

ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 36, n° 3 (1986), p. 167-181

http://www.numdam.org/item?id=AIF_1986__36_3_167_0

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INTERPOLATING SEQUENCES OF COMPLEX HYPERPLANES IN THE UNIT BALL OF $\mathbb{C}^n$

by

PASCAL J. THOMAS

This paper gives a sufficient condition for the existence of a solution to the following problem :

Given a sequence of complex hyperplanes, $\{L_j\}_{j \in \mathbb{Z}_+}$, all intersecting $\mathbb{B}^n$ (the unit ball of $\mathbb{C}^n$), and given a sequence of holomorphic functions $\{f_j\}_{j \in \mathbb{Z}_+} \subseteq H^\infty(\mathbb{B}^{n-1})$ is there a function $f \in H^\infty(\mathbb{B}^n)$ such that $f|_{L_j} \equiv f_j \circ \phi_j^{-1}$, $j \in \mathbb{Z}_+$, where ϕ_j is a complex-linear map from $\mathbb{B}^{n-1}$ onto $L_j \cap \mathbb{B}^n$? If there is such an f , we shall say that $\{L_j\}_{j \in \mathbb{Z}_+}$ is *interpolating*.

Notations. — If $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $w = (w_1, \dots, w_n) \in \mathbb{C}^n$, then $z \cdot \bar{w} = \sum_{j=1}^n z_j \bar{w}_j$ and $|z| = (z \cdot \bar{z})^{1/2}$ (modulus of z),

$$z^* = \frac{z}{|z|} \in \partial \mathbb{B}^n = \{z : |z| = 1\}.$$

For all $j \in \mathbb{Z}_+$, a_j = point of smallest modulus in L_j (a_j is the center of the ball $L_j \cap \mathbb{B}^n$). Equivalently,

$$L_j = \{z \in \mathbb{C}^n : (z - a_j) \cdot \bar{a}_j = 0\} \quad (a_j \neq 0).$$

For all $j \in \mathbb{Z}_+$,

$$U_j = \left\{ z \in \mathbb{B}^n : \left| \frac{\bar{a}_j \cdot (a_j - z)}{|a_j| (1 - z \cdot \bar{a}_j)} \right| < \delta_0 \right\}.$$

Key-words: Interpolating sequences — Bounded holomorphic functions — Carleson measures — Extension of functions.

THEOREM 1. — Given a sequence $\{L_j\}$ as above, it is interpolating if the following sufficient conditions are met:

$$(B) \sum_{j \in \mathbf{Z}_+} \frac{(1 - |a_j|^2)(1 - |a_k|^2)}{|1 - a_j \cdot \bar{a}_k|^2} \leq M < \infty$$

and

$$(U) \text{ for all } j, k \in \mathbf{Z}_+, j \neq k, \text{ then } U_j \cap U_k = \emptyset.$$

Remarks. — 1) By applying an element of the unitary group, we can send any a_j to a point of the form $(a, 0)$, $a \in \mathbf{B}^1$. Then

$$U_j = \left\{ (z_1, z_2) : \left| \frac{z_1 - a}{1 - \bar{a} z_1} \right| < \delta_0 \right\}.$$

Since the definition of U_j is rotation-invariant, we see that for all j , U_j is a tube surrounding the hyperplane L_j , of radius commensurate to $1 - |a_j|$.

In particular, for $\epsilon > 0$ small enough, U_j contains any set of the form $\{z \in \mathbf{B}^n : \exists w \in L_j : d_H(z, w) < \epsilon\}$, where

$$d_H(z, w) = \left(1 - \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - z \cdot \bar{w}|^2} \right)^{1/2}$$

is the "hyperbolic" distance, invariant under automorphism of $\mathbf{B}^n$. The regions U_j are not automorphism-invariant, but condition (U) implies in particular that the lines are separated in the metric d_H , so that if $j \neq k$, we can find $f \in H^\infty(\mathbf{B}^n)$ such that $f|_{L_j} \equiv 1$ and $f|_{L_k} \equiv 0$ (explicit computation omitted).

2) Trivially, if $\{L_j\}_{j \in \mathbf{Z}_+}$ is interpolating, then the sequence $\{a_j\}_{j \in \mathbf{Z}_+}$ associated to it is.

In [3], Berndtsson gives a sufficient condition for a sequence $\{a_j\}_{j \in \mathbf{Z}_+}$ to be interpolating:

$$\prod_{j: j \neq k} |\phi_{a_j}(a_k)| \geq \epsilon > 0,$$

where $\phi_a(z)$ is the automorphism of $\mathbf{B}^n$ defined in ([7], 2.2.1, p. 25):

$$\phi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - z \cdot \bar{a}},$$

$P_a(z) = (z \cdot \bar{a}/|a|^2)a$ is the projection of z onto the complex line through a and 0 , $Q_a(z) = z - P_a(z)$ is the projection of z onto the complex hyperplane through 0 orthogonal to a , and $s_a = (1 - |a|^2)^{1/2}$.

$|\phi_{a_j}(a_k)|^2 = d_H(a_j, a_k)^2$, so that the convergence of the above product is equivalent to (B) together with the requirement that the points a_j are separated, i.e. $d_H(a_j, a_k) \geq \delta > 0$ for $j \neq k$. (U) implies, of course, that a_j are separated. We are now ready for the following

DEFINITION. — Given a function $f_k: L_k \rightarrow \mathbf{C}$, define an extension $\tilde{f}_k: \mathbf{B}^n \rightarrow \mathbf{C}$ by

$$\tilde{f}_k = f_k \circ \phi_{a_k} \circ Q_{a_k} \circ \phi_{a_k}.$$

This definition makes sense, since

$$\begin{aligned} \phi_{a_k}(L_k) &= \phi_{a_k}^{-1}(L_k) = \{z: \phi_{a_k}(z) \cdot \bar{a}_k = |a_k|^2\} \\ &= \left\{ z: 1 - \frac{1 - |a_k|^2}{1 - z \cdot \bar{a}_k} = |a_k|^2 \right\} \\ &= \{z: z \cdot \bar{a}_k = 0\} = \text{Range}(Q_{a_k}), \end{aligned}$$

and consequently $\phi_{a_k}(\text{R}(Q_{a_k})) = L_k$, so $\tilde{f}_k$ is indeed defined on $\mathbf{B}^n$. Furthermore,

$$\begin{aligned} \tilde{f}_k|_{L_k} &= f_k \circ \phi_{a_k} \circ Q_{a_k}|_{\text{R}(Q_{a_k})} \circ \phi_{a_k}|_{L_k} \\ &= f_k \circ \phi_{a_k} \circ \phi_{a_k}|_{L_k}, \text{ since } Q \text{ is a projection,} \\ &= f_k, \text{ since } \phi = \phi^{-1}. \end{aligned}$$

In other words, $\tilde{f}_k \circ \phi_{a_k} = (f_k \circ \phi_{a_k}) \circ Q_{a_k}$, i.e. first we pull back the situation to the case where f_k is defined on a complex hyperplane through 0 , and extend it trivially to be independent of the last coordinate.

Clearly, $\|\tilde{f}_k\|_{H^\infty(\mathbf{B}^n)} = \|f_k\|_{H^\infty(L_k)}$; (f_k is what was denoted in the introduction $f_k \circ \phi_k^{-1}$).

3) Suppose that for all $j \in \mathbf{Z}_+$, $a_j = (\alpha_j, 0)$, $\alpha_j \in \mathbf{B}^1$. Then all the L_j are parallel, $L_j = \{z_1 = \alpha_j\}$, and $\{L_j\}$ is an interpolating sequence if and only if $\{\alpha_j\}_{j \in \mathbf{Z}_+}$ is an interpolating sequence in $\mathbf{B}^1$.

Conditions (U) reduces to

$$\left| \frac{\alpha_j - \alpha_k}{1 - \alpha_j \bar{\alpha}_k} \right| \leq c < 1 \quad \text{for } j \neq k,$$

and condition (B) reduces to:

$$\sum_{j:j \neq k} \frac{(1 - |\alpha_j|^2)(1 - |\alpha_k|^2)}{|1 - \alpha_j \bar{\alpha}_k|^2} \leq c.$$

In the case $n = 1$, it is well known (see Carleson [4] or Garnett [5]) that if the points are separated (i.e. (U)), then (B) $\Leftrightarrow \{\alpha_j\}$ is interpolating, so from that point of view the result is sharp.

4) Of course the points a_j cannot cluster at any interior point of B^n . We will, without loss of generality, remove a finite number of hyperplanes from our sequence and henceforth assume $|a_j| \geq 1/2$, $j \in \mathbf{Z}_+$, for technical reasons.

The main step in the proof of the theorem is the following:

PROPOSITION 1. — *Under the assumptions (U) and (B), there exist two positive constants C_1 and C_2 , and analytic functions $\{F_k\}_{k \in \mathbf{Z}_+}$ such that*

$$(i) \quad \forall z \in B, \sum_k |F_k(z)| \leq c_1$$

$$(ii) \quad \forall k \in \mathbf{Z}, |F_k|_{L_k} \geq c_2$$

$$(iii) \quad \forall j \neq k, |F_k|_{L_j} \leq \frac{c_2}{2}$$

(the F_k are "pseudo P. Beurling functions").

Proof of the Theorem (assuming Proposition 1). — We will show that one can construct from the F_k true P. Beurling functions, i.e. $E_k(z)$ verifying:

$$(i) \quad \forall z \in B, \sum_k |E_k(z)| \leq c < \infty$$

$$(ii)' \quad E_k|_{L_k} \equiv 1$$

$$(iii)' \quad E_k|_{L_j} \equiv 0, \quad j \neq k.$$

Then our interpolating function will be $f = \sum_k \tilde{f}_k(z) E_k(z)$.
 $f|_{L_k} = \tilde{f}_k|_{L_k} = f_k$, and $\|f\|_\infty \leq c(\sup_k \|\tilde{f}_k\|_\infty) = c \sup_k \|f_k\|_\infty < \infty$.

To construct the E_k :

First let $G_k = \frac{F_k}{(F_k|_{L_k})^\sim}$ where $\sim$ is the extension discussed above.

Then $\sum_k |G_k(z)| \leq c_1/c_2$, $G_k|_{L_k} \equiv 1$, $|G_k|_{L_j}| \leq \frac{1}{2}$, $j \neq k$.

Let $H_k = G_k \prod_{j:j \neq k} (1 - G_j)$.

Since every factor is bounded below by $1/2$,

$$\left| \prod_{j:j \neq k} (1 - G_j) \right| \geq e^{-2c_1/c_2} \text{ on } L_k \text{ and } |H_k|_{L_k}| \geq e^{-2c_1/c_2},$$

while $H_k|_{L_j} \equiv 0$, $j \neq k$.

$$\forall z \in B, \sum_k |H_k(z)| \leq e^{c_1/c_2} \sum_k |G_k(z)| \leq \frac{c_1}{c_2} e^{c_1/c_2}.$$

Finally, let $E_k = H_k/(H_k|_{L_k})^\sim$;

$E_k|_{L_j} \equiv 0$, $j \neq k$, $E_k|_{L_k} \equiv 1$, and $\sum_k |E_k(z)| \leq \frac{c_1}{c_2} e^{3c_1/c_2}$, q.e.d.

Proof of Proposition 1. — Let

$$F_k(z) = (1 - |a_k|^2 / 1 - z \cdot \bar{a}_k)^p \prod_{\substack{j:j \neq k \\ |1 - a_k \cdot \bar{a}_j| \leq C_0(1 - |a_k|^2)}} \phi_{a_j}(z) \cdot \bar{a}_j$$

where $p \geq 4$ and $C_0 = C_0(\delta_0) > 1$ will be specified, and following [3],

$$W(a_k, z) = \exp - \sum_j \left[\left(\frac{1 + z \cdot \bar{a}_j}{1 - z \cdot \bar{a}_j} - \frac{1 + a_k \cdot \bar{a}_j}{1 - a_k \cdot \bar{a}_j} \right) \frac{(1 - |a_j|^2)(1 - |a_k|^2)}{1 - |a_j \cdot \bar{a}_k|^2} \right].$$

Convergence of the infinite product will be proved below. Note that $|\phi_{a_j}(z) \cdot \bar{a}_j| \leq |\phi_{a_j}(z)| |a_j| \leq 1$, so

$$|F_k(z)| \leq 2^{p-4} (1 - |a_k|^2 / |1 - z \cdot \bar{a}_k|)^4 |W(a_k, z)|.$$

The main step in the proof of [3] is that

$$\forall z \in B, \sum_k (1 - |a_k|^2 / |1 - z \cdot \bar{a}_k|)^4 |W(a_k, z)| \leq M_1$$

so $\sum_k |F_k(z)| \leq 2^{p-4} M_1 = c_1$, which proves (i).

Proof of (iii). — Case 1: j is such that

$$|1 - a_j \cdot \bar{a}_k| \leq C_0 (1 - |a_k|^2).$$

Then $\phi_{a_j}(z) \cdot \bar{a}_j = (a_j - z) \cdot \bar{a}_j / 1 - z \cdot \bar{a}_j = 0$ for $z \in L_j$ is a factor in the infinite product, so $|F_k(z)| = 0 \leq c_2/2$.

Case 2: j is such that $|1 - a_j \cdot \bar{a}_k| \geq C_0 (1 - |a_k|^2)$.

LEMMA 1. — If $\{L_k\}_{k \in \mathbf{Z}_+}$ satisfy (U), and $z \in L_j$, $j \neq k$, then $C_3 |1 - z \cdot \bar{a}_k| \geq |1 - a_j \cdot \bar{a}_k|$, where C_3 is a constant depending only on δ_0 .

Thus for all $z \in L_j$,

$$\frac{1 - |a_k|^2}{|1 - z \cdot \bar{a}_k|} \leq \frac{C_3 (1 - |a_k|^2)}{|1 - a_j \cdot \bar{a}_k|} \leq \frac{C_3}{C_0} = \frac{1}{2}$$

if we pick $C_0 = 2C_3$.

So for $z \in L_j$, $|F_k(z)| \leq (1/2)^p |W(a_k, z)|$. But

$$\begin{aligned} |W(a_k, z)| &= \left(\exp - \sum_j \frac{1 - |z \cdot \bar{a}_j|^2}{|1 - z \cdot \bar{a}_j|^2} \frac{(1 - |a_j|^2)(1 - |a_k|^2)}{|1 - a_j \cdot \bar{a}_k|^2} \right) \\ &\times \left(\exp \sum_j \frac{(1 - |a_j|^2)(1 - |a_k|^2)}{|1 - a_j \cdot \bar{a}_k|^2} \right) \leq e^M \text{ (see [3])}. \end{aligned}$$

So it will be enough to take

$$p \geq \log_2 \left(\frac{2e^M}{C_2} \right) \quad \text{to get (iii).}$$

Proof of (ii). — First note that

$$F_k|_{L_k} \equiv W(a_k, z) \prod_{\substack{j: j \neq k \\ |1 - a_j \cdot \bar{a}_k| \leq C_0(1 - |a_k|^2)}} \phi_{a_j}(z) \cdot \bar{a}_j$$

$z \in L_k \subset U_k$, hence $z \notin U_j$, so

$$|\phi_{a_j}(z) \cdot \bar{a}_j| = \left| \frac{(a_j - z) \cdot \bar{a}_j}{1 - z \cdot \bar{a}_j} \right| \geq \delta_0 |a_j| \geq \frac{\delta_0}{2};$$

each term in the infinite product is bounded below, so we only have to consider

$$\begin{aligned} \sum_{\substack{j: |1 - a_j \cdot \bar{a}_k| \leq C_0(1 - |a_k|^2) \\ j \neq k}} |1 - \phi_{a_j}(z) \cdot \bar{a}_j| \\ = \sum_{\substack{j: |1 - a_j \cdot \bar{a}_k| \leq C_0(1 - |a_k|^2) \\ j \neq k}} \frac{1 - |a_j|^2}{|1 - z \cdot \bar{a}_j|}. \end{aligned}$$

By Lemma 1, exchanging k and j ,

$$C_3 |1 - z \cdot \bar{a}_j| \geq |1 - a_k \cdot \bar{a}_j|.$$

Thus our sum is

$$\begin{aligned} &\leq C_3 \sum_{j: |1 - a_j \cdot \bar{a}_k| \leq C_0(1 - |a_k|^2)} \frac{1 - |a_j|^2}{|1 - a_k \cdot \bar{a}_j|} \\ &\leq C_3 \sum_j \frac{C_0(1 - |a_k|^2)}{|1 - a_j \cdot \bar{a}_k|} \frac{(1 - |a_j|^2)}{|1 - a_k \cdot \bar{a}_j|} \\ &\leq C_3 C_0 M, \end{aligned}$$

so the infinite product in F_k converges and is bounded below by $e^{-(2/\delta_0)C_0C_3M}$.

On the other hand,

$$\begin{aligned} |W(a_k, z)| &\geq \exp - \sum_j \frac{1 - |z \cdot \bar{a}_j|^2}{|1 - z \cdot \bar{a}_j|^2} \frac{(1 - |a_j|^2)(1 - |a_k|^2)}{1 - |a_j \cdot \bar{a}_k|^2} \\ &\geq \exp - \sum_j \frac{1 - |z \cdot \bar{a}_j|^2}{|1 - z \cdot \bar{a}_j|^2} \frac{C_3}{|1 - a_k \cdot \bar{a}_j|} \frac{(1 - |a_j|^2)(1 - |a_k|^2)}{1 - |a_j \cdot \bar{a}_k|^2} \end{aligned}$$

by lemma 1.

LEMMA 2. — Given any two points $a_j, a_k \in \mathbf{B}^n$, $z \in L_k$, then

$$\frac{1 - |z \cdot \bar{a}_j|^2}{|1 - z \cdot \bar{a}_j|} \leq 18 \frac{1 - |a_k \cdot \bar{a}_j|^2}{|1 - a_k \cdot \bar{a}_j|}.$$

Thus

$$|W(a_k, z)| \geq \exp - \sum_j 18 C_3 \frac{(1 - |a_j|^2)(1 - |a_k|^2)}{|1 - a_j \cdot \bar{a}_k|^2} = e^{-18 C_3 M}.$$

So we may take $c_2 = e^{-2M(c_0/\delta_0 + 9)C_3}$, which concludes the proof of (ii).

Proof of the Lemmas

Proof of Lemma 1. — Choose coordinates so that $a_j = (a, 0)$. Let $a_k = (b_1, b')$, $b' \in \mathbf{C}^{n-1}$. $a_k \notin U_j$ means

$$|b_1 - a| \geq \delta_0 |1 - b_1 \bar{a}|,$$

so it will be enough to show

$$C |1 - a \bar{b}_1 - z' \cdot \bar{b}'| \geq |b_1 - a|,$$

for $z = (a, z') \in L_j \cap B$, i.e.

$$|z'|^2 \leq 1 - |a|^2.$$

$$\begin{aligned} |1 - a \bar{b}_1 - z' \cdot \bar{b}'| &\geq |1 - a \bar{b}_1| - \sqrt{1 - |a|^2} \sqrt{1 - |b_1|^2} \\ &= \frac{|b_1 - a|^2}{|1 - a \bar{b}_1| + \sqrt{1 - |a|^2} \sqrt{1 - |b_1|^2}}. \end{aligned}$$

However,

$$1 - |a|^2 \leq 2(1 - |a|) \leq 2|1 - b_1 \bar{a}| \leq \frac{2}{\delta_0} |b_1 - a|$$

and

$$\begin{aligned} 1 - |b_1|^2 &\leq 2(1 - |b_1|) \leq 2(1 - |a| + |b_1 - a|) \\ &\leq 2 \left(1 + \frac{1}{\delta_0}\right) |b_1 - a|. \end{aligned}$$

So the last expression is

$$\geq \frac{|b_1 - a|^2}{\left(\frac{1}{\delta_0} + \sqrt{\frac{2}{\delta_0} \cdot 2 \left(1 + \frac{1}{\delta_0}\right)}\right) |b_1 - a|}$$

and $C_3 = (\delta_0^2 / (1 + 2\sqrt{1 + \delta_0}))^{-1}$ will do.

Proof of Lemma 2. — Note first that

$$\frac{1 - |z \cdot \bar{a}_j|^2}{|1 - z \cdot \bar{a}_j|} \leq (1 + |z \cdot \bar{a}_j|) \frac{1 - |z \cdot \bar{a}_j|}{|1 - z \cdot \bar{a}_j|} \leq 2.$$

So that if $1 - |a_k \cdot \bar{a}_j|^2 / |1 - a_k \cdot \bar{a}_j| \geq 1/9$, we have

$$\frac{1 - |z \cdot \bar{a}_j|^2}{|1 - z \cdot \bar{a}_j|} \leq 2(9) \frac{1 - |a_k \cdot \bar{a}_j|^2}{|1 - a_k \cdot \bar{a}_j|}, \text{ q.e.d.}$$

If on the contrary

$$(1 - |a_k \cdot \bar{a}_j|^2) \leq \frac{1}{9} |1 - a_k \cdot \bar{a}_j|,$$

then

$$(1 - |a_k|^2) \leq \frac{1}{9} |1 - a_k \cdot \bar{a}_j|.$$

So

$$\begin{aligned} |1 - z \cdot \bar{a}_j|^{1/2} &\geq |1 - a_k \cdot \bar{a}_j|^{1/2} - |1 - z \cdot \bar{a}_k|^{1/2} \\ &= |1 - a_k \cdot \bar{a}_j|^{1/2} - (1 - |a_k|^2)^{1/2} \geq \left(1 - \frac{1}{3}\right) |1 - a_k \cdot \bar{a}_j|^{1/2}; \end{aligned}$$

and ([3], lemma 5)

$$\begin{aligned} 1 - |z \cdot \bar{a}_j|^2 &\leq 2(1 - |z \cdot \bar{a}_j|) \leq 4(1 - |z \cdot \bar{a}_k| + 1 - |a_k \cdot \bar{a}_j|) \\ &\leq 4(1 - |a_k|^2 + 1 - |a_k \cdot \bar{a}_j|^2). \end{aligned}$$

Hence

$$\begin{aligned} \frac{1 - |z \cdot \bar{a}_j|^2}{|1 - z \cdot \bar{a}_j|} &\leq \frac{4(1 - |a_k|^2 + 1 - |a_k \cdot \bar{a}_j|^2)}{\left(\frac{2}{3}\right)^2 |1 - a_k \cdot \bar{a}_j|} \\ &\leq \frac{(9)(4)(2)(1 - |a_k \cdot \bar{a}_j|^2)}{4|1 - a_k \cdot \bar{a}_j|}, \text{ q.e.d.} \end{aligned}$$

More Remarks. — 5) The interpolation problem is invariant under automorphisms of the ball. Condition (U) is not. An optimal (but not very practical) statement of the theorem would be: if there exists

$\psi \in \text{Aut}(B)$ such that $\{\psi(L_j)\}_{j \in \mathbb{B}_+}$ satisfies (B) and (U), then $\{L_j\}_{j \in \mathbb{Z}_+}$ is an interpolating sequence.

It is natural to ask whether the theorem can be proved if one substitutes for (U) the weaker, invariant requirement that the hyperplanes L_j be separated in the metric d_H . Unfortunately, it seems to require some new idea, since U_j is precisely the region where $|\phi_{a_j}(z) \cdot \bar{a}_j|$ is small.

6) Amar [1] has put to use (essentially) the same infinite product $P(z) = \prod_{j \in \mathbb{Z}_+} \phi_{a_j}(z) \cdot \bar{a}_j$ to prove similar results; specifically, if $f_j \in H^\infty$, $f \in \text{BMOA}$ is obtained, and if f_j verify:

$$(H^p) \quad \sum_{j \in \mathbb{Z}_+} (1 - |a_j|^2) \int_{L_j} |f_j|^p d\lambda_{2n-2} < \infty$$

where $p \geq 1$, and $d\lambda_{2n-2}$ is $2n - 2$ -dimensional Lebesgue measure on L_j , then $f \in H^p(B^n)$ is obtained.

This is done by solving a certain $\bar{\partial}$ problem, namely, if g is a C^∞ solution to the interpolation problem, let $f = g + uP$ with $\bar{\partial}u = -(1/P) \bar{\partial}g$. One then needs:

$$(US) \quad \exists \delta_0, \delta_1 > 0 \text{ such that } \forall z \in U_k(\delta_0), \quad \prod_{j: j \neq k} |\phi_{a_j}(z) \cdot \bar{a}_j| \geq \delta_1.$$

Clearly, $(US) \implies (B)$, and by Remark 5, $(US) \implies (U)$ (cf. [1], lemma 2.1). Applying (US) to $z = a_k$, one sees that it implies in fact

$$(P) \quad \forall k \in \mathbb{Z}_+, \quad \sum_{j: j \neq k} \frac{(1 - |a_k \cdot \bar{a}_j^*|^2)(1 - |a_j|^2)}{|1 - a_k \cdot \bar{a}_j|^2} \leq c.$$

With the help of lemmas 1 and 2, one can show that (U) and (P) $\Leftrightarrow$ (US).

Under those assumptions, one can use Berndtsson's L^∞ solution to the $\bar{\partial}$ equation [2] to obtain an interpolating $f \in H^\infty$, but one has to require a further condition involving "C1 measures" (see [2]), which is also more restrictive than (B), and not equivalent to (P). It gives rise to unwieldy computation, even for $n = 2$.

But we are now in a position to strengthen Amar's results; Theorem 1 implies that under (US), bounded data can be interpolated by a bounded function, and we have:

THEOREM 2. — *If $\{L_k\}_{k \in \mathbb{Z}_+}$ verifies (U) and (B), and $\{f_k\}$ verifies (H^p) , then there exists $f \in H^p(B)$ such that*

$$f|_{L_k} = f_k, \quad \forall k \in \mathbb{Z}_+ \quad (1 \leq p < \infty).$$

Note that, since $\sum_k (1 - |a_k|^2) \int_{L_k} \cdot d\lambda_{2n-2}$ is a Carleson measure in B^n , condition (H^p) must be verified if there is an interpolating function f .

Theorem 2 is a consequence of :

LEMMA 4. — *If there are P. Beurling functions for a sequence of hyperplanes $\{L_k\}$, then it is H^p -interpolating.*

This implies in particular that any H^∞ -interpolating sequence will be H^p -interpolating, since one can show it will necessarily have P. Beurling functions (follow Varopoulos' proof [9] or [5], p. 298).

Proof of lemma 4. — Let $f(z) = \sum_{k \in \mathbb{Z}_+} \hat{f}_k(z) E_k(z)$, where

E_k are the P. Beurling functions and $\hat{f}_k|_{L_k} = f_k$.

Let $S = \partial B^n$, $d\sigma = 2n - 1$ -dimensional Lebesgue measure on S

$$\begin{aligned} \int_S |f|^p d\sigma &= \int_S \left| \sum_k \hat{f}_k E_k \right|^p d\sigma \\ &\leq \int_S \left(\sum_k |\hat{f}_k|^p \right) \left(\sum_k |E_k|^q \right)^{p/q} d\sigma \\ &\leq c \sum_k \int_S |\hat{f}_k|^p d\sigma, \quad (\text{where } 1/p + 1/q = 1). \end{aligned}$$

It is enough to show that, for an appropriate choice of $\hat{f}_k$, the last series is convergent (which will retroactively prove that the integrals we wrote down were making sense).

Let $\hat{f}_k(z) = (1 - |a_k|^2 / |1 - z \cdot \bar{a}_k|)^{2n} \tilde{f}_k(z)$; $\hat{f}_k|_{L_k} = \tilde{f}_k|_{L_k}$, but $\hat{f}_k$ drops off more rapidly away from L_k .

$$\int_S |\hat{f}_k(z)|^p d\sigma(z) = \int_S \left(\frac{1 - |a_k|^2}{|1 - z \cdot \bar{a}_k|} \right)^{2pn} |f_k|^p \circ \phi \circ Q \circ \phi(z) d\sigma(z)$$

where $\phi = \phi_{a_k}$, $Q = Q_{a_k}$. Since $\phi(S) = S$, we make the change of variable $w = \phi(z)$, to get

$$\int_S |\hat{f}_k|^p d\sigma = \int_S |1 - w \cdot \bar{a}_k|^{2pn} |f_k|^p \circ \phi \circ Q(w) J_\phi(w) d\sigma(w)$$

where $J_\phi(w)$ is the real Jacobian of $\phi|_S$ at w .

The Jacobian matrix of ϕ as a map from $\mathbf{B}^n$ to $\mathbf{B}^n$ can be computed with no difficulty (e.g. in the case $a_k = (0, a)$) and the real Jacobian of ϕ as a map from $\mathbf{B}^n$ to $\mathbf{B}^n$ is

$$\begin{aligned} & (1 - |a_k|^2)^{n+1} / |1 - w \cdot \bar{a}_k|^{2(n+1)}. \\ |J_\phi(w)| &= \left(\frac{\partial |\phi(w)|}{\partial |w|} \right)^{-1} \frac{(1 - |a_k|^2)^{n+1}}{|1 - w \cdot \bar{a}_k|^{2(n+1)}} \\ &= \left(\frac{1 - |a_k|^2}{|1 - w \cdot \bar{a}_k|^2} \right)^{-1} \frac{(1 - |a_k|^2)^{n+1}}{|1 - w \cdot \bar{a}_k|^{2(n+1)}} \\ &= \frac{(1 - |a_k|^2)^n}{|1 - w \cdot \bar{a}_k|^{2n}}. \end{aligned}$$

So

$$\begin{aligned} \int_S |\hat{f}_k|^p d\sigma &= (1 - |a_k|^2)^n \int_S |1 - w \cdot \bar{a}_k|^{2n(p-1)} |f_k|^p \circ \phi \circ Q(w) d\sigma(w) \\ &\leq 2^{2n(p-1)} (1 - |a_k|^2)^n \int_S |f_k|^p \circ \phi \circ Q(w) d\sigma(w) \\ &= 2^{2n(p-1)} (1 - |a_k|^2)^n \int_{R(Q)} |f_k|^p \circ \phi(w') d\lambda_{2(n-1)}(w'), \end{aligned}$$

where $d\lambda_{2(n-1)}$ is $2n-2$ -dimensional Lebesgue measure on $R(Q)$, because $|f_k|^p \circ \phi \circ Q$ is a function depending on $n-1$ variables only. Notice that

$$\phi_{a_k} : R(Q_{a_k}) \cong \mathbf{B}^{n-1}(0, 1) \longrightarrow L_k \cong \mathbf{B}^{n-1}(0, (1 - |a_k|^2)^{1/2})$$

is given by $\phi_{a_k}(z) = a_k - s_{a_k} z (z \cdot \bar{a}_k = 0!)$ so that ϕ simply induces a dilation with ratio $(1 - |a_k|^2)^{1/2}$ and

$$\begin{aligned} & \int_{R(Q_k)} |f_k|^p \circ \phi(w') d\lambda_{2(n-1)}(w') \\ &= (1 - |a_k|^2)^{-(n-1)} \int_{L_k} |f_k|^p(w'') d\lambda_{2(n-1)}(w''), \end{aligned}$$

hence $\int_S |\hat{f}_k|^p d\sigma \leq C(n, p) (1 - |a_k|^2) \int_{L_k} |f_k|^p d\lambda_{2(n-1)}$, which by (H^p) is a term in a convergent series, q.e.d.

7) In the other direction (finding *necessary* conditions), the "trivial" result cannot be improved.

Namely, if $\{L_j\}$ is an interpolating sequences of hyperplanes, then $\{a_j\}$ is an interpolating sequence of points, so they must satisfy Varopoulos's necessary condition (cf. [10]):

$$(V) \quad \sum_{j \in \mathbf{Z}_+} \left(\frac{(1 - |a_j|^2)(1 - |a_k|^2)}{|1 - a_j \cdot \bar{a}_k|^2} \right)^n \leq C$$

where C is a constant (independent of k).

On the other hand, using the fact that $\bigcup_{j \in \mathbf{Z}_+} L_j$ must be a zero-set for an H^∞ function, and Skoda's Blaschke condition for the Nevanlinna class [8] (which cannot be quantitatively improved for H^∞ , cf. Hakim & Sibony [6], or again [3]), we find:

$$(S) \quad \sum_{j \in \mathbf{Z}_+} (1 - |a_j|^2)^n \leq C.$$

(S) is a consequence of (V) (which is the invariant version of (S)). No stronger condition of the same type can be substituted for (S) without some geometrical requirement (e.g. all L_j are parallel!), as shown by:

PROPOSITION 2. — *For all $n \geq 1$, for all $\epsilon > 0$, there is an interpolating sequence of C -hyperplanes, $\{L_j\}_{j \in \mathbf{Z}_+}$ in $\mathbf{B}^n$ such that*

$$(6) \quad \sum_{j \in \mathbf{Z}_+} (1 - |a_j|)^{n-\epsilon} = +\infty.$$

Proof. — We shall use as "centers" of the hyperplanes L_j the points a_j given by Berndtsson ([3], Theorem 4) which satisfy (6) (refer to [3] for the precise details of the construction).

Berndtsson shows that there are "pseudo P. Beurling functions", $F_j \in H^\infty(\mathbf{B}^n)$ satisfying (i) and:

$$(ii)'' \quad F_j(a_j) = 1$$

$$(iii)'' \quad |F_j(a_k)| \leq 1/2, \quad j \neq k.$$

Since in fact

$$F_j(z) = \left(\frac{1 - |a_j|^2}{1 - z \cdot \bar{a}_j} \right)^{n+1}$$

we have (ii) since $F_j|_{L_j} \equiv 1$,

LEMMA 5. — *With Berndtsson's choice of a_j , we also have:*

$$(iii) \quad |F_j(z)| \leq \frac{1}{2}, \quad z \in L_k, \quad j \neq k.$$

Proposition 2 then follows in the same way as Theorem 1 (with $c_2 = 1$).

Proof of Lemma 5. — Recall that $1 - R_m \ll r_m$ are two sequences of positive numbers, and that Berndtsson's sequence is indexed $a_j^m, m \in \mathbb{Z}_+, 1 \leq j \leq C_m$.

$$|1 - a_j^m \cdot \bar{a}_k^m| \geq 100(1 - R_m), \quad j \neq k,$$

and

$$|1 - a_j^m \cdot \bar{a}_k^n| \geq 50 \max(r_m, r_n), \quad m \neq n.$$

If $z \in L_{a_k^m}$,

$$1 - z \cdot \bar{a}_k^m = 1 - |a_k^m|^2 = 1 - R_m^2.$$

For $j \neq k$,

$$2(|1 - z \cdot \bar{a}_j^m| + |1 - z \cdot \bar{a}_k^m|) \geq |1 - a_j^m \cdot \bar{a}_k^m|$$

so

$$|1 - z \cdot \bar{a}_j^m| \geq \frac{1}{2}(100)(1 - R_m) - (1 - R_m^2) \geq 20(1 - R_m^2),$$

so that

$$|F_{a_j^m}(z)| \leq \frac{1}{20^{n+1}} \leq \frac{1}{2}.$$

For $F_{a_k^n}, n \neq m$, things are even easier:

$$\begin{aligned} |1 - z \cdot \bar{a}_k^n| &\geq \frac{1}{2} |1 - a_j^m \cdot \bar{a}_k^n| - (1 - R_m^2) \\ &\geq \frac{50}{2} \max(r_n, r_m) - (1 - R_m^2) \\ &\geq 10(1 - R_m^2), \quad \text{q.e.d.} \end{aligned}$$

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Manuscrit reçu le 21 mai 1985

révisé le 3 juillet 1985.

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