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FINE AND QUASI CONNECTEDNESS IN NONLINEAR POTENTIAL THEORY

by
D.R. ADAMS and J.L. LEWIS

I. DESCRIPTION OF RESULTS

1.

If E is an arbitrary subset of the Euclidean space $\mathbf{R}^n$, let $B_{\alpha,p}(E)$ denote the Bessel capacity of E, $0<\alpha<\infty, 1<p<\infty$, i.e. $\inf\{\|f\|_p^p : f \in L_+^p(\mathbf{R}^n),\quad G_\alpha * f \geqslant 1 \quad \text{on} \quad E\}$. Here $L^p(\mathbf{R}^n)$ is the usual Lebesgue space of p-th power summable functions, $L_+^p(\mathbf{R}^n)$ the non-negative elements, $\|f\|_p$ the usual norm of f in L^p, and $G_\alpha * f$ the convolution over $\mathbf{R}^n$ of f with the Bessel kernel G_α – the L_+^1 function on $\mathbf{R}^n$ whose Fourier transform is $(1+|\xi|^2)^{-\alpha/2}, \xi \in \mathbf{R}^n$. The reader might want to consult any one of serveral sources for the various properties of $B_{\alpha,p}$ and the associated non-linear potentials; see especially [16], [15], [4], [12], [13]. In particular, we will need the following : if $B_{\alpha,p}(E)<\infty$, then there exists a Borel measure μ, supported on $\overline{E}$ = closure of E, such that

$$\mu(\overline{E}) = B_{\alpha,p}(E) = \|G_\alpha * \mu\|_{p'}^{p'}, p' = p/(p-1), \tag{1}$$

$$G_\alpha * (G_\alpha * \mu)^{p'-1}(x) \geqslant 1, B_{\alpha,p} - \text{a.e. } x \in E, \tag{2}$$

$$G_\alpha * (G_\alpha * \mu)^{p'-1}(x) \leqslant M, \text{ for all } x \in \mathbf{R}^n, \tag{3}$$
M a constant depending only on α, p and n.

Key-words : Fine topology, Capacity, Quasi topology, Fine boundary.

If $G_\alpha * (G_\alpha * \nu)^{p'-1}(x) \leqslant M$, for all x,
and $G_\alpha * (G_\alpha * \nu)^{p'-1}(x) \geqslant 1$, for all $x \in E$, where
ν is a Borel measure with finite total variation $\|\nu\|_1$,
then $B_{\alpha,p}(E) \leqslant M \cdot \|\nu\|_1$. (4)

Our main interest in this note is a study of the connectedness of subsets of $\mathbf{R}^n$ in the fine and quasi topologies associated with $B_{\alpha,p}$ – the (α, p)-fine topologies and the (α, p)-quasi topologies; see definitions below. When $\alpha = 1$ and $p = 2$, $B_{1,2}$ is equivalent (same null sets) to the classical Newtonian capacity for $n \geqslant 3$ and to logarithmic capacity for $n = 2$. Hence in this case, the (1,2)-fine topology and the (1,2)-quasi topology coincide with the familiar fine and quasi topologies of classical potential theory, i.e. the theory developed extensively by M. Brelot, O. Frostman, H. Cartan, G. Choquet, B. Fuglede and others ; see [5]. Our main results are Theorems 1 and 2 and their corollaries, below. They extend the corresponding results of Fuglede [7], [8] (see especially Theorem 2 of [8]) and Lyons [14], who treated the case $\alpha = 1$, $p = 2$.

Our methods, however, are completely different ; they are akin to those of geometric measure theory. For when $p \neq 2$, there is so far no adequate theory of balayage of measures (one of the principal tools of the classical theory) since there is no maximum (minimum) principle in this case. Lyons uses a variant of a lemma called Hall's lemma in [6]. This lemma, for the case $\alpha = 1$, $p = 2$, says that the projection onto the unit sphere of the set where a given Newtonian potential is greater than one has surface area at most an absolute constant times the value of the potential at the origin. Unfortunately, this lemma does not generalize to nonlinear potentials for all values of α and p under consideration. In particular, simple examples can be constructed to show that it fails for any α, p satisfying : $\alpha p > 1$, $p > 2 - (\alpha/n)$, and $n - \alpha p > (n-1)(p-1)$.

In nos. 2-7 below, the (α, p)-fine and (α, p)-quasi topologies and some of their elementary properties are discussed as well as the main results. The proofs are given in Section II. It might be noted that the proof of Theorem 1 relies on the Kellogg property (Proposition 1), which, for all $p > 1$, is a recent result of

Hedberg and T. Wolff [13]. Throughout, the letter c will denote various constants depending only on α, p and n, not necessarily the same constant in any single string of estimates.

2. The (α, p)-fine topologies.

A set $E \subset \mathbf{R}^n$ is (α, p)-thin at x iff

$$\int_0^1 [r^{\alpha p-n} B_{\alpha,p} (E \cap B(x, r))]^{p'-1} \frac{dr}{r} < \infty, \tag{5}$$

where $1 < p \leqslant n/\alpha$, $p' = p/(p-1)$, and $B(x, r)$ is an open ball in $\mathbf{R}^n$ centered at x of radius r. E is termed (α, p)-fat at x iff E is not (α, p)-thin at x. V is an (α, p)-fine neighborhood of the point x iff $x \in V$ and the complement $V^c = \mathbf{R}^n \setminus V$ is (α, p)-thin at x. The (α, p)-fine topology consists of those sets V which are an (α, p)-fine neighborhood of each of its points, i.e. V^c is (α, p)-thin at every point of V. This topology has been studied by Meyers in [17]. Frequently, when the pair (α, p) is understood, we will drop them from the notation. The same convention will be adopted in discussing the quasi topologies.

Now we set $b(E) = \{x : E$ is (α, p)-fat at $x\}$, the (α, p)-base of the set E, $e(E) = b(E)^c$, the (α, p)-fine exterior of E, and $\widetilde{E}$ = closure of E in the (α, p)-fine topology.

PROPOSITION 1. – (a) *Kellogg Property* : $B_{\alpha,p} (E \cap e(E)) = 0$. (b) *Choquet Property* : *For every* $\epsilon > 0$, *there exists an open set* G *containing* $e(E)$ *such that* $B_{\alpha,p} (E \cap G) < \epsilon$.

For the proof of Proposition 1, see [13]. Note that it easily follows that $B_{\alpha,p} (E) = 0$ iff $b(E) = \emptyset$.

PROPOSITION 2. – *For any open ball* $B(x, r)$,

$$B_{\alpha,p} (\widetilde{E} \cap B(x, r)) = B_{\alpha,p} (E \cap B(x, r)) = B_{\alpha,p} (b(E) \cap B(x, r)).$$

Furthermore, $\widetilde{E} = E \cup b(E)$.

Proof. – Following Meyers [17], we can deduce from [12] (Theorem 2) or from [4] (Theorems 5.1 and 5.2) that

$$B_{\alpha,p}\,((E \cup b(E)) \cap B(x\,,r)) = B_{\alpha,p}\,(E \cap B(x\,,r))\,.$$

Similarly, by also noting that $E \subset b(E) \cup (E \cap e(E))$, it follows easily that

$$B_{\alpha,p}\,(E \cap B(x\,,r)) = B_{\alpha,p}\,(b(E) \cap B(x\,,r))\,.$$

We conclude by showing $\widetilde{E} = E \cup b(E)$. In fact, $b(E)$ is nothing more than the set of $(\alpha\,,p)$-fine limit points of E (i.e. x is an $(\alpha\,,p)$-fine limit point of E iff whenever V is a finely open neighborhood of x, then $(V - \{x\}) \cap E \neq \emptyset$). Indeed, if E is fat at x, then so is $(V - \{x\}) \cap E$, hence it is not empty; conversely, if E is thin at x, then $(E \cup b(E))^c \cup \{x\}$ is a finely open neighborhood of x which does not meet E except possibly at x.

Note that when $\alpha = 1$, $p = 2$, $b(E)$ agrees with the notion of "the base of a set" as given in [7] and [8].

3. The $(\alpha\,,p)$-quasi topologies.

In [8], Fuglede has shown how to construct a quasi topology on $\mathbf{R}^n$ with a given countably subadditive set function. This quasi topology is "almost" a topology in the sense that it is closed under countable unions and finite intersections only. When we apply this idea to the set function $B_{\alpha,p}$, we naturally call the resulting quasi topology, the $(\alpha\,,p)$-quasi topology. A set E is an $(\alpha\,,p)$-quasi open set iff for every $\epsilon > 0$ there exists an open set $G \supset E$ such that $B_{\alpha,p}\,(G \backslash E) < \epsilon$. Note that if E is quasi open and $B_{\alpha,p}\,(N) = 0$, then $E \cup N$ is quasi open.

PROPOSITION 3. – *The $(\alpha\,,p)$-fine topology and the $(\alpha\,,p)$-quasi topology are compatible in the sense of Fuglede* [8] *for all $(\alpha\,,p)$, $1 < p \leqslant n/\alpha$. In particular :*

(a) *if* E *is quasi open, then* $E = H \cup N$, *where* H *is finely open and* $B_{\alpha,p}\,(N) = 0$;

(b) *if* E *is finely open, then* E *is quasi open.*

The proof of Proposition 3 follows from Proposition 1 and a straightforward adaptation of the corresponding arguments of [8]. In particular, (b) is just the Choquet Property. The set H of (a) can be taken to be the (α, p)-fine interior of E, i.e. $E^{c\sim c}$.

4. The (α, p)-fine boundary.

The boundary of E in the fine topology will be denoted by $\partial_f E = \tilde{E} \cap E^{c\sim}$. Since $b(E) \cap b(E^c) \subset \partial_f E$ and $b(\partial_f E) \subset b(E) \cap b(E^c)$, we have by Proposition 2, $B_{\alpha,p}(\partial_f E) = B_{\alpha,p}(b(E) \cap b(E^c))$. Also note that if both E and E^c are finely open then $\partial_f E = \phi$. This is true in the quasi topology only modulo null sets. In fact in Section 7, we prove

PROPOSITION 4. — *Let* G *be an open set in* $\mathbf{R}^n$ *and suppose* E *is a subset of* G *that is both* (α, p)*-quasi open and* (α, p)*-quasi closed relative to* G, *then* $B_{\alpha,p}(G \cap \partial_f E) = 0$.

In section 8, we prove

THEOREM 1. — *For* $1 < \alpha p \leqslant n$, *there is a constant* $C = C(\alpha, p, n)$ *such that for any set* E

$$\min \{B_{\alpha,p}(E \cap Q), B_{\alpha,p}(E^c \cap Q)\} \leqslant C \cdot B_{\alpha,p}(Q \cap \partial_f E), \quad (6)$$

for all open cubes Q *contained in* $\mathbf{R}^n$.

Theorem 1 and Proposition 4 together give

COROLLARY 1. — *If* G *is open and connected, then* G *is connected in the* (α, p)*-quasi topology provided* $\alpha p > 1$.

Remarks. — (i) The above corollary is false for $\alpha p \leqslant 1$ since we can disconnect $\mathbf{R}^n$ with $n-1$ dimensional hypersurfaces and they all have capacity zero when $\alpha p \leqslant 1$. See [16].

(ii) Note that if G is open and connected, then G is finely connected for all (α, p), $1 < p \leqslant n/\alpha$. This is a consequence of two facts : G must be connected in the ordinary density topology of $\mathbf{R}^n$ (for Lebesgue n-measure) and the fine topologies are all (strictly) smaller than the ordinary density topology. The first fact is proven explicitly in [11]; see also [19]. It also follows immediately from Lemma 1 of Section 8. For the second, note that if E is thin at x then E has ordinary Lebesgue density zero at x.

5. Arcwise connectedness.

A closed continuous path γ joining x to y is called a coordinate path iff any compact subset of $\gamma - \{x, y\}$ is contained in a finite union of line segments parallel to the coordinate axes. We will be interested in arcwise connectedness using only coordinate paths, hence the next theorem generalizes a result of Lyons [14].

THEOREM 2. – *Let* $x \in \mathrm{E}$ *and suppose that* E *is* (α, p)*-finely open. Then for* $\alpha p > 1$ *there is an* (α, p)*-finely open neighborhood* V *of* x, $\mathrm{V} \subset \mathrm{E}$, *with the property that any* $y, z \in \mathrm{V}$ *can be joined by a coordinate path in* E *of length at most* $c \cdot |z - y|$; c *is a constant depending only on* α, p *and* n.

COROLLARY 2. – *If* E *is* (α, p)*-finely open and* (α, p)*-finely connected, then for* $\alpha p > 1$, E *is arcwise connected.*

Remark. – (iii) The above corollary is false for $\alpha p \leqslant 1$. From the open ball $\mathrm{B}(0, 1)$ remove a closed (exponential) cusp K with vertex at the origin and also remove from $\mathrm{B}(0, 1/2) \backslash \mathrm{K}$ the $n - 1$ dimensional surfaces of distance $1/m$ from K, $m = 2, 3, \ldots$. The resulting subset of $\mathrm{B}(0, 1)$ together with the origin, call it D, will be (α, p)-finely open and (α, p)-finely connected for $\alpha p \leqslant 1$, but will not be arcwise connected. Indeed $\mathrm{D} \backslash \{0\}$ is an open connected set, hence a finely connected set as noted in remark (ii). Also any finely open set containing the origin must contain points of $\mathrm{D} \backslash \{0\}$.

6. Further results.

A function $\phi : \mathbf{R}^n \longrightarrow \mathbf{R}$ is (α, p)-quasi continuous iff for every $\epsilon > 0$ there is an open set G such that $B_{\alpha,p}(G) < \epsilon$ and ϕ restricted to G^c is continuous there. Equivalently, ϕ is quasi continuous iff the inverse image of every open set is a quasi open set. Thus 1_E = characteristic function of the set E, is quasi continuous iff E is both quasi open and quasi closed, and from Proposition 4, iff $B_{\alpha,p}(\partial_f E) = 0$. Now if we further restrict attention to $\alpha p > 1$ then we can use Theorem 1 which gives: for $\alpha p > 1$, 1_E is (α, p)-quasi continuous iff $\min\{B_{\alpha,p}(E), B_{\alpha,p}(E^c)\} = 0$. The following proposition deals with the case $\alpha p \leqslant 1$. It is discussed in Section 10.

PROPOSITION 5. – *For* $\alpha p \leqslant 1$, 1_E *is* (α, p)*-quasi continuous iff*

$$\lim_{r \to 0} \frac{B_{\alpha,p}(E \cap B(x, r))}{B_{\alpha,p}(B(x, r))} = 1_E(x) \tag{7}$$

locally (α, p)*-quasi almost uniformly, i.e. for every compact set* K *and every* $\epsilon > 0$ *there is a subset* $e \subset K$ *such that* $B_{\alpha,p}(e) < \epsilon$ *and the convergence is uniform on* $K \backslash e$.

II. THE DETAILS

7. Proof of proposition 4.

For each $\epsilon > 0$, there are open sets O_1 and O_2 in G such that $O_1 \supset E$, $O_2 \supset G \backslash E$ and $B_{\alpha,p}(O_1 \backslash E) < \epsilon$, $B_{\alpha,p}(O_2 \cap E) < \epsilon$. Hence $B_{\alpha,p}(O_1 \cap O_2) < 2\epsilon$. But then $F = (O_1 \cap O_2)^{\sim}$ has capacity less than 2ϵ by Proposition 2. Since $B_{\alpha,p}(\partial_f E \cap G) = B_{\alpha,p}(b(E) \cap b(G \backslash E))$, we can conclude the proof by showing that $b(E) \cap b(G \backslash E)$ is contained in F except possibly for a set of capacity zero. Since $E = H \cup N$ where H is finely open in G and $B_{\alpha,p}(N) = 0$, it follows that $H \backslash F$

is finely open and that $G\backslash E$ is thin at every point of $H\backslash F$. So if $x \in b(E) \cap b(G\backslash E)$ then $x \in (G\backslash H) \cup F$. Or except for a set of capacity zero $x \in (G\backslash E) \cup F$. Applying the same argument to $G\backslash E$ implies that $x \in E \cup F$ except for a set of capacity zero. Hence $B_{\alpha,p}$ — a.e. $x \in b(E) \cap b(G\backslash E)$ must lie in F.

8. Proof of Theorem 1 and Corollary 1.

For $1 < \alpha p < n$, set

$$R = \left\{ x : \lim_{r\to 0} \frac{H^n [E^c \cap B(x, r)]}{r^n} = 0 \right\}$$

and

$$S = \left\{ x : \lim_{r\to 0} \frac{H^n [E \cap B(x, r)]}{r^n} = 0 \right\}.$$

Then $R = E$ and $S = E^c$, H^n — a.e.. Here we are using H^d for d-dimensional Hausdorff measure in $\mathbf{R}^n$, $d \in \mathbf{Z}^+$. Let $\partial^* E = R^c \cap S^c$ be the measure theoretic boundary of E. If $x \in E\backslash R$, then E^c is fat at x since otherwise the ratio $r^{\alpha p - n} B_{\alpha,p}(E^c \cap B(x, r))$ tends to zero with r. So either $x \in b(E) \cap b(E^c)$ or E is thin at x. But the last condition can only hold for a set of capacity zero by the Kellogg Property. Thus $E \subset R \cup \partial_f E$, $B_{\alpha,p}$ — a.e.. Similarly, $E^c \subset S \cup \partial_f E$, $B_{\alpha,p}$ — a.e.. Hence by the subadditivity of $B_{\alpha,p}$, (6) will follow upon showing

$$\min \{B_{\alpha,p}(R \cap Q), B_{\alpha,p}(S \cap Q)\} \leqslant C \cdot B_{\alpha,p}(\partial^* E \cap Q), \quad (8)$$

since $\partial^* E \subset \partial_f E$.

To prove (8), we need some preliminary lemmas. Let e_i denote the coordinate directions in $\mathbf{R}^n$, $i = 1, \ldots, n$ and p_i the projection of $\mathbf{R}^n$ onto the vector space V_i generated by $e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n$. The following lemma appears in [10], 4.5.11.

LEMMA 1. — *For* H^{n-1} *a.e.* $y \in V_i$ *the following statement holds : if* $u \in p_i^{-1}(y) \cap S$ *and* $v \in p_i^{-1}(y) \cap R$, *then there exists* $w \in p_i^{-1}(y) \cap \partial^* E$ *lying on the line segment from* u *to* v.

LEMMA 2. — *Let* Q' *be a coordinate cube with side length* r *and suppose that*

$$H^{n-1}\,[p_i\,(\partial^*\,E \cap Q')] \leqslant n^{-2}\,2^{-n-2}\,r^{n-1} \qquad (9)$$

for $1 \leqslant i \leqslant n$. *Then there is a set* $L \subset Q'$ *with*

$$H^n(L) \geqslant [1 - 2^{-n-2}]\,r^n$$

and with the property that any $x\,,y \in L$ *can be joined by a coordinate path* $\gamma \subset Q'$ *with* $H^1(\gamma) \leqslant 8nr$ *and either* $\gamma \subset R$ or $\gamma \subset S$.

Note that in Lemma 2, $L \subset R$ or $L \subset S$ since $R \cap S = \emptyset$.

Proof. — Set $G_i = \{y \in Q' :$ either $p_i^{-1}(p_i(y)) \cap Q' \subset R$ or $p_i^{-1}(p_i(y)) \cap Q' \subset S\}$. By Lemma 1 and (9), we have

$$H^n(G_i) \geqslant (1 - n^{-2}\,2^{-n-2})\,r^n\,.$$

Thus

$$H^n\Big(\bigcap_{i=1}^{n} G_i\Big) \geqslant (1 - n^{-1}\,2^{-n-2})\,r^n\,.$$

By Fubini's Theorem, we can obtain sets $K_1\,,K_2\,,\ldots,K_{n-1}$ with the properties

(a) $K_1 \subset K_2 \subset \cdots \subset K_{n-1} = p_n\Big(\bigcap_{i=1}^{n} G_i\Big)$.

(b) K_m, for $1 \leqslant m \leqslant n-1$, lies inside an m-dimensional plane parallel to the vector space generated by $e_1\,,\ldots,e_m$,

(c) $H^m(K_m) \geqslant [1 - n^{-1}\,2^{-n-2}]\,r^m\,,\ 1 \leqslant m \leqslant n-1$.

We set $K_n = \bigcap_{i=1}^{n} G_i$. Now let $L_1 = K_1$ and define L_m, $1 < m \leqslant n$, inductively by

$$L_m = \{y \in K_m : p_m(y) = p_m(z) \text{ for some } z \in L_{m-1}\}.$$

From (c) it follows that

$$H^n(L_n) \geqslant (1 - 2^{-n-2})\,r^n\,.$$

We conclude by showing that Lemma 2 is valid with $L \equiv L_n$. First suppose $y\,,z \in L_n$ and $p_n(y)\,,p_n(z) \in L_1$. Then $p_n(y)\,,p_n(z)$ lie on a line segment parallel to e_1, so there is a $w \in p_1^{-1}(p_1(y)) \cap p_n^{-1}(p_n(z))$. Let γ be the union of the line segments from y to w and from w to z. Since $y, z \in \bigcap_{i=1}^{n} G_i$, it follows that $H^1(\gamma) \leqslant 2r$, and either $\gamma \subset R \cap Q'$ or $\gamma \subset S \cap Q'$.

For the induction, suppose we have shown that for some m, $1 \leqslant m \leqslant n-2$, whenever $p_n(z), p_n(y) \in L_m$, then y can be joined to z by a coordinate path γ with $H^1(\gamma) \leqslant 4mr$, and either $\gamma \subset R \cap Q'$ or $\gamma \subset S \cap Q'$. If $p_n(y), p_n(z) \in L_{m+1}$, then by definition there exists $y_1, z_1 \in L_n$ with

$$p_{m+1}(p_n(y)) = p_{m+1}(p_n(y_1)), p_{m+1}(p_n(z))$$
$$= p_{m+1}(p_n(z_1)) \text{ and } p_n(z_1), p_n(y_1) \in L_m .$$

Thus, $p_n(y), p_n(y_1)$ and $p_n(z), p_n(z_1)$ respectively lie on lines parallel to e_{m+1}, and consequently, there exist y_2, z_2 with $y_2 \in p_{m+1}^{-1}(p_{m+1}(y)) \cap p_n^{-1}(p_n(y_1))$ and

$$z_2 \in p_{m+1}^{-1}(p_{m+1}(z)) \cap p_n^{-1}(p_n(z_1)).$$

Let γ_y be the union of the line segments from y to y_2 and y_2 to y_1. γ_z is defined similarly for z, z_2, z_1. Then γ_y joins y to y_1 and either $\gamma_y \subset R$ or $\gamma_y \subset S$ since $y, y_1 \in \bigcap_{i=1}^{n} G_i$ and similarly for γ_z. By the induction hypothesis, there exists γ_1 joining y_1 to z_1 with $H^1(\gamma_1) \leqslant 4mr$ and either $\gamma_1 \subset R \cap Q'$ or $\gamma_1 \subset S \cap Q'$. Let $\gamma = \gamma_1 \cup \gamma_y \cup \gamma_z$. Then γ joins y to z, $H^1(\gamma) \leqslant 4(m+1)r$, and either $\gamma \subset R \cap Q'$ or $\gamma \subset S \cap Q'$, which concludes the induction.

Remark. – (iv) The argument of Lemma 2 is due to B. Davis.

(*Proof of Theorem 1 continued*). – We may assume

$$Q = \{y : |y_i| < 1/2\}, y = (y_1, \ldots, y_n)$$

and $H^n(R \cap Q) \leqslant H^n(S \cap Q)$. From [16] and [18], we have : if $1 < \alpha p < n$, then for any set $F \subset \mathbf{R}^n$ there are constants c_k, $k = 1, 3$, such that

$$H^{n-1}[p_i(F)]^{\frac{n-\alpha p}{n-1}} \leqslant c_1 B_{\alpha,p}[p_i(F)] \leqslant c_2 B_{\alpha,p}[F]. \qquad (10)$$

Now if $H^{n-1}[p_i(\partial^* E \cap Q)] \geqslant n^{-2} 2^{-n-2}$ for some i, then (10) implies that (8) holds for some large constant. Therefore, we may assume that the above is false for $i = 1, \ldots, n$. Fix an $x \in R$ and let $\{Q_k\}$, $k = 0, 1, 2, \ldots$, be a sequence of open dyadic coordinate cubes of Q with

(i) $x \in \bar{Q}_{k+1} \subset \bar{Q}_k \subset \cdots \subset \bar{Q}_0 = Q$

(ii) Q_k has side length 2^{-k}, $k = 0, 1, 2, \ldots$.

The claim now is that there exists a k and an i such that

$$H^{n-1}(p_i[Q_k \cap \partial^* E]) \geqslant n^{-2}\, 2^{-n-2}\, 2^{-(n-1)k}. \tag{11}$$

To see this, suppose (11) is false. Then by Lemma 2, there is an $L^k \subset Q_k$ with $H^n(L^k) \geqslant [1 - 2^{-n-2}]2^{-nk}$ and either $L^k \subset R$ or $L^k \subset S$, $k = 0, 1, 2, \ldots$. Since

$$H^n(R \cap Q) \leqslant H^n(S \cap Q),\ L^0 \subset S.$$

Also $H^n[L^1 \cap L^0] \geqslant [1 - 2^{-n-2}]2^{-n} - 2^{-n-2} > 0$, and since $L^1 \cap L^0 \subset S$, it follows that $L^1 \subset S$. By induction, we must have $L^k \subset S$, for all k. Also since

$$H^n(Q_k \cap S) \geqslant H^n(L^k) \geqslant 2^{-nk}\,[1 - 2^{-n-2}],$$

we must have

$$\varliminf_{r \to 0} r^{-n} H^n(B(x, r) \cap S) > 0.$$

But this is impossible since $S = E^c$, H^n — a.e. and $x \in R$.

Now let μ be the measure of (1) – (3) for the set $Q \cap \partial^* E$. Then $G_\alpha * (G_\alpha * \mu)^{p'-1} \geqslant 1$, $B_{\alpha,p}$ — a.e. on $Q \cap \partial^* E$. We next claim that

$$G_\alpha * (G_\alpha * \mu)^{p'-1}(x) \geqslant c^*, \text{ for all } x \in R \cap Q \tag{12}$$

where c^* depends only on α, p and n. To see this, let $\mu_1 = \mu$ restricted to $B(x, \sqrt{n}\, 2^{2-k})$. Then for $y \in Q_k = Q_k(x)$, it is easily seen that

$$G_\alpha * (G_\alpha * \mu)^{p'-1}(y) \leqslant c[G_\alpha * (G_\alpha * \mu_1)^{p'-1}(y) + G_\alpha * (G_\alpha * \mu)^{p'-1}(x)]. \tag{13}$$

Now if (12) is false, choose a constant c_0 sufficiently small so that $G_\alpha * (G_\alpha * \mu)^{p'-1}(x) \leqslant c_0$ for some $x \in R \cap Q$ and such that (13) yields

$$G_\alpha * (G_\alpha * \mu_1)^{p'-1}(y) \geqslant c,\ B_{\alpha,p} \text{ a.e. } y \in Q_k \cap \partial^* E,$$

for some $c > 0$. From (3) and (4) it follows that

$$B_{\alpha,p}(Q_k \cap \partial^* E) \leqslant c \cdot \mu(B(x, \sqrt{n}\, 2^{2-k})). \tag{14}$$

From [4], we always have the lower estimate

$$G_\alpha * (G_\alpha * \mu)^{p'-1}(x) \geqslant c \int_0^1 [r^{\alpha p-n}\, \mu(B(x\,,r))]^{p'-1} \frac{dr}{r}, \quad (15)$$

hence it follows that

$$G_\alpha * (G_\alpha * \mu)^{p'-1}(x) \geqslant c[2^{(n-\alpha p)k}\, \mu(B(x\,,\sqrt{n}\,2^{2-k}))]^{p'-1}. \quad (16)$$

So putting (10), (11), (14), (16) together, we conclude

$$G_\alpha * (G_\alpha * \mu)^{p'-1}(x) \geqslant c > c_0$$

for c_0 small enough. This is a contradiction, hence (12) must hold. Since $x \in R \cap Q$ is arbitrary it follows from (4) that

$$B_{\alpha,p}(R \cap Q) \leqslant c \cdot \|\mu\|_1 = c \cdot B_{\alpha,p}(\partial^* E \cap Q),$$

which proves Theorem 1 for $\alpha p < n$.

For $\alpha p = n$, we must make some modifications. The first inequality in (10) is now no longer valid. Instead we use : if $\alpha p = n$, then there exists constants c_k, $k = 1, 2, 3$, depending only on α, p, n such that for any set $F \subset \mathbf{R}^n$,

$$[\log(c_1/H^{n-1}(F))]^{1-p} \leqslant c_2 \cdot B_{\alpha,p}(F) \quad (17)$$

when $H^{n-1}(F) \leqslant 1$ and

$$H^{n-1}(F) \leqslant c_3 \cdot B_{\alpha,p}(F) \quad (18)$$

when $H^{n-1}(F) \geqslant 1$. (17) and (18) follow in the standard way; see [1] (Theorem 3), [3], [16]. Due to the logarithm in (17), we also modify our construction of the cubes Q_k. Now choose open coordinate cubes in Q so that

(i′) $Q_k \cap Q_{k+1} \neq \emptyset$ and $\operatorname{dist}(x\,, Q_k) = 2^{-k-1}$,

(ii) Q_k has side length 2^{-k}, $k = 0, 1, 2, \ldots$,

(iii) the center of each Q_k lies on a line parallel to one of the coordinate axes.

Finally we replace (16) with

$$G_\alpha * (G_\alpha * \mu)^{p'-1}(x) \geqslant c \cdot k\, \mu(A_k(x))^{p'-1},$$

where $A_k(x) = \{y : 2^{-k-1} < |x-y| < 2^{-k+1}\}$. This again follows from (15). The argument for $\alpha p = n$ is now easily constructed as before.

Proof of corollary 1. – Since G is open and connected, we can find open cubes Q_k, $k = 1, 2, \dots$ such that (i) $Q_k \subset G$ for all k, (ii) $Q_k \cap Q_{k+1} \neq \emptyset$ for all k, (iii) $\bigcup_k Q_k = G$. Now if $G = A \cup B$, $A \cap B \neq \emptyset$, A and B quasi open, then by Proposition 4, $B_{\alpha,p}(Q_k \cap \partial_f A) = 0$ for all k, consequently by Theorem 1, $\min\{B_{\alpha,p}(A \cap Q_k), B_{\alpha,p}(B \cap Q_k)\} = 0$ for all k. (ii) now easily implies that if $B_{\alpha,p}(A \cap Q_k) = 0$ then $B_{\alpha,p}(A \cap Q_{k+1}) = 0$, consequently $B_{\alpha,p}(A) = 0$. Thus either A or B must have capacity zero, $\alpha p > 1$, and hence G must be quasi connected.

9. Proof of Theorem 2.

Let ϵ be a small positive number to be fixed later. Then there exists a $\delta = \delta(x, \epsilon) > 0$ such that

$$r^{\alpha p-n} B_{\alpha,p}(E^c \cap B(x, r)) < \epsilon \tag{19}$$

for $0 < r \leqslant \delta$. Let $Q(x, r) = \{y : |y_i - x_i| < r,\ i = 1, \dots, n\}$ and let V^* be the set of all $y \in E \cap [Q(x, \delta/n^2) - \{x\}]$ for which

$$\sup_{0<s\leqslant |x-y|} \{s^{\alpha p-n} B_{\alpha,p}(E^c \cap B(y, s))\} < 4^{n-\alpha p}\epsilon. \tag{20}$$

Clearly $V = V^* \cup \{x\}$ is contained in E. We claim first of all that V is a finely open neighborhood of x. Indeed, suppose $y \in V - \{x\}$. Then an easy argument using (19) and (20) shows the existence of $\delta_1 = \delta_1(y, \epsilon) < \delta/2n^2$ such that whenever $z \in B(y, \delta_1)$ then

$$\sup_{|z-y|<s\leqslant |x-z|} s^{\alpha p-n} B_{\alpha,p}(E^c \cap B(z, s)) < 4^{n-\alpha p}\epsilon.$$

Thus if $z \in B(y, \delta_1) \cap V^c$, then either $z \in E^c$ or

$$\sup_{0<s\leqslant |z-y|} s^{\alpha p-n} B_{\alpha,p}(E^c \cap B(z, s)) \geqslant 4^{n-\alpha p}\epsilon. \tag{21}$$

Given t, $0 < t < \delta_1$, let μ be the measure satisfying (1) – (3) for the set $B(y, t) \cap E^c$. Then if $z \in B(y, t/2) \cap E$ satisfies (21) it follows as in the proof of (12) that $G_\alpha * (G_\alpha * \mu)^{p'-1}(z) \geqslant c\epsilon$ and thereupon that $B_{\alpha,p}(V^c \cap B(y, t/2)) \leqslant \frac{c}{\epsilon} B_{\alpha,p}(E^c \cap B(y, 2t))$.

Integrating this inequality, we see that V^c is thin whenever $y \in V - \{x\}$. A similar argument shows V^c is thin at x, and our first claim is proved.

Next we define R, S relative to E, E^c as in the proof of Theorem 1. Note now that $R = E$. Let $y, z \in V$ and let Q_0 be a parallel subcube of $Q(x, \delta/n^2)$ whose side length r is proportional to $|y - z|$ and with $y, z \in Q_0$. Using the bisection method, divide Q_0 into parallel subcubes and choose Q_k, $k = 1, 2, \ldots$, with

(a) $y \in \overline{Q}_{k+1} \subset \overline{Q}_k \subset \cdots \subset \overline{Q}_0$

(b) Q_k has side length $2^{-k} r$, $k = 1, 2, \ldots$.

Then from (19), (20), (10) and the fact that $\partial^* E \subset E^c$, we see for ϵ sufficiently small that the hypotheses of Lemma 2 are satisfied with Q_k replacing Q'. By Lemma 2 it then follows for ϵ sufficiently small that there exists $L^k \subset Q_k \cap E$ with the property that any two points of L^k can be joined by a coordinate path contained in $E \cap Q_k$ with length at most $4n(2^{-k} r)$, $k = 0, 1, 2, \ldots$. Also, $H^n(L^k) \geqslant (1 - 2^{-n-2})(2^{-k} r)^n$. Choose $y_k \in L^k \cap E$. Clearly $L^k \cap L^{k+1} \neq \emptyset$ hence y_{k+1} can be joined to y_k by a coordinate path $\sigma_k \subset E \cap Q_0$ with $H^1(\sigma_k) \leqslant 4n\, 2^{-k} r$. Now set $\sigma = \bigcup_{k=0}^{\infty} \sigma_k$, then σ joins y to $y_0 \in Q_0$ and $H^1(\sigma) \leqslant cr$. A similar argument shows that we can join y_0 to z by a coordinate path γ with $H^1(\gamma) \leqslant cr$. Then $\gamma \cup \sigma \subset E$, $H^1(\gamma \cup \sigma) \leqslant cr \leqslant |z - y|$, and $\gamma \cup \sigma$ joins y to z. Since y, z are arbitrary in V the proof of Theorem 2 is complete provided $\alpha p < n$. The modifications required to deal with $\alpha p = n$ are similar to those outlined in the proof of Theorem 1, and will hence be omitted.

10. Proof of Proposition 5.

If we interpret $B_{\alpha,p}$ for $\alpha = 0$ as Lebesgue n-measure, then Proposition 5 is just a standard fact in measure theory when $\alpha = 0$ (which is a simple consequence of Lebesgue's differentiation theorem and Egoroff's theorem) since then "1_E quasi continuous" translates into "E measurable" by Luzin's theorem. Hence we are proposing to extend this to $0 < \alpha \leqslant 1/p$.

Note first of all that the local quasi almost uniform convergence of (7) easily implies that 1_E is quasi continuous since $B_{\alpha,p}(E \cap B(x, r))$ is a continuous function of x when $\alpha p \leqslant 1$. (This last fact is no longer true for $\alpha p > 1$ since then $n - 1$ dimensional surfaces have positive capacity.)

The necessity in Proposition 5 follows from

LEMMA 3. — *Let* $\alpha p \leqslant 1$. *If* 1_E *is* (α, p)*-quasi continuous, then*

$$\lim_{r \to 0} \frac{B_{\alpha,p}(E \cap B(x, r))}{B_{\alpha,p}(B(x, r))} = 1_E(x), \; B_{\alpha,p} - \text{a.e. } x, \qquad (22)$$

and

LEMMA 4. — *If the sequence* $\{f_k\}$ *of* (α, p)*-quasi continuous functions converges* $B_{\alpha,p}$ — a.e. *to an* (α, p)*-quasi continuous function* f, *then* $\{f_k\}$ *converges to* f *locally* (α, p)*-quasi almost uniformly.*

Remarks. — (v) Lemma 3 trivially holds for $\alpha p > 1$ by Theorem 1 ; in fact, $E = \mathbf{R}^n$ or $\emptyset$ modulo null sets in this case.

(vi) Fernström [9] has shown that there is a Borel set of positive $B_{\alpha,p}$ capacity for which the limit in (22) is zero for all x.

The proof of Lemma 3 is contained in a more general differentiation result established in [2]. There the analogue of Lebesgue's theorem was obtained for (α, p)-quasi continuous ϕ for which

$$\int_0^\infty B_{\alpha,p}(\{y \in B(x, r) : |\phi(y)| > \lambda\})\, \lambda^{p-1}\, d\lambda < \infty.$$

The proof follows the standard format — prove weak type estimates for the corresponding maximal function and then utilize the density of $C_0(\mathbf{R}^n) =$ continuous functions on $\mathbf{R}^n$ with compact support.

Lemma 4 follows the usual argument except now we need : if $\{E_k\}$, $k = 1, 2, \ldots$ is a non-increasing sequence of (α, p)-quasi closed and bounded sets for which $B_{\alpha,p}(\bigcap_k E_k) = 0$, then $\lim_{k \to \infty} B_{\alpha,p}(E_k) = 0$. This is a special case of a result due to Fuglede [8] (for any countably subadditive set function and any non-increasing sequence of quasi compact sets).

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