must verify

$$K(sx_1,tx_2,s^\alpha t^\beta x_3) = s^{-1-\alpha}t^{-1-\beta}K(x_1,x_2,x_3).$$

In such a case, see [8] theorem 2.1, K is determined by its values on the set $\Omega = \{-1,+1\} \times S^1$. Moreover, a formula for the integration in « polar coordinates » holds

$$\int_{\mathbf{R}^3} f(x)\,dx = \int_\Omega \int_0^\infty \int_0^\infty f(su_1,tu_2,s^\alpha t^\beta u_3)s^{1+\alpha}t^{1+\beta}M(u)\,\frac{ds\,dt}{st}\,du$$

where $M(u)$ is a function on Ω bounded between two positive constants. If K is odd in the first two variables and $H_u f$ stands for the Hilbert transform along the surface $(su_1,tu_2,|s|^\alpha|t|^\beta u_3)$

$$\text{p.v. } K*f(x) = \lim_{\varepsilon\to 0}\int_\Omega K(u)M(u)\left(\int_\varepsilon^\infty \int_\varepsilon^\infty f(x_1-su_1,x_2-tu_2,\; x_3 - s^\alpha t^\beta u_3)\,\frac{ds\,dt}{st}\right)du$$

$$= \frac{1}{4}\int_\Omega K(u)M(u)H_u f(x)\,du$$

thus, if $Tf = \text{p.v. } K*f$,

$$\|Tf\|_p \leqslant \frac{1}{4}\int_\Omega |K(u)|\,|M(u)|\,\|H_u f\|_p\,du.$$

Now, it is easy to see that the L^p-bound of H_u is independent of u. It is enough to change f into $\tilde{f}(x) = f\left(\frac{x_1}{u_1},\frac{x_2}{u_2},\frac{x_3}{u_3}\right)$ above. Then, we have proved the following:

Corollary 5. — *Let* K *be a measurable function on* $\mathbf{R}^3$ *such that*

i) $K(sx_1,tx_2,s^\alpha t^\beta x_3) = s^{-1-\alpha}t^{-1-\beta}K(x)$

ii) K *is odd in* x_1, x_2.

iii) $\int_\Omega |K(u)|\,du < +\infty$. *Then,* $\|Tf\|_p \leqslant C_p\|f\|_p$, $1 < p < \infty$, *where* $Tf = \text{p.v. } K*f$.

An example of a kernel K verifying the preceding conditions is:

$$K(x_1,x_2,x_3) = \operatorname{sgn}(x_1x_2)\,\frac{|x_1|^{\alpha-1}|x_2|^{\beta-1}}{|x_1|^{2\alpha}|x_2|^{2\beta} + |x_3|^2}.$$

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