ANNALES DE L'INSTITUT FOURIER

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*Annales de l'institut Fourier*, tome 36, nº 1 (1986), p. 97-104

<http://www.numdam.org/item?id=AIF_1986__36_1_97_0>

Ann. Inst. Fourier, Grenoble
36, 1 (1986), 97 - 104

# TRANSLATION INVARIANT FORMS ON $L^p(G)$ $(1 < p < \infty)$

by Jean BOURGAIN

## 1. Introduction.

It was proved by G.H. Meisters and W.M. Schmidt in [4] that translation invariant linear forms on the complex Lebesgue space $L^2(G)$, G a compact Abelian group with a finite number of components, are necessarily continuous and consequently scalar multiples of the Haar measure. Their result was generalized by B. Johnson (see [2]) to the case of Abelian groups G such that $G/C$ is polythetic, where C is the connected component of the identity. We will not consider such group theoretical refinements here and restrict ourselves essentially to the circle group case. It should be mentioned however that for instance the Hilbert space $L^2(D)$, $D = \{1, -1\}^{\mathbf{N}}$ the Cantor discontinuum, does admit discontinuous translation invariant linear forms (see [4]). In both [2], [4] the space $L^2(G)$ is identified with the square summable Fourier transforms. The fact that the Hausdorff-Young inequalities for $p \neq 1$ are only in one direction, makes the proof of an $L^p(G)$ analogue (obtained here) more delicate. Our basic result states that any mean zero element $f$ of $L^p(G)$, $1 < p < \infty$, has a representation $f = \sum_{j=1}^{J} (f_j - \tau(a_j)f_j)$ with $f_j$ in $L^p(G)$ and $a_j$ in $G (1 \leqslant j \leqslant J)$. For fixed $a \in G$, $\tau(a)$ in the usual translation operator defined by $(\tau(a)f)(a') = f(a + a')$. Moreover, one may fix $J = \max([p], [p']) + 1$, $[\ ]$ standing for the integer part and $p' = \dfrac{p}{p-1}$. This result solves a problem raised by A. Connes

*Key-words* : Translation invariant form – Hausdorff-Young inequalities – Interpolation space

[1]. For $1 < p < 3$, the value for J can not be lowered. It seems more difficult to decide the sharpness of our theorem in case $3 \leq p$. We adopt in our presentation the general situation of a translation invariant function lattice on G, solving partially the broader problem suggested in [4].

## 2. Proof of main result.

Denote $\lambda$ the normalized Haar-measure of G and $\Gamma$ its (discrete Abelian) dual. Assume X a translation invariant Banach lattice of complex-valued functions on the group G, thus $\|f\|_X = \|f\| = \|\tau(a)f\|$ whenever $f \in X$ and $a \in G$, and X generated by $\Gamma$. We suppose that X is $p$-convex and $p'$-concave for some $1 < p \leq 2$ with constants $M^{(p)}(X) = M_{(p')}(X) = 1$ (see [3] for definitions and basic theory). In this setting, we may formulate the main theorem obtained in [5] and characterizing X as a certain interpolation space.

PROPOSITION 1 (cf [5], p. 268). – *Define* $X_1 = L^2(G)$ *and* $X_0$ *as the set of those complex valued* $f$ *in* $L^0(G)$ *such that* $|f|^{1-\theta}\,|g|^\theta \in X$ *whenever* $g \in X_1$ *and* $\|f\|_{X_0} < \infty$ where $\theta$ *is defined by* $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{1}$ *and*

$$\|f\|_{X_0} = \sup\{\|\,|f|^{1-\theta}\,|g|^\theta\,\|_X^{1/1-\theta}\ ;\ \|g\|_{X_1} \leq 1\}.$$

*Then the space* $X_0$ *equipped with* $\|\ \|_{X_0}$ *is a translation invariant Banach lattice on* G *and* X *identifies with the complex interpolation space* $[X_0, X_1]_\theta$.

Next fact which we recall is a consequence of the essential uniqueness of the Haar measure for compact Abelian groups and the connectedness assumption on G (see [4], p. 413 for the details).

PROPOSITION 2. – *Let* $\chi \neq 0$ *in* $\Gamma$ *and* J *a fixed positive integer. Consider the homomorphism from* $G^J = G + \overset{J}{\ldots} + G$ *on* $\pi^J$ ($\pi$ *= circle group) defined by*

$$h(a_1, \ldots a_J) = (\chi(a_1), \ldots, \chi(a_J)).$$

*If then* $f$ *is any integrable complex valued function on* $\pi^J$, *we have the equality*

$$\int_{\pi^J} f(\theta)\, d\theta = \int_{G^J} (fh)(\bar{a})\, d\bar{a} \quad \bar{a} = (a_1, \dots, a_J).$$

Consider first a polynomial $f$ on G, i.e. $f$ has a finite support, such that $\hat{f}(0) = \int_G f(a)\lambda(da) = 0$. For $i = 1, \dots, J$ and $\bar{a} = (a_1 \dots a_J)$ in $G^J$, we define

$$f_j[\bar{a}] = \sum_{\chi \neq 0} \hat{f}(\chi) \frac{1 - \overline{\chi(a_j)}}{|1 - \chi(a_1)|^2 + \dots + |1 - \chi(a_J)|^2} \chi \tag{1}$$

which, as a consequence of Prop. 2, makes sense for almost all $\bar{a}$ in $G^J$. Moreover, by construction, we have the identity

$$f = \sum_{j=1}^{J} \{f_j[\bar{a}] - \tau(a_j) f_j[\bar{a}]\}. \tag{2}$$

Proposition 3. – *With the notations introduced above and* $J > p'$, *we have the inequality*

$$\int_{G^J} \sum_{j=1}^{J} \|f_j[\bar{a}]\|_X \, d\bar{a} \leqslant c\, \|f\|_X \tag{3}$$

*where* $c$ *is a constant not depending on* $f$.

Prop. 3 implies in particular the existence of some $\bar{a} \in G^J$ for which, besides (2), $\sum_{j=1}^{J} \|f_j[\bar{a}]\|_X \leqslant c\, \|f\|_X$. Since X is reflexive, a standard approximation and compactness argument enables us then to write each $f \in X$ in the form $f = \sum_{j=1}^{J} \{f_j - \tau(a_j) f_j\}$ for some $f_j \in X$, $a_j \in G\,(1 \leqslant j \leqslant J)$. A more careful analysis of the preceding shows that $\bar{a} = (a_1, \dots, a_J)$ may be chosen in a subset (depending on $f$) of $G^J$ with measure 1.

Clearly a translation invariant linear form on X has each element $f - \tau(a)f$ in its null space and therefore coincides with a multiple of the (continuous) linear functional $f \longrightarrow \hat{f}(0)$. The space

$L^q(G)$ for $1 < q < \infty$ is $q$-convex and $q$-concave, hence satisfies the $p$-convexity, $p'$-concavity hypothesis with $p = \min(q, q')$. As a corollary, the statement appearing in the introduction follows.

COROLLARY 4. – *Any* $f$ *in* $L^q(G)$ $(1 < q < \infty)$ *can be written in the form*

$$f = \sum_{j=1}^{J} \{f_j - \tau(a_j) f_j\}$$

*where* $f_j \in L^q(G)$, $\|f_j\|_q \leqslant c_q \|f\|_q$, $a_j \in G (1 \leqslant j \leqslant J)$ *and* $J = \max([q], [q']) + 1$.

To show (3), it is of course enough to verify that

$$\int_{G^J} \|f_1[\bar{a}]\|_X \, d\bar{a} = \int_{G^J} \|f * \Omega_{\bar{a}}\|_X \, d\bar{a} \leqslant c \|f\|_X \tag{4}$$

where for $\bar{a} \in G^J$, $\Omega_{\bar{a}}$ is the distribution defined by

$$\hat{\Omega}_{\bar{a}}(\chi) = \frac{1 - \overline{\chi(a_1)}}{|1 - \chi(a_1)|^2 + \cdots + |1 - \chi(a_J)|^2}.$$

Thus

$$\hat{\Omega}_{\bar{a}}(\chi) = \frac{1 - \operatorname{Re}\chi(a_1) + i \operatorname{Im}\chi(a_1)}{2[J - \operatorname{Re}\chi(a_1) - \cdots - \operatorname{Re}\chi(a_J)]} = \frac{1}{2J} \sum_{s=0}^{\infty} [\hat{\alpha}_{\bar{a}}^s(\chi) + i \hat{\beta}_{\bar{a}}^s(\chi)]$$

where

$$\alpha_{\bar{a}}^s = \Sigma_\chi (1 - \operatorname{Re}\chi(a_1)) \left[\frac{\Sigma \operatorname{Re}\chi(a_j)}{J}\right]^s \chi$$

and

$$\beta_{\bar{a}}^s = \Sigma_\chi \operatorname{Im}\chi(a_1) \left[\frac{\Sigma \operatorname{Re}\chi(a_j)}{J}\right]^s \chi.$$

Considers the operators $A_s : X \longrightarrow L^1_X(G^J)$ (resp. $B_s$) defined by $A_s f = f * \alpha_{\bar{a}}^s$ (resp. $B_s f = f * \beta_{\bar{a}}^s$). Since $X = [X_0, X_1]_\theta$ (obtained in Prop. 1) implies also $L^1_X = [L^1_{X_0}, L^1_{X_1}]_\theta$ (see [6], Th. 5.2.1.), we may estimate

$$\|A_s\| \leqslant (\|A_s\|_{X_0 \to L^1_{X_0}})^{1-\theta} (\|A_s\|_{X_0 \to L^1_{X_1}})^\theta \tag{5}$$

and similarly $\|B_s\|$. In order to obtain Prop. 3 it will be sufficient to prove the majoration

$$\sum_{s=0}^{\infty} (\|A_s\| + \|B_s\|) \leqslant c. \tag{6}$$

The norms appearing are estimated from (5). We restrict ourselves to $\|B_s\|$, which behaves worse than $\|A_s\|$ and determines the condition on $J$. Clearly, since $X_0$ is translation invariant

$$\|B_s\|_{X_0 \to L^1_{X_0}} \leqslant \|B_s\|_{X_0 \to L^\infty_{X_0}} \leqslant \|\beta_a^s\|_{M(G)}.$$

The latter quantity is further dominated by

$$\left\| \sin\theta_1 \left[ \frac{\sum_{j=1}^{J} \cos\theta_j}{J} \right]^s \right\|_{A(\Pi^J)}$$

$$= (2J)^{-s} \left\| \sin\theta_1 \left[ 2\cos\theta_1 + \sum_{j=2}^{J} (e^{i\theta_j} + e^{-i\theta_j}) \right]^s \right\|_{A(\Pi^J)}$$

and by multi-nomial expansion of the $s$-power, can be estimated by

$$(2J)^{-s} \sum_{k+s_2+t_2+\ldots+s_J+t_J \leqslant s} \frac{s!}{k!\, s_2!\, t_2! \ldots s_J!\, t_J!} \|\sin\theta\,(2\cos\theta)^k\|_{A(\Pi)}$$

$$\leqslant (2J)^{-s} \sum_{k=0}^{s} \frac{s!}{k!\,(s-k)!} (2J-2)^{s-k}\, 2^k/k^{1/2} \leqslant C(J)\, s^{-1/2}$$

since

$$\|\sin\theta(\cos\theta)^k\|_A \leqslant 2^{-k} \sum_{j=0}^{k-1} \left| \binom{k}{j} - \binom{k}{j+1} \right| \leqslant ck^{-1/2}.$$

Thus we proved that

$$\|B_s\|_{X_0 \to L^1_{X_0}} \leqslant C(J) s^{-1/2}. \tag{7}$$

Since $X_1 = L^2(G)$, it follows from Parseval's formula that

$$\|B_s\|_{X_1 \to L^1_{X_1}} \leqslant \|B_s\|_{X_1 \to L^2_{X_1}} \leqslant \sup_{\chi \in \Gamma \setminus \{0\}} \left\| \operatorname{Im} \chi(a_1) \left[\frac{\Sigma \operatorname{Re} \chi(a_j)}{J}\right]^s \right\|_{L^2(G^J)}.$$

But, by proposition 2, this expression can be computed on $\Pi^J$ as

$$\left\| \sin\theta_1 \left[\frac{\sum_{j=1}^{J} \cos\theta_j}{J}\right]^s \right\|_{L^2(\Pi^J)}$$

$$= J^{-s} \left\{ \sum_{s_1 + \ldots + s_J = s} \frac{2s!}{2s_1 \ldots 2s_J!} \left(\int \sin^2\theta \cos^{2s_1}\theta\right) \prod_{j \geqslant 2} \int \cos^{2s_j}\theta \right\}^{1/2}$$

$$\leqslant C(J)\, J^{-s} \left\{ \sum_{s_1 + \ldots + s_J = s} \frac{2s!}{2s! \ldots 2s_J!} \frac{1}{s_1 s_1^{1/2}} \prod_{j \geqslant 2} \frac{1}{s_j^{1/2}} \right\}^{1/2}$$

$$\leqslant C(J) J^{-s} s^{J/4} \left(\frac{2s!}{(2s + J + 1)!}\right)^{1/2} \left\{ \sum_{t_1 + \ldots + t_J = 2s + J + 1} \frac{(2s + J + 1)!}{t_1! \ldots t_J!} \right\}^{1/2}$$

$$\leqslant C(J)\, J^{-s} s^{J/4} s^{-J/2 - 1/2} J^{s + (1 + J)/2}$$

from where

$$\|B_s\|_{X_1 \to L^1_{X_1}} \leqslant C(J)\, s^{-(J/4 + 1/2)}. \tag{8}$$

Applying (5), (7), (8) and the equation $1/p = \theta/2 + (1 - \theta)/1$, we get the majoration

$$\|B_s\| \leqslant C(J)\, s^{-(1-\theta)1/2} s^{-\theta(J/4 + 1/2)} = c(J)\, s^{-1/2 - J/2p'}.$$

Consequently, the series (6) will converge provided $J > p'$. This concludes the proof.

## 3. Sharpness of the result.

It is reasonable to ask whether the condition $J > \max([q], [q'])$ stated in Corollary 4 is best possible. We show that this is the case for $1 < q \leqslant 2$. As will be clear from what follows, an application of this argument to for instance a Rudin-Shapiro type polynomial shows that always J has to be at least 3. However, for $q \geqslant 3$, our method does not seem to determine whether the condition $J > q$ is necessary.

Take $G = \pi$. Fixing an integer N, the function $f(x) = N^{-1/q'} \sum_{\substack{|n| \leqslant N \\ n \neq 0}} e^{inx}$ has $\|f\|_q \leqslant c$. Assume now $f$ represents as

$$f = \sum_{j=1}^{J} [f_j - \tau(\theta_j) f_j]$$

then for $|n| \leqslant N$, $n \neq 0$

$$N^{-1/q'} \leqslant \left( \sum_{j=1}^{J} |\hat{f}_j(n)| \right) \max_j |1 - e^{in\theta_j}| .$$

Suppose we have $r$ integers $n_1, n_2, \ldots, n_r$. One can then find a set of integers S, $\operatorname{card}(S) > \frac{N}{2r}$ and such that $S - S = \{a - b \mid a \in S, b \in S\}$ satisfies

$$(S - S) \setminus \{0\} \subset \{n \in Z \mid |n| \leqslant N, n \neq 0\} \setminus \{n_1, \ldots, n_r\}.$$

Next, an entropy argument for the group $\pi^J$ allows to find some $n_{r+1}$ in $S - S$, $n_{r+1} \neq 0$, such that

$$\max_{1 \leqslant j \leqslant J} |1 - e^{in_{r+1}\theta_j}| \leqslant c\, (\operatorname{card} S)^{-1/J} .$$

Thus $|n_{r+1}| \leqslant N$, $n_{r+1} \neq n_1, \ldots, n_r$ and

$$\max_j |1 - e^{in_{r+1}\theta_j}| \leqslant c \left(\frac{N}{r}\right)^{-1/J} .$$

The inequality

$$N^{-1/q'} \leqslant c \left( \sum_{j=1}^{J} |\hat{f}_j(n_r)| \right) \left(\frac{N}{r}\right)^{-1/J} \qquad (r = 1, 2, \ldots) \tag{9}$$

gives $J \geqslant q'$. Suppose $q'$ an integer and $J = q'$. Then (9) becomes

$$\sum_{j=1}^{J} |\hat{f}_j(n_r)| > r^{-1/q'} \qquad (r = 1, 2, \ldots)$$

which, by the Hausdorff-Young inequality $(\Sigma\,|\hat{f}(n)|^{q'})^{1/q'} \leqslant \|f\|_q$, implies

$$\Sigma_j \|f_j\|_q > c(\log N)^{1/q'}.$$

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Manuscrit reçu le 29 juin 1984
révisé le 10 janvier 1985.

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