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An $L^p$-version of a theorem of D.A. Raikov


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AN $L^p$-VERSION OF A THEOREM
OF D. A. RAIKOV

by Gero FENDLER(*)

1. Introduction.

Let $G$ be a locally compact group, for $p \in (1, \infty)$, let $Pf_p(G)$ denote the closure of $L^1(G)$ in the convolution operator norm on $L^p(G)$. Denote by $W_p(G)$ the dual of $Pf_p(G)$ which is contained in the space of pointwise multipliers of the Figà-Talamanca Herz space $A_p(G)$. (See [5], [8], [9] for all this.) It is shown in these notes that on the unit sphere of $W_p(G)$ the weak * (i.e. the $\sigma(W_p, Pf_p)$ topology and the $A_p$-multiplier topology coincide ($u_\beta \to u$ in the latter if $\|(u_\beta - u)v\| \to 0$ for each $v \in A_p(G)$).

If $p = 2$ and $G$ is amenable then $W_2(G)$ is just the Fourier-Stieltjes algebra of $G$, denoted $B(G)$, and $A_2(G)$ is the Fourier algebra of $G$. From this point of view the above enunciation is an $L^p$-version of a theorem of D.A. Raikov, which asserts that on the positive face of the unit sphere of $B(G)$ the weak * topology coincides with the topology of uniform convergence on compact sets (since $A_p(G)$ always contains functions which take the value one on a given compact set the latter topology is clearly weaker than the $A_p(G)$ multiplier topology; and on norm bounded sets obviously stronger than the weak * topology).

The proof is based on a technique of G.C. Rota [10], first used in harmonic analysis by E.M. Stein; our application is close to the work of M. Cowling [3]. On the other hand this paper continues the line of studies taken up by E.E. Granirer and M. Leinert in [7].

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2. An estimate for the $L^p$-operator norm of the sum of two "spectrally disjoint" operators.

If $R$ and $S$ are two commuting normal (of course bounded) operators on an Hilbert space $H$ then, via the Gelfand transform, $R$ and $S$ correspond to some continuous functions on a locally compact space $X$; further $R, S$ are spectrally disjoint, if the supports of those functions are disjoint. It then follows easily that $\| R + S \| = \max \{ \| R \|, \| S \| \}$; we remark that there exists an orthogonal projection $P$ with $PR = R = RP$ and $(1 - P)S = S = S(1 - P)$.

From this:

$$\| (R+S)\xi \| = \| (R+S)(P+1-P)\xi \|$$

$$= \| PRP\xi + (1-P)S(1-P)\xi \|$$

$$= (\| PRP\xi \|^2 + \| (1-P)S(1-P)\xi \|^2)^{1/2}$$

$$\leq (\| R \|^2 \| P\xi \|^2 + \| S \|^2 \| (1-P)\xi \|^2)^{1/2}$$

$$\leq \max \{ \| R \| , \| S \| \} \left( \| P\xi \|^2 + \| (1-P)\xi \|^2 \right)^{1/2}$$

$$\leq \max \{ \| R \| , \| S \| \} \| \xi \| \text{ for all } \xi \in H.$$  

Now let $(X, \mu)$ be a $\sigma$-finite measure space; an operator $T$ acting on all $L^p$-spaces will be called special if:

i) $Tf \geq 0$ if $f \geq 0$

ii) $\| Tf \|_p \leq \| f \|_p$ $f \in L^p(X, \mu), 1 \leq p \leq \infty$

iii) $T1_X = 1_X$

iv) $\int_X Tf(x) \overline{g(x)} \, d\mu(x) = \int_X f(x) \overline{Tg(x)} \, d\mu(x)$ for $f, g \in L^2(X, \mu)$.

Those operators will serve as a substitute for orthogonal projections, since by a method due to G.C. Rota they may be seen as conditional expectations on a certain measure space.
We begin with the following observation:

**Proposition.** Let $(Y, \mathcal{S}, \nu)$ be a $\sigma$-finite measure space, $\mathcal{S}_1 \subset \mathcal{S}$ a sub-$\sigma$-algebra of $\mathcal{S}$ such that $(Y, \mathcal{S}_1, \nu)$ is again $\sigma$-finite, (which ensures the existence of a conditional expectation operator $E_1$ with respect to $\mathcal{S}_1$).

Then we have for $\xi, \eta \in L^p(Y, \mathcal{S}, \nu)$:

$$\|E_1 \xi + (1 - E_1) \eta\|_p \leq (\|\xi\|_p^p + 2 \|\eta\|_p^p)^{1/r}$$

where $r = p$ if $1 \leq p \leq 2$ and $r = p'$, the index conjugate to $p$, if $2 \leq p \leq \infty$.

**Proof.** Clearly

1) $\|E_1 \xi + (1 - E_1) \eta\|_1 \leq \|\xi\|_1 + 2 \|\eta\|_1$

2) $\|E_1 \xi + (1 - E_1) \eta\|_2 \leq \|\xi\|_2 + \|\eta\|_2 \leq \|\xi\|_2^2 + 2 \|\eta\|_2^2$

3) $\|E_1 \xi + (1 - E_1) \eta\|_{\infty} \leq \|\xi\|_{\infty} + 2 \|\eta\|_{\infty}$

and the assertion follows from interpolation between 1) and 2) (resp. 2) and 3)) on mixed $L^p(L^q)$-spaces (see [1]).

Let $(X, \mu)$ and $T$ be as above. Define $Y = X \times X$ and endow $Y$ with the usual product $\sigma$-algebra denoted $\mathcal{S}$. We define a measure $\nu$ on $Y$ by requiring that

$$\nu(S_0 \times S_1) = \int_X \chi_{S_0}(x) T \chi_{S_1}(x) \, d\mu(x)$$

(whenever $S_0, S_1$ are measurable subsets of $X$).

Denote by $\mathcal{S}_1$ and $\mathcal{S}_0$ the $\sigma$-algebras of sets $X \times S (S \subset X$ measurable), respectively of sets $S \times X (S \subset X$ measurable), further denote by $E_1, E_0$ the corresponding conditional expectation operators. For a measurable function $\xi$ on $X$ we define for $x = (x_0, x_1) \in Y$

$$\xi^i(x_0, x_1) = \xi(x_i) \quad i = 0, 1.$$
Further:

\[ E_0(\xi^i) = \begin{cases} (T\xi)^0 & \text{if } i = 1 \\ \xi^0 & \text{if } i = 0 \end{cases} \]

\[ E_1(\xi^0) = (T\xi)^1. \]

For a proof of these facts we refer the reader to the book of E.M. Stein [11].

**Proposition.** — Let \((X, \mu)\) be a \(\sigma\)-finite measure space, \(T\) a special operator and \(1 < p < \infty\). Then for \(\xi_1, \xi_2 \in L^p(X, \mu)\):

\[ \|T^2\xi_1 + (1 - T^2)\xi_2\| \leq (\|\xi_1\|_p^r + 2\|\xi_2\|_{p'}^r)^{1/r}, \quad \text{with } r = \min(p, p'). \]

**Proof.** — We apply the above procedure to \(T\), then

\[ \|T^2\xi_1 + (1 - T^2)\xi_2\| = \|E_0((T\xi_1)^1 + \xi_2^0 - (T\xi_2)^1)\| \leq \|(T\xi_1)^1 + \xi_2^0 - (T\xi_2)^1\| \leq \|E_1(\xi^0) + (1 - E_1)(\xi_2^0)\| \leq (\|\xi_1\|_p^r + 2\|\xi_2\|_{p'}^r)^{1/r}. \]

**Corollary.** — Let \(R, S\) be bounded operators on \(L^p(X, \mu)\), then we have

\[ \|T^2R + (1 - T^2)S\| \leq (\|R\|_r^r + 2\|S\|_{r'}^{1/r}). \]

3. The weak* topology on the unit sphere of \(W_p(G)\).

Let \(G\) be a locally compact group, with a fixed left Haar measure \(dg\) and modular function \(\Delta\). Let \(L^p(G), 1 \leq p \leq \infty\), denote the usual Lebesgue spaces with respect to \(dg\) and for functions \(f, h\) on \(G\) let be defined \(f^* h(x) = \int_G f(g) h(g^{-1} x) \, dg\),

\[ f^\sim(g) = f(g^{-1}) \Delta(g^{-1}), \quad f^* = f^\sim, \quad f^p(g) = f(g^{-1}). \]

For this section let now \(p \in (1, \infty)\) be fixed and let \(A_p(G)\) (as in [8]) be the algebra of functions \(u\) on \(G\) which can be represented as \(u = \sum v_n \ast w_n^\vee\), where

\[ \sum_n \|v_n\|_p \cdot \|w_n\|_{p'<\infty} \cdot \frac{1}{p} + \frac{1}{p'} = 1. \]
The norm on $A_p$ is defined as the inf $\Sigma \|v_n\|_p', \|w_n\|_p$ taken over all such representations of $u$.

If $f$ is an element of $L^1(G)$ then on one hand $w \mapsto f \ast w$ defines a convolution operator on $L^p(G)$ and on the other $u \mapsto \int_G f(g) u(g) \, dg$ a continuous linear functional on $A_p(G)$. From $\langle f, v \ast w^\vee \rangle = \langle f \ast w, v \rangle$ it follows that the corresponding norms of $f$ coincide.

Let $Pf_p(G)$ denote the closure of $L^1(G)$ in the algebra of convolution operators on $L^p(G)$ and $W^p(G)$ the dual space of $Pf_p(G)$, which is contained in $L^\infty(G)$, and in which $A_p(G)$ is norm non-increasingly embedded.

If $t$ is a nonnegative (almost everywhere) function with $\|t\|_1 = 1$ then $t \ast t^\sim$, as a convolution operator, is almost a special operator, except that $(G, dg)$ might not be $\sigma$-finite.

Let $U_\alpha$ be an open relatively compact neighborhood base at the identity $e$ of $G$. If $V_\alpha = V_\alpha^{-1}$ are open neighborhoods of $e$ such that $V_\alpha^2 \subset U_\alpha$ then $\tau_\alpha = \lambda(V_\alpha)^{-1} \chi_{V_\alpha}$, where $\lambda(V)$ denotes the Haar measure of $V$ and $\chi_{V}$ its characteristic function, $t_\alpha = \tau_\alpha \ast \tau_\alpha^\sim$ and $e_\alpha = t_\alpha \ast t_\alpha$ are approximate identities for $L_1(G)$, $e_\alpha$ being the square of a “special” operator. This last fact we seem really to need in the proof of the following

**Lemma.** — Let $e_\alpha = t_\alpha \ast t_\alpha$ be as above, if $u_\beta$ is a net in $W^p(G)$ such that $u_\beta \rightharpoonup u_0$ in the weak* topology of $W_p(G)$ and if $\|u_\beta\|_{W_p} \rightharpoonup \|u_0\|_{W_p}$, then for $\epsilon > 0$ there exist $\beta_0, \alpha_0$ such that

i) $\|e_{\alpha_0} \ast u_\beta - u_\beta\|_{W_p} \leq \epsilon$ for all $\beta > \beta_0$

and

ii) $\|e_{\alpha_0} \ast u_0 - u_0\|_{W_p} \leq \epsilon$.

**Proof.** — Clearly ii) is a consequence of i), so it is enough to prove i) and we may assume that $\|u_0\| = 1$. We suppose now that there is a net $u_\beta$ which converges to $u_0$ as described in the lemma and an $\epsilon > 0$ such that for all $\alpha_0, \beta_0$ there exists $\beta > \beta_0$ with

$\|e_{\alpha_0} \ast u_\beta - u_\beta\| > \epsilon$.

We shall derive a contradiction.
Let $0 < \eta < \epsilon/2$, to be specified later, and choose $f \in L^1(G)$ with
$$\|f\|_{\|f\|_p} = 1, \langle f, u_0 \rangle \geq 1 - \eta,$$
then choose $\alpha_0$ with
$$\|e_{\alpha_0} * f - f\|_{\|f\|_p} \leq \eta$$
and $\beta_0$ with
$$|\langle u_\beta, e_{\alpha_0} * f \rangle - \langle u_0, e_{\alpha_0} * f \rangle| \leq \eta,$$
$$\|u_\beta\| \leq 1 + \eta \text{ for all } \beta > \beta_0.$$ We may now fix $\beta > \beta_0$ with
$$\|e_{\alpha_0} * u_\beta - u_\beta\|_{\|u\|_w} > \epsilon$$
and find $g \in L^1(G), \|g\|_{\|g\|_p} = 1$, with
$$\langle e_{\alpha_0} * u_\beta - u_\beta, g \rangle > \epsilon - \eta$$
i.e. $\langle u_\beta, (e_{\alpha_0} * f) - (1 - e_{\alpha_0})*g \rangle = \langle u_\beta, (1 - e_{\alpha_0})*(-g) \rangle > \epsilon - \eta$.

Now, the supports of $t_{\alpha_0}, f, g$ are contained in a $\sigma$-finite open subgroup $G_0$ of $G$. Since for an $L^1(G)$ function $h$ with support in $G_0 : \|h\|_{\|h\|_p(G_0)} = \|h\|_{\|h\|_p(G)}$, we may apply the estimation of the corollary of the last section to $e_{\alpha_0} * f - \lambda g + \lambda e_{\alpha_0} * g$, where $\lambda > 0$:
$$\|e_{\alpha_0} * f + (1 - e_{\alpha_0}) * (-\lambda g)\| \leq (\|f\|_r + 2 - \lambda g\|_r)^{1/r} = (1 + 2\lambda r)^{1/r}.$$ So on one hand
$$\langle u_\beta, e_{\alpha_0} * f + (1 - e_{\alpha_0}) * (-\lambda g) \rangle \leq \|u_\beta\| (1 + 2\lambda r)^{1/r}$$
$$\leq (1 + \eta) (1 + 2\lambda r)^{1/r},$$
and on the other
$$|\langle u_\beta, e_{\alpha_0} * f + (1 - e_{\alpha_0}) * (-\lambda g) \rangle| = |\langle u_0, f \rangle + \langle u_0, e_{\alpha_0} * f - f \rangle + \langle u_\beta - u_0, e_{\alpha_0} * f \rangle + \lambda \langle e_{\alpha_0} * u_\beta - u_\beta, g \rangle| \geq 1 - 3\eta + \lambda \epsilon/2.$$ But $1 - 3\eta + \lambda \epsilon/2 \leq (1 + \eta)(1 + 2\lambda r)^{1/r}$ cannot hold for all $\eta \in (\epsilon/2, 0), \lambda > 0$.

We thank the referee for pointing out to us the following implication of the lemma (due to M. Cowling, theorem 3 of [3]; see [4] for a different proof).
COROLLARY. — Translations act continuously on $W_p(G)$.  

Proof. — For $h \in G$ let $u(g) = u(h^{-1} g)$ and $u_h(g) = u(gh)$, $g \in G$.

We first consider left translations, if $u$ is in $W_p(G)$, $\varepsilon > 0$ then we find, by the lemma, an element $e$ of $L^1(G)$ with

$$\|e \ast u - u\|_{W_p} \leq \varepsilon.$$ 

Then

$$\|h u - u\|_{W_p} \leq \|h u - h(e \ast u)\|_{W_p} + \|h(e \ast u) - e \ast u\|_{W_p}$$

$$+ \|e \ast u - u\|_{W_p}$$

$$\leq \|u - e \ast u\|_{W_p} + \|h e - e\|_{L^1} \|u\|_{W_p} + \|e \ast u - u\|_{W_p}$$

$$\leq 3\varepsilon$$ if $h$ is in a neighborhood $V$ of the identity, chosen such that $\|h e - e\|_{L^1} \leq \varepsilon \|\|u\|_{W_p}^{-1}$ for all $h \in V$.

From $\|f\|_{P_{f_p}} = \|f^\sim\|_{P_{f_p}^\sim}$, for $f \in L^1(G)$, we infer that $\|u\|_{W_p} = \|u^\sim\|_{W_p^\sim}$, for $u \in W_p(G)$, and hence the continuity of right translations, on $W_p$, follows from that of left translations on $W_p^\sim$.

It has been proved by Herz [8], that for $v \in A_p(G)$ and $u \in W_p(G)$ the pointwise product $u \cdot v$ is in $A_p(G)$ and $\|u \cdot v\|_{A_p} \leq \|u\|_{W_p} \|v\|_{A_p}$. 

We say that a net $u_\beta \in W_p(G)$ converges to $u \in W_p$ in the $A_p$-multiplier topology, if, for all $v \in A_p$, $u_\beta v \longrightarrow uv$ in $A_p$ norm.

THEOREM. — On the unit sphere $S = \{u \in W_p/\|u\|_{W_p} = 1\}$ of $W_p(G)$ the weak * and the $A_p$-multiplier topology coincide.

Proof. — Let $u_\beta, u \in S$ be such that $u_\beta \longrightarrow u$ in the weak * topology. Let $e_\alpha = t_\alpha * t_\alpha$ be as in the lemma. Then for $v \in A_p(G)$

$$\|u_\beta v - uv\| \leq \|(u_\beta - e_\alpha_0 * u_\beta)v\| + \|[e_\alpha_0 * (u_\beta - u)]v\|$$

$$+ \|([e_\alpha_0 * u - u])v\|$$

$$\leq \varepsilon \|v\| + \|[e_\alpha_0 * (u_\beta - u)]v\| + \varepsilon \|v\|,$$

when $\beta \geq \beta_0$, where $\alpha_0, \beta_0$ are chosen according to the lemma.

Since $t_\alpha_0 \in L^1(G) \cap L^\infty(G)$ has compact support we may
apply lemma 6 of [7] and find $\beta_1 > \beta_0$ such that for $\beta > \beta_1$ 
$\|e_{a_0} * (u_\beta - u)\| \leq \epsilon$.

For the converse it is sufficient to note that $u_\beta \to u$ uniformly on compact sets, whenever $u_\beta \to u$ in the $A_p$-multiplier topology and $\|u\|_{A_p}$ is bounded. So, for a compact set $K$, let $v \in A_p(G)$ be a function which takes the value one on $K$ (e.g. take $v = \lambda(U)^{-1} \chi_v \chi_K - \chi_U$, where $U$ is open, relatively compact) then

$$\sup_{g \in K} |(u_\beta - u)(g)| \leq \|(u_\beta - u) v\|_\infty \leq \|(u_\beta - u) v\|_{A_p} \to 0.$$  

The following corollary is of interest with respect to the problems considered in [6]. To state it, let, for a compact set $K \subset G$, $A_p^K(G) = \{v \in A_p(G)/\text{supp } v \subset K\}$. This space we consider as a subspace of $W_p(G)$.

**COROLLARY.** — On the unit sphere of $(A_p^K(G), \| \cdot \|_{W_p})$ the weak * and the norm topology coincide.

**Proof.** — Let $u_\beta, u \in A_p^K(G)$ be such that $u_\beta \to u$ in the weak * topology and $\|u_\beta\|_{W_p} = 1 = \|u\|_{W_p}$. Then, for $v \in A_p^K(G)$ which is constant one on $K$,

$$\|u_\beta - u\|_{W_p} = \|(u_\beta - u) v\|_{W_p} \leq \|(u_\beta - u) v\|_{A_p} \to 0$$

by our theorem. The converse is evident.

4. Addendum.

When the paper was already finished we realized that, by our method, we can improve a theorem of E.E. Granirer, theorem 3 of [6], which we think to be central in the cited paper.

Let $MA_p(G)$ be the algebra of (continuous, bounded) functions on $G$ which pointwise multiply $A_p(G)$ into itself and let for $u \in MA_p(G)$ 

$$\|u\|_{MA_p} = \sup \{\|uv\|_{A_p}/\|v\|_{A_p} = 1\}.$$  

**THEOREM.** — Let $u \in MA_p(G)$ be such that $u(g) = \|u\|_{MA_p}$ for an $g \in G$. If $u_\beta$ is a net in $MA_p(G)$ such that

$$\|u_\beta\|_{MA_p} \to \|u\|_{MA_p}$$
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and \( u_\beta \longrightarrow u \) in the \( \sigma(\text{MA}_p(G), L^1(G)) \)-topology then \( u_\beta \longrightarrow u \) in the \( \text{A}_p \)-multiplier topology.

To prove this theorem we need an auxiliary result for whose proof we use that we admit complex scalars for our linear spaces.

**Proposition.** — The linear span of \( \{ v \in \text{A}_p(G) / v(e) = \| v \|_{\text{A}_p}, v \text{ has compact support} \} \) is norm dense in \( \text{A}_p(G) \).

**Proof.** — The dual space of \( \text{A}_p(G) \) is the ultra weak operator topology closure of \( \text{Pf}_p(G) \) in the space of bounded operators on \( L^p(G) \), the duality is given by

\[
\langle T, u \rangle = \sum_{n=1}^{\infty} \int_G T w_n(g) v_n(g) \, dg
\]

when \( u = \sum_{n=1}^{\infty} v_n * w_n \in \text{A}_p(G), T \in \text{A}_p(G)^* \) (see [9]).

By theorem 4.1 and theorem 9.4 of [2] we have

\[
e^{-1} \| T \| \leq \sup \{ \langle Tf, f^* \rangle / f \in L^p(G), \| f \|_p = 1 \},
\]

where \( f^* = |f|^{p-1} \exp(-i \arg(f(.))) \) is the unique element of \( L^p(G) \) with \( \langle f, f^* \rangle = 1 \) and norm one.

If we approximate \( f \in L^p(G) \) by \( f \cdot \chi_K \), where \( K \subseteq G \) is a suitable compact set, in the \( L^p \)-norm, then \( (f \chi_K)^* = f^* \chi_K \) approximates \( f^* \) in \( L^p \)-norm. This is why we can restrict the supremum to be taken over the elements \( f \in L^p(G) \) with compact support and norm one.

If \( f \in L^p(G) \) has compact support then \( v = f^* \cdot f^* \) will have compact support too, and if \( \| f \|_p = 1 \) then,

\[
1 = \| f \|_p \cdot \| f^* \|_{p'} \geq \| v \|_{\text{A}_p} \geq \| v \|_\infty = f^* \cdot f^*(e) = \| f \|_p^p = 1.
\]

Hence for any \( T \in \text{A}_p(G)^* \):

\[
e^{-1} \| T \| \leq \sup \{ \langle T, v \rangle / v(e) = \| v \|_{\text{A}_p}, v \text{ has compact support} \},
\]

and the proposition follows by an application of the Hahn-Banach theorem.
Proof of the theorem. — We may assume $\|u\|_{MA_p} = 1$ and, since translations are isometries of $MA_p(G)$, we may further assume $u(e) = \|u\|_{MA_p} = 1$.

Since there exists $\beta_0$ such that $\sup \{ \|u_\beta\|_{MA_p}/\beta \geq \beta_0 \} < \infty$ it suffices, by the above proposition, to show $u_\beta v \longrightarrow uv$ when $v$ has compact support, say $K$, and $v(e) = \|v\|_{A_p} = 1$. Now, the $u_\beta v$ and $uv$ are elements of $A^p_K(G)$, and on this space the $W_p$-norm is equivalent to the $A_p$-norm (this follows from proposition 1 of [6] and proposition 3 of [8]). Thus we must only show $\|u_\beta v - uv\|_{W_p} \longrightarrow 0$.

Clearly, $u_\beta v \longrightarrow uv$ in the weak* topology of $A^p_K(G)$, and, if we can show that $\lim \|u_\beta v\|_{W_p} = \|uv\|_{W_p}$, then the corollary of the last section finishes the proof.

But,

$$1 = u(e)v(e) \leq \|uv\|_{W_p} \leq \lim \inf \|u_\beta v\|_{W_p}$$
and

$$1 = u(e)v(e) = \|u\|_{MA_p} \|v\|_{A_p} = \lim \|u_\beta\|_{MA_p} \|v\|_{A_p} \geq \lim \sup \|u_\beta v\|_{A_p} \geq \lim \sup \|u_\beta v\|_{W_p}$$

from which $\lim \|u_\beta v\|_{W_p} = 1 = \|uv\|_{W_p}$ follows.

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