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ON HOLOMORPHICALLY SEPARABLE COMPLEX SOLV-MANIFOLDS

by

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0. Introduction.

Let G be a connected complex Lie group, H a closed complex subgroup, and consider the complex homogeneous space $X = G/H$. We are interested in understanding when X is a Stein manifold. For example, if G is semi-simple and H is unipotent, then X is quasi-affine but not Stein. Thus, even when $\mathcal{O}(X)$ separates points, X may not be Stein. On the other hand this special case is well-understood : *If* G *is reductive, e.g. semi-simple, then* X *is Stein if and only if* H *is reductive.* This theorem of Matsushima [8] has an important analytic interpretation. Let K be a maximal compact subgroup of G and let K/L be a minimal K-orbit in X. Then X is Stein if and only if K/L is totally real in X and $X = (K/L)^{\mathbf{C}}$.

The idea of considering K-orbits is also decisive in the case where X is a Lie group, i.e. $H = (e)$: X *is Stein if and only if* K *is totally real* [9]. In the general setting it is certainly a reasonable strategy to attempt to find the compact real submanifolds M in X and relate the induced structure on M to that of X. For example, this works quite nicely in the nilpotent case.

If G is nilpotent, then it is easy to reduce function-theoretic considerations to the case of discrete isotropy, i.e. $X = G/\Gamma$, where Γ is discrete. In this case there is a uniquely determined connected real subgroup $G_{\mathbf{R}}(\Gamma)$ of G with $\Gamma < G_{\mathbf{R}}(\Gamma)$ and $M := G_{\mathbf{R}}(\Gamma)/\Gamma$ compact. It follows that X is Stein if and only if M is totally real

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in X. As a result, if $X = G/H$ is the homogeneous space of a nilpotent Lie group, then X is Stein if and only if it is holomorphically separable [3].

The goal of the present paper is to study the *solvable* case. One might think that this is only a slightly more general situation than the nilpotent setting. However, there are substantial difficulties :

1) While nilpotent groups are essentially algebraic, solvable groups are not and discrete subgroups may have a transcendental nature ;

2) Given a discrete group Γ in G solvable, it is possible that $G_{\mathbf{R}}(\Gamma)$ does not exist as in the nilpotent case ;

3) Even when $G_{\mathbf{R}}(\Gamma)$ exists and $M = G_{\mathbf{R}}(\Gamma)/\Gamma$ is totally real in $X = G/\Gamma$, it is possible that X is not Stein or not even Kähler [2] [7]. Recently J. Loeb [7] gave a necessary and sufficient condition for G/Γ to be Kähler in terms of the eigenvalues of the adjoint representation of $\mathfrak{g}_{\mathrm{R}}$.

To circumvent the above problems we use the results of Loeb and the relationship between G and Γ and their algebraic closures. We would like to thank D. Snow for drawing our attention to a result of Hochschild and Mostow, which considerably shortened our original proof.

For a precise statement of the results it is convenient to introduce some notation. First, $X = G/H$ is called a *complex solv-manifold* if G is a connected complex solvable Lie group and H is a closed complex subgroup. We denote by H° the identity component of H and define $N_G(H) := \{g \in G \mid gHg^{-1} = H\}$ to be the normalizer of H in G. If $\dim_{\mathbf{C}} X = n$ and $\mathcal{O}(X)$ contains functions $f_1, \ldots, f_n$ which are generically independent, i.e. $df_1 \wedge \ldots \wedge df_n \not\equiv 0$ then we say that $\mathcal{O}(X)$ (or simply : X) has *maximal rank.*

THEOREM. – *Let* $X = G/H$ *be a complex solv-manifold with* $\mathcal{O}(X)$ *having maximal rank. Then* X *is a Stein manifold. Moreover there exists a subgroup* $\widetilde{H}$ *of* H *with the following properties :*

1) $\widetilde{H}$ *has finite index in* H *and* $\pi_1(G/\widetilde{H})$ *is nilpotent.*

2) If $G = G' \cdot N_G(H^{\circ})$, *then* $\widetilde{H}$ *contains the commutator group* H' *of* H.

1. Preliminary remarks.

Before going into the details of the proof we would like to list some standard facts about solvable Lie groups which will be frequently used in the text and to give references for them.

1) Every connected subgroup of a simply connected solvable Lie group L is simply connected and closed in L ([13], p. 243). If H is a connected complex subgroup of a connected and simply connected solvable complex Lie group G, then $G/H = \mathbf{C}^n$ ([6]).

2) The complex algebraic closure of a solvable (resp. nilpotent) Lie group $N \subset GL(m, \mathbf{C})$ is solvable (resp. nilpotent) ([13], p. 110).

3) Let G be a connected and simply connected solvable complex Lie group. By a theorem of Hochschild and Mostow ([5], Theorem 3.1) there exists a solvable linear algebraic complex Lie group

$$\overline{G} = (\mathbf{C}^*)^k \ltimes G,$$

containing G as a Zariski-dense (and topologically closed) normal subgroup. The group $\overline{G}$ is called a *regular algebraic hull of* G. Note that the commutator groups of $\overline{G}$ and G coincide: $\overline{G}' = G'$ (cf. [1]). For every closed complex subgroup H of G the complex manifolds $\overline{G}/H$ and $(\mathbf{C}^*)^k \times G/H$ are biholomorphically equivalent. In particular $\pi_1(G/H)$ is nilpotent iff $\pi_1(\overline{G}/H)$ is nilpotent.

4) Let G be a connected solvable linear algebraic group with Lie algebra $\mathfrak{g}$ and $H \subset G$ a complex subgroup with Lie algebra $\mathfrak{h}$. Let X denote the Graßmann-manifold of $\dim \mathfrak{h}$-dimensional complex linear subspaces of $\mathfrak{g}$. Via the Plückerembedding $X \longrightarrow \mathbf{P}^M$ the group $GL(\mathfrak{g})$ operates in a natural way as a group of collineations of $\mathbf{P}^M$, stabilizing X. The adjoint representation $\mathrm{Ad} : G \longrightarrow GL(\mathfrak{g})$ is algebraic (cf. [1]) and induces in the above manner an algebraic representation of G on $\mathbf{P}^M$. Since $N_G(H^\circ)$ is the stabilizer of $\mathfrak{h} \in X$, it is an algebraic subgroup of G. Note that G' is unipotent and $G'(\mathfrak{h}) = G'/N_G(H^\circ) \cap G' = \mathbf{C}^m$.

5) If G is a connected solvable linear algebraic group and H is an algebraic subgroup of G, then G/H is a Stein manifold. In fact $G/H = \mathbf{C}^n \times (\mathbf{C}^*)^m$ ([6], [11]).

2. Proof of the Theorem.

We begin by remarking that if X is homogeneous and $\mathcal{O}(X)$ has maximal rank then X is Kähler.

LEMMA 1. – *If* $X = G/H$ *and* $\mathcal{O}(X)$ *has maximal rank, then there exists a regular holomorphic map* $F : X \longrightarrow \mathbf{C}^N$ *with discrete fibers. In particular* X *is a Kähler manifold.*

Proof. – Take $x \in G/H$ and a holomorphic map $F : G/H \longrightarrow \mathbf{C}^m$ with $\mathrm{rank}_x F = \dim G/H$. Let A_i, $i \in I \subset \mathbf{N}$, be the irreducible components of the singular set A of F. Given $x_i \in A_i$ the set $B_i := \{g \in G \mid g(x_i) \in A\}$ is a nowhere dense analytic set in G, and we take $g_0 \in G \setminus \bigcup_{i \in I} B_i$. Then the singular set of

$$F \times F \circ g_0 : G/H \longrightarrow \mathbf{C}^m \times \mathbf{C}^m$$

is contained in A and of lower dimension than A. After finitely many steps we get a regular map as desired. □

Let L be an arbitrary connected solvable complex Lie group. We call $h \in L$ a *regular element of* L if the complex algebraic closure of $\{\mathrm{Ad}\, h^n \mid n \in \mathbf{Z}\}$ in $\mathrm{Gl}(\mathfrak{L})$ contains a maximal torus of the algebraic closure $(\mathrm{Ad}\, L)_a$ of $\mathrm{Ad}\, L$ in $\mathrm{GL}(\mathfrak{L})$. The importance of the role of regular elements in a solvable Lie group was pointed out by Mostow in [10]. If $h \in L$ is a regular element of L, then h lies on a 1-parameter subgroup of L. If L is simply connected, then this 1-parameter subgroup is uniquely determined by h (Mostow [10]). Note that every Zariski-dense subgroup of a connected solvable linear algebraic group L contains a regular element of L.

In the following context G always denotes a connected and simply connected solvable complex Lie group and $\overline{G}$ denotes a regular algebraic hull of G.

Recall the basic theorem of Matsushima and Morimoto [8]: *Let* $E \longrightarrow B$ *be a holomorphic fiber bundle with fiber* F *and complex Lie group* S *as structure group. Assume that* S/S° *is finite and* F *and* B *are Stein. Then* E *is Stein.* We now use this to reduce our problem to the case of discrete isotropy. If $X = G/H$ and H is discrete, we refer to X as being "*complex-parallelizable*".

PROPOSITION 2. – *Assume that the theorem is true for every complex parallelizable homogeneous solv-manifold with maximal rank. Then it is true in general.*

Proof. – We consider the holomorphic normalizer-fibration

$$\overline{G}/H = (\mathbf{C}^*)^k \times G/H \longrightarrow \overline{G}/N_{\overline{G}}(H^\circ).$$

If $\overline{G} = \overline{G}' \cdot N_{\overline{G}}(H^\circ)$, then $\overline{G}/N_{\overline{G}}(H^\circ)$ is an orbit of the unipotent group $\overline{G}'$ and biholomorphically equivalent to $\mathbf{C}^m$, cf. remark 4. The bundle $\overline{G}/H \longrightarrow \overline{G}/N_{\overline{G}}(H^\circ) = \mathbf{C}^m$ splits analytically :

$$\overline{G}/H = \mathbf{C}^m \times N_{\overline{G}}(H^\circ)/H,$$

and, since $N_{\overline{G}}(H^\circ)/H$ has maximal rank, our assumption yields that $N_{\overline{G}}(H^\circ)/H$ is Stein. Then $\overline{G}/H$ and G/H are also Stein manifolds. There is a subgroup $\widetilde{H}$ of H of finite index containing H' such that $\pi_1(N_{\overline{G}}(H^\circ)/\widetilde{H})$ is nilpotent. The relation

$$\pi_1(N_{\overline{G}}(H^\circ)/\widetilde{H}) = \pi_1(\overline{G}/\widetilde{H}) = \mathbf{Z}^k \times \pi_1(G/\widetilde{H})$$

forces $\pi_1(G/\widetilde{H})$ to be nilpotent.

Now we assume that $\overline{G} \neq \overline{G}' \cdot N_{\overline{G}}(H^\circ)$. Denote by $N_{\overline{G}}^\circ$ the identity component of $N_{\overline{G}}(H^\circ)$ and note that

$$\overline{G}/\overline{G}' \cdot N_{\overline{G}}^\circ = (\mathbf{C}^*)^r \times \mathbf{C}^s$$

is a non-trivial linear algebraic abelian group. Thus there is a *connected* closed complex subgroup I of $\overline{G}$,

$$G' \cdot N_{\overline{G}}^\circ \subset I,$$

such that $\overline{G}/I = \mathbf{C}$ or $\overline{G}/I = \mathbf{C}^*$. The group $\hat{H} := H \cap N_{\overline{G}}^\circ$ has finite index in H and the holomorphic fibration

$$\overline{G}/\hat{H} \longrightarrow \overline{G}/I$$

splits analytically, (see [4]).

Since $I/\hat{H}$ has maximal rank, we may assume by induction on $\dim \overline{G}/H$ that $I/\hat{H}$ is a Stein manifold and that there exists a subgroup $\widetilde{H} \subset \hat{H}$ of finite index with $\pi_1(I/\widetilde{H})$ nilpotent. Then $\overline{G}/\hat{H}$ and G/H are Stein and $\pi_1(G/\widetilde{H})$ is nilpotent. □

In view of Proposition 2 it remains to study the case where $\Gamma := H$ is a discrete subgroup of G and to show that G/B is Stein and that Γ contains a normal nilpotent subgroup $\widetilde{\Gamma}$ of finite index with $\Gamma/\widetilde{\Gamma}$ abelian. The following theorem is decisive for the proof of this fact.

THEOREM (Loeb [7]). – *Let* L *be a connected complex Lie group and* L_R *a real form of* L. *Let* $\Gamma \subset L_R$ *be a cocompact discrete subgroup of* L_R. *Then the following statements are equivalent:*

i) L/Γ *is a Kähler manifold*

ii) L/Γ *is a Stein manifold*

iii) For every element x *in the Lie algebra* $\mathfrak{l}_R$ *of* L_R, *the eigenvalues of the derivation* $\operatorname{ad} x : \mathfrak{l}_R \longrightarrow \mathfrak{l}_R$ *are purely imaginary.* □

We use this result for the proof of

LEMMA 3. – *Let* L *be a connected and simply connected solvable complex Lie group and* $\Gamma \subset L$ *a discrete subgroup with* $\mathcal{O}(L/\Gamma)$ *having maximal rank. If* Γ *contains a regular element of* L, *then there exists a nilpotent subgroup* $\widetilde{\Gamma}$ *of* Γ *with* $\Gamma' \subset \widetilde{\Gamma}$ *and* $\Gamma/\widetilde{\Gamma}$ *a finite (abelian) group.*

Proof. – Denote by $S_{\mathbf{R}}$ (resp. S) the minimal connected real (resp. complex) subgroup of L' containing Γ'. Take a regular element $\gamma \in \Gamma$ of L which is of course automatically a regular element of the regular algebraic hull $\bar{L}$ of L. There is exactly one 1-parameter real (resp. complex) subgroup $A_{\mathbf{R}}$ (resp. A) of L containing γ. Hence the real-algebraic closure of $\mathbf{Z}_\gamma := \{\gamma^n \mid n \in \mathbf{Z}\}$ in $\bar{L}$ contains $A_{\mathbf{R}}$ and since it normalizes the real-algebraic closure $S_{\mathbf{R}}$ of $S \cap \Gamma$, we get the connected and simply connected subgroup $M_{\mathbf{R}} := A_{\mathbf{R}} \cdot S_{\mathbf{R}}$ of L. Of course this construction only works if $\mathbf{Z}\gamma \simeq \mathbf{Z}$. But otherwise $\operatorname{Ad} \mathbf{Z}\gamma \subset GL(\mathfrak{l})$ would be finite and $\mathfrak{l}$ would already be nilpotent.

The group $M_{\mathbf{R}}$ is a real form of the complex Lie group $M := A \cdot S$, because $M/\Gamma \cap M_{\mathbf{R}} \subset L/\Gamma \cap M_{\mathbf{R}}$ has maximal rank and $M_{\mathbf{R}}/\Gamma \cap M_{\mathbf{R}}$ is compact. Now we apply the theorem of Loeb. By Lemma 1 we know that $M/\Gamma \cap M_{\mathbf{R}}$ is a Kähler manifold. Hence $\operatorname{Spec} \operatorname{Ad} g_{|\mathfrak{m}_{\mathbf{R}}} \subset S^1$ for every $g \in M_{\mathbf{R}}$. Since $\exp^{-1}(\Gamma \cap S_{\mathbf{R}}) \subset \mathfrak{s}_{\mathbf{R}}$

contains a uniform discrete lattice of $\mathfrak{s}_{\mathbf{R}}$ which is stabilized by $\mathrm{Ad}\,\gamma_{|\mathfrak{s}_{\mathbf{R}}}$, we may assume that $\mathrm{Ad}\gamma_{|\mathfrak{s}_{\mathbf{R}}} \in \mathrm{SL}(n,\mathbf{Z})$, $n = \dim_{\mathbf{R}} \mathfrak{s}_{\mathbf{R}}$. But if $g \in \mathrm{SL}(n,\mathbf{Z})$ with Spec $g \subset S^1$, then g^m is unipotent for some $m \in \mathbf{N}^+$: Since Spec $g^k \subset S^1$ for all $k \in \mathbf{N}$, we have only finitely many possibilities for the characteristic polynomials $\chi(g^k)$ in $\mathbf{Z}[X]$. Hence $Y := \bigcup_{k \in \mathbf{N}} \mathrm{Spec}\, g^k$ is a finite set and there exists $m \in \mathbf{N}^+$ with $\alpha^m = 1$ for all $\alpha \in Y$. Applying this to our situation we see that $\mathrm{Ad}\,\gamma^m{}_{|\mathfrak{m}_{\mathbf{R}}}$ is unipotent for some $m \in \mathbf{N}$. By our definition of regularity the element γ^m is also a regular element of L; therefore we may take $m = 1$.

Since $\mathrm{Ad}\,\mathbf{Z}\gamma \subset \mathrm{Ad}\,\overline{\mathrm{L}}$ operates as a group of unipotent transformations on $\mathfrak{m}$, so does $(\mathrm{Ad}\,\mathbf{Z}\gamma)_a$, the algebraic closure of $\mathrm{Ad}\,\mathbf{Z}\gamma$ in $\mathrm{Ad}\,\overline{\mathrm{L}}$. But $(\mathrm{Ad}\,\mathbf{Z}\gamma)_a$ contains a maximal torus T of $\mathrm{Ad}\,\overline{\mathrm{L}}$ and it follows that $\mathrm{Ad}\,\overline{\mathrm{L}}_{|\mathfrak{m}}$ is unipotent. Define $\mathrm{L}_a(\Gamma)$ to be the identity component of the algebraic closure of Γ in $\overline{\mathrm{L}}$ and $\widetilde{\Gamma} := \Gamma \cap \mathrm{L}_a(\Gamma)$. The fact $\mathrm{L}_a(\Gamma)' = \mathrm{L}_a(\Gamma') \subset \mathrm{S} \subset \mathrm{M}$ implies that $\mathrm{Ad}\,\mathrm{L}_a(\Gamma)$ is unipotent. As a consequence $\mathrm{L}_a(\Gamma)$ is nilpotent. Note that $\Gamma' \subset \widetilde{\Gamma}$ and $\Gamma/\widetilde{\Gamma}$ is finite. □

As an immediate consequence of this lemma we get the following result:

LEMMA 4. – *Let* L *be a connected solvable linear algebraic group and* $\Gamma \subset \mathrm{L}$ *a Zariski-dense discrete subgroup with* L/Γ *having maximal rank. Then* L *is nilpotent and* L/Γ *is a Stein manifold.*

Proof. – We consider the universal covering $\pi : \mathrm{L}_0 \longrightarrow \mathrm{L}$ of L and denote by $\overline{\mathrm{L}}_0$ a regular algebraic hull of the simply connected solvable complex Lie group L_0. With $\Gamma_0 := \pi^{-1}(\Gamma)$ we have $\mathrm{L}/\Gamma = \mathrm{L}_0/\Gamma_0$. Since Γ is Zariski-dense in L, it contains a regular element γ of L and every element of $\pi^{-1}(\gamma) \subset \Gamma_0$ is regular with respect to L_0. By lemma 3 we get a nilpotent subgroup $\widetilde{\Gamma}_0 \subset \Gamma_0$ of finite index and $\widetilde{\Gamma} := \pi(\widetilde{\Gamma}_0)$ is a nilpotent subgroup of finite index in Γ. Obviously $\widetilde{\Gamma}$ is Zariski-dense in L and as a consequence L is nilpotent. The theorem of Gilligan and Huckleberry [3] tells us that L/Γ is Stein. □

Finally we complete the proof of the theorem:

By proposition 2 we may assume that $\Gamma := \mathrm{H}$ is discrete. Moreover it is sufficient to prove the theorem for the solvmanifold

$\overline{G}/\Gamma$. Denote by L_a the identity component of the algebraic closure of Γ in $\overline{G}$ and consider the holomorphic fibration

$$\overline{G}/\Gamma \longrightarrow \overline{G}/L_a\Gamma$$

with connected fiber $L_a/\Gamma \cap L_a$. Since $\overline{G}/L_a\Gamma$ is Stein (cf. remark 5) and since $L_a/\Gamma \cap L_a$ is Stein (Lemma 4), it follows from the theorem of Matsushima and Morimoto that $\overline{G}/\Gamma$ is Stein. Since $\widetilde{\Gamma} := L_a \cap \Gamma$ is nilpotent, of finite index in Γ and contains Γ', the proof ot the theorem is complete. □

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