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TOPOLOGICAL TRIVIALITY OF VERSAL UNFOLDINGS OF COMPLETE INTERSECTIONS

by James DAMON (*)

1. Introduction.

In this paper, we continue an investigation which has been pursued by a number of authors [18], [29], [24], and [4, I, II]. We are concerned with when a versal unfolding of a germ of a mapping is topologically trivial (i.e. topologically a product mapping) along certain parameter subspaces. We shall consider this problem for the case of weighted homogeneous germs $f_0 : k^s, 0 \to k^t, 0$ with $s > t$ which have finite singularity type [21] (here $k = \mathbf{R}$ or $\mathbf{C}$ and the germs may be C^∞, real analytic, or holomorphic). Such germs define complete intersections with isolated singularities.

In [4, II], the problem of topological triviality along the direction of maximal weight was reduced to proving the surjectivity of any one of a sequence of linear maps τ_i or τ_i^+ which measure the failure of relations between certain deformations of f_0 to lift to the versal unfolding. Here we shall be principally interested in verifying that certain classes of germs satisfy the conditions which imply that τ_1 or τ_1^+ is surjective.

Let $N(f_0)$ denote the space of non-trivial infinitesimal deformations of f_0 (e.g. $= m_X T_X^1$ if $f_0^{-1}(0) = X$). Then, a sufficient condition that τ_1 be surjective in « most weights » is that $N(f_0)^*$ (the dual with respect to k) is a principal $Q(f_0)$-module [4, II, thm. 6.5] ($Q(f_0)$ is the local algebra of f_0 and by principal we mean that $N(f_0)^*$ is generated by one element). One of our concerns is to investigate when this condition doesand does not hold (e.g. it fails for large families of curve singularities in k^3 and for

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intersections of pairs of generic quadrics in k^n, $n > 4$; however, it holds for a number of infinite families of surface singularities in k^4).

We will analyze the behavior of N( ) in two situations. One is when the germ F defining the singularity is obtained by adjoining a power to a lower dimensional singularity. This is analogous to the Thom-Sebastiani construction $(f(x) \to F(x,z) = f(x) + z^{\ell})$ for isolated hypersurface singularities [27]. We derive a formula for the Milnor number $\mu(F)$ and the structure of the Jacobian algebra $\mathscr{J}(F)$ and N(F) in terms of the corresponding objects before adjoining powers (§ 3,4). Secondly, we consider complete intersections obtained by intersecting a hypersurface singularity with a non-singular quadric. For such singularities we show (§ 5) that Gorenstein properties of the Jacobian algebra are related to N( )* being principal.

Using these results, we establish (§ 6-8) that for infinite families of surface singularities in k^4 beginning with the exceptional uni-modal and simple elliptic singularities, τ_1^+ is surjective. In contrast with this, there are counter-examples for unimodal curve singularities where this fails. Also, in addition to these algebraic methods, we illustrate how geometric methods can be applied for the case of intersections of pairs of generic quadrics in k^n. Taken together these methods yield:

THEOREM. — *For the infinite families of surface singularities beginning with the exceptional unimodal and simple elliptic surface singularities in k^4 (table 2) and for the complete intersections in $k^n (n \geqslant 4)$ defined by pairs of nonsingular quadrics:*

1) *the unfoldings versal in non-maximal weight are topologically versal so that the versal unfoldings are topologically trivial along the direction of maximal weight,*

2) *for unimodal germs these unfoldings are finitely determined and for the families of surface singularities these unfoldings are topologically stable as germs.*

These results can be contrasted with the results for topological stability using the Whitney conditions to determine the Thom-Mather strata [20], [21], [26], (e.g. Bruce [2], Bruce-Giblin [3], Wall [28]). Together with other results (e.g. Greuel [14]), they lead to some surprising comparisons for topological stability which will be discussed elsewhere.

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2. Preliminaries.

In this section, we will establish the notation to be used throughout this paper (it is the same as in [4, I, II]).

We consider germs $f_0 : k^s, 0 \to k^t, 0$ which may be C^∞ or real analytic when $k = \mathbf{R}$ or holomorphic when $k = \mathbf{C}$. We denote the set of such germs (for a fixed category) by $\mathscr{C}_{s,t}$. We assign local coordinates $\underset{\sim}{x}$ for k^s and $\underset{\sim}{y}$ for k^t. Then, we let $\mathscr{C}_x$ denote the algebra of germs (in the appropriate category) $k^s, 0 \to k$. This algebra has maximal ideal m_x. If $\underset{\sim}{u}$ denotes local coordinates for k^q, then an unfolding of f_0 with parameters $\underset{\sim}{u}$ is a germ f: $k^{s+q}, 0 \to k^{t+q}, 0$ such that $f(x,u) = (\bar{f}(x,u),u)$ and $\bar{f}(x,0) = f_0(x)$. We let $\mathscr{C}_{x,u}$ denote the algebra of germs $k^{s+q}, 0 \to k$. It has maximal ideal $m_{x,u}$. The germ f_0 induces the algebra homomorphism $f_0^* : \mathscr{C}_y \to \mathscr{C}_x$, and this often will not be explicitly indicated. For example, $m_y.\mathscr{C}_x$ will denote the ideal in $\mathscr{C}_x$ generated by m_y (i.e., $f_0^* m_y.\mathscr{C}_x$). We let $\theta(f_0)$ denote the module of vector fields $\zeta : k^s, 0 \to Tk^t$ such that $\pi \circ \zeta = f_0$ ($\pi : Tk^t \to k^t$ is the projection).

Then, $\theta(f_0)$ is the free $\mathscr{C}_x$-module generated by $\left\{\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_t}\right\}$. We let $\varepsilon_i = \frac{\partial}{\partial y_i}$ and $\varepsilon_i' = \frac{\partial}{\partial x_i}\cdot$ In general, the R module generated by a set of elements $\{u_1, \ldots, u_n\}$ will be denoted by $R\{u_1, \ldots, u_n\}$. If the number n is clear from context, then this module will be denoted by $R\{u_i\}$. Similarly, the vector space spanned by the $\{u_i\}$ will be denoted by $\langle u_1, \ldots, u_n\rangle$ or $\langle u_i\rangle$ if n is understood. Then, $\theta(f_0) \simeq \mathscr{C}_x\{\varepsilon_i\}$. Also, we let $\theta_s = \theta(\mathrm{id}_{k^s})(\simeq \mathscr{C}_x\{\varepsilon_i'\})$, and similarly for θ_t.

The extended $\mathscr{K}$-tangent space of f_0 is defined as

$$T\mathscr{K}_e.f_0 = \mathscr{C}_x\left\{\frac{\partial f_0}{\partial x_i}\right\} + m_y.\mathscr{C}_x\{\varepsilon_i\}.$$

Then, we define

$$\tilde{N}(f_0) = \theta(f_0)/T\mathscr{K}_e.f_0.$$

This is a $\mathscr{C}_x$-module which is annihilated by $m_y.\mathscr{C}_x$; thus it is a module over $Q(f_0) = \mathscr{C}_x/m_y.\mathscr{C}_x$, the *local algebra of* f_0. We also let $N(f_0) = m_x.\tilde{N}(f_0)$. This is a $Q(f_0)$ sub-module. If f_0 has rank 0, then $N(f_0) \simeq m_x\theta(f_0)/T\mathscr{K}_e.f_0$. We may enlarge the ideal $m_y.\mathscr{C}_x$ to include the $t \times t$ minors of df_0. This enlarged ideal $J(f_0)$ is the *Jacobian ideal of* f_0. The algebra $\mathscr{J}(f_0) = \mathscr{C}_x/J(f_0)$ is the *Jacobian algebra of* f_0. Then, by Cramer's rule it follows that $\tilde{N}(f_0)$ is also a $\mathscr{J}(f_0)$-module.

If $\dim_k \tilde{N}(f_0) < \infty$, then f_0 is said to be of *finite singularity type* (this is equivalent by a theorem of Mather [19, III] to f_0 being finitely $\mathscr{K}$-determined). Also, Mather has shown [19, IV] that this implies when $s \geqslant t$ that f_0 defines a complete intersection with isolated singularity (and is equivalent to it when $k = \mathbf{C}$). In the case $s > t$, we let $X = f_0^{-1}(0)$. Then in the notation of algebraic geometry, $\tilde{N}(f_0) = T^1_X$ and $Q(f_0) = \mathcal{O}_X$.

$\mathrm{Hom}_k(N(f_0),k) \simeq N(f_0)^*$ is also a $Q(f_0)$-module via $g.\beta(v) = \beta(g.v)$ for $g \in Q(f_0)$, $\beta \in N(f_0)^*$ and $v \in N(f_0)$. By $N(f_0)^*$ being a *principal* $Q(f_0)$-module, we mean that it is generated as a $Q(f_0)$-module by one element. This will be the appropriate generalization for us of the notion of a finite dimensional local k-algebra Q being a 0-dimensional Gorenstein algebra. In turn, this means that there is a k-linear functional $\varphi : Q \to k$ such that the pairing on Q, $(g,h) \to \varphi(g.h)$, is non-singular. The non-singularity condition is equivalent to knowing that for the maximal ideal m of Q, $\dim_k \mathrm{Ann}(m) = 1$ ($\mathrm{Ann}(m) = \{h \in Q : h.m = 0\}$) *and* φ *does not vanish identically on* $\mathrm{Ann}(m)$. The ideal $\mathrm{Ann}(m)$ is called the *socle* of Q. It is also easy to see that being Gorenstein with linear functional φ is equivalent to Q^* being generated as a Q-module by φ.

When we restrict consideration to germs f_0 which are weighted homogeneous we can assign weights $wt(x_i) = a_i$ and $wt(y_i) = d_i$ so that if $f_0 = (f_{01}, \ldots, f_{0t})$, then f_{0i} is a weighted homogeneous polynomial of weighted degree $= d_i$. We assume all $a_i, d_i > 0$. For vector fields, we let $wt(\varepsilon_i) = -wt(y_i)$ and say that $\zeta = \Sigma h_i\varepsilon_i$ is weighted homogeneous with $wt(\zeta) = \ell$ if each h_i is weighted homogeneous and $h_i = 0$ or $wt(h_i\varepsilon_i) \,(\overset{\mathrm{def}}{=} wt(h_i) + wt(\varepsilon_i)) = \ell$ for each i. For weighted homogeneous f_0, the generators of $T\mathscr{K}_e.f_0$ are weighted homogeneous; thus, if f_0 has finite singularity type, there is an *induced* weighting on $N(f_0)$. We denote the subspace of terms of weight $= m$ by $N(f_0)_m$. The maximum m for which $N(f_0)_m \neq 0$ is denoted by max wt and the subspace, by $N(f_0)_{\max}$. The germ f_0 will be called *uni-maximal* if $\dim_k N(f_0)_{\max} = 1$. Also,

$N(f_0)^*$ has a weighting such that if $h : N(f_0) \to k$ vanishes except on $N(f_0)_m$, then $wt(h) = \max wt - m$; thus, $wt(g.h) = wt(g) + wt(h)$ for weighted homogeneous $g \in Q(f_0)$.

Remark 2.1. – It is easy to see using Nakayama's lemma that if $N(f_0)^*$ is principal then $\dim_k N(f_0)_{\max} = 1$ and the projection $p : N(f_0) \to N(f_0)_{\max} \xrightarrow{\simeq} k$ is a generator for $N(f_0)^*$.

Remark 2.2. – It can also be shown that whether $N(f_0)^*$ is principal is determined generically by the weights $\{a_i\}$ and $\{d_i\}$. If there is one germ f_0 with the given weights for which $N(f_0)^*$ is principal, then it will be true for a Zariski open subset of such germs with the same weights.

3. Adjoining Powers to Complete Intersections.

We begin to investigate an f_0 which is obtained from a lower dimensional singularity by adjoining powers of new variables. In the hypersurface case, Thom and Sebastiani considered the germ $f(x) + g(z)$ formed from germs f and g defining isolated hypersurface singularities. In the case of complete intersections defining isolated singularities, there is no general operation of this type. We restrict our consideration to adjoining to $f(x)$ a z^ℓ in some coordinate. Even for this, the situation is far more complicated than the hypersurface case. For example, $f(x) + z^2$ has the same properties as f when f defines an isolated hypersurface singularity. However, in the complete intersection case, adjoining a square to a simple singularity can give a new singularity having moduli.

We begin with an analytic germ $f : k^s, 0 \to k^t, 0$ which is a complete intersection defining an isolated singularity (i.e. f has finite singularity type). We write $f = (f_1, f_2)$ where $f_1 : k^s, 0 \to k^{t-1}, 0$ and $f_2 : k^s, 0 \to k, 0$; and we form $F(x,z) = (f_1, f_2 + z^\ell) : k^{s+1}, 0 \to k^t, 0$. We ask when F also has finite singularity type. This is answered by

Proposition 3.1. – *Let* F *be obtained by adjoining a power to* $f = (f_1, f_2)$ *as above so that* $F(x,z) = (f_1(x), f_2(x) + z^\ell)$. *Then* F *has finite singularity type if and only if both* f *and* f_1 *have finite singularity type.*

Proof. – Recall by [19, IV], g has finite singularity type iff $\dim_k \mathscr{J}(g) < \infty$, where as in (§ 2) $\mathscr{J}(g) \xrightarrow{\simeq} \mathscr{C}_x/J(g)$ is the Jacobian algebra of g, and $J(g)$ its Jacobian ideal.

First suppose f and f_1 have finite singularity type, then $J(f_1) \supset m_x^r$ for some $r > 0$. From the definition of $J(F)$ we see $z^{\ell-1}J(f_1) \subset J(F)$, so $m_z^{\ell-1} m_x^r \mathscr{C}_{x,z} \subset J(F)$. Thus,

$$m_x^r(f_2+z^\ell) \equiv m_x^r f_2 \bmod J(F).$$

Hence, $m_x^r.f_2 \in J(F)$, and so $m_x^r.J(f) \subset J(F)$. Also, by assumption $m_x^q \subset J(f)$ for some $q > 0$; thus, $m_q^{r+q} \subset J(F)$ and $m_x^{r+q}.\mathscr{C}_{x,z} \subset J(F)$.

Lastly, $f_2 \in m_x$ so $f_2 m_x^{r-1} \subset m_x^r$ and $z^{\ell-1} f_2 m_x^{r-1} \subset J(F)$. Also,

$$z^{2\ell-1} m_x^{r-1} \equiv -f_2 z^{\ell-1} m_x^{r-1} \bmod J(F).$$

Thus, $z^{2\ell-1} m_x^{r-1} \subset J(F)$. It then follows by induction that $z^{m\ell-1} m_x^{r-m} \subset J(F)$. If we let $N = \max\{r+q, r\ell-1\}$, then

$$m_{x,z}^N \subset \left(m_x^{r+q} + \sum_{m=1}^{r} z^{m\ell-1} m_x^{r-m}\right)\mathscr{C}_{x,z} \subset J(F).$$

Conversely suppose that F has finite singularity type. We easily see that the image of $J(F)$ in $\mathscr{C}_{x,z}/m_z.\mathscr{C}_{x,z} \xrightarrow{\simeq} \mathscr{C}_x$ is $J(f)$. Thus, $J(f)$ has finite codimension and f has finite singularity type.

For f_1, we consider the complexification of F (which is analytic) if $k = \mathbf{R}$, and still denote it by F. Thus, we may assume $k = \mathbf{C}$. The complexified F still satisfies $m_{x,z}^N \subset J(F)$; and if the real f_1 is not of finite singularity type, neither will the complexified one be. Thus, suppose f_1 is not of finite singularity type (with $k = \mathbf{C}$). Then, there is an analytic subspace V_1, containing 0 and of dimension $\geqslant 1$, such that $J(f_1)$ vanishes on V_1. Let V_2 denote the analytic hypersurface defined by $f_2 + z^\ell = 0$. As $\dim_{\mathbf{C}} V_1 \times \mathbf{C} \geqslant 2$, $\dim_{\mathbf{C}}(V_1 \times \mathbf{C}) \cap V_2 \geqslant 1$ and it has a component W containing 0. Then $F|W = 0$; and each $t \times t$ minor of dF vanishes on W for it is a sum of terms each containing a $(t-1) \times (t-1)$ minor of df_1 as a factor. Thus $J(F)$ vanishes on W contradicting the fact that F has finite singularity type.

□

Two special cases of interest are given by the corollaries.

COROLLARY 3.2. – *Let $f: k^s, 0 \to k^2, 0$ have finite singularity type, with $f = (f_1, f_2)$. Then $F(x,z) = (f_1(x), f_2(x)+z^\ell)$ has finite singularity type iff f_1 defines an isolated hypersurface singularity.*

Also, let $g(z)$ be defined by two Pham-Brieskorn polynomials

$$g(z) = \left(\sum_{i=1}^{k} a_i z_i^{\ell_i}, \sum_{i=1}^{k} b_i z_i^{\ell_i}\right)$$

and let $A_{ij} = \det \begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix}$. Suppose $f(x) : k^s, 0 \to k^2, 0$ has finite singularity type then :

COROLLARY 3.3. — i) *For* g *to have finite singularity type it is necessary and sufficient that* $A_{ij} \neq 0$, $i \neq j$ *(or* $g \not\equiv 0$ *if* $k=1$*).* ii) $f(x) + g(z)$ *has finite singularity type if and only if* $g(z)$ *does and* $b_j f_1 - a_j f_2$ *defines an isolated singularity for* $1 \leqslant j \leqslant k$.

Proof. — By performing a linear change of coordinates in the target we see that $f(x) + g(z)$ is equivalent to a germ with, say, $z_k^{\ell_k}$ appearing in only one coordinate function. It is obtained by adjoining a power. Then, the result follows by an easy induction argument.

□

4. Milnor Number, $\mathscr{J}$, and $\tilde{\mathrm{N}}$ after Adjoining Powers.

Let $f : k^s, 0 \to k^t, 0$ be a weighted homogeneous polynomial germ (with only positive weights) so that $f = (f_1, f_2)$ as in the preceding section. If f and f_1 have finite singularity type, then we can adjoin a power and define $F(x,z) = (f_1(x), f_2(x) + z^{\ell})$, which has finite singularity type. Here we will compute the Milnor number $\mu(F)$ and determine the structure of $\mathscr{J}(F)$ and $\tilde{N}(F)$ (and hence of $N(F)$). To compute the Milnor number, we have the following formula (M. Giusti has indicated he is also aware of this formula).

PROPOSITION 4.1. — *Let* f *be a weighted homogeneous polynomial germ as above with both* f *and* f_1 *of finite singularity type. If* F *is obtained as above by adjoining* z^{ℓ} *then*

$$\mu(F) = (\ell - 1)\mu(f) + \ell\mu(f_1). \tag{4.1}$$

Proof. — The proof of this result uses a formula for μ of weighted homogeneous complete intersections f due to Greuel-Hamm [15]. For f as above,

$$\mu(f) = \sum_{\rho=1}^{r} \frac{d_1 \ldots d_t}{w_1 \ldots w_s} d_{\kappa_\rho}^{s-t} \operatorname{res}_{\eta=0}(R_\rho(\eta))$$

where $w_i = wt(x_i)$, $d_i = wt(y_i)$ and the d_{κ_ρ} are the set of distinct representatives of possibly repeated integers $d_1, \ldots, d_t$, so there is a total of r distinct weights. Also,

$$\mathrm{R}_\rho(\eta) = \frac{1}{\eta+1} \prod_{v=1}^{s} \left(\eta + 1 - \frac{w_v}{d_{\kappa_\rho}}\right) \prod_{k=1}^{t} \left(\eta + \frac{d_{\kappa_\rho} - d_k}{d_{\kappa_\rho}}\right)^{-1}$$

and $\mathrm{res}_{\eta=0}$ denotes the residue at $\eta = 0$. Applying the formula to $\mathrm{F}(x,z)$ we obtain

$$\mu(\mathrm{F}) = \sum_{\rho=1}^{r} \frac{d_1 \ldots d_t}{w_1 \ldots w_s \left(\frac{d_t}{\ell}\right)} d_{\kappa_\rho}^{s+1-t} \,\mathrm{res}_{\eta=0}\left(\mathrm{R}_\rho(\eta) . \left(\eta + 1 - \frac{\frac{d_t}{\ell}}{d_{\kappa_\rho}}\right)\right).$$

(Note: we can choose weights d_i so that $\ell | d_t$).

Then, taking d_t/d_{κ_ρ} inside res ( ) for $\mu(f)$ we have

$$(4.2) \quad \mu(\mathrm{F}) - (\ell - 1)\mu(f) = \sum_{\rho=1}^{r} \frac{d_1 \ldots d_{t-1}}{w_1 \ldots w_s} d_{\kappa_\rho}^{s-(t-1)}$$
$$\mathrm{res}_{\eta=0}\left(\mathrm{R}_\rho(\eta)\left(\ell\left[\eta + 1 - \frac{\frac{d_t}{\ell}}{d_{\kappa_\rho}}\right] - (\ell-1)\frac{d_t}{d_{\kappa_\rho}}\right)\right).$$

Now,

$$\ell\left[\eta + 1 - \frac{\frac{d_t}{\ell}}{d_{\kappa_\rho}}\right] - (\ell - 1)\frac{d_t}{d_{\kappa_\rho}} = \ell\left(\eta + 1 - \frac{d_t}{d_{\kappa_\rho}}\right) = \ell\left(\eta + \frac{d_{\kappa_\rho} - d_t}{d_{\kappa_\rho}}\right).$$

Also,

$$\mathrm{R}_\rho(\eta)\left(\eta + \frac{d_{\kappa_\rho} - d_t}{d}\right) = \frac{1}{\eta + 1}$$
$$\cdot \prod_{v+1}^{s} \left(\eta + 1 - \frac{w_v}{d_{\kappa_\rho}}\right) \prod_{k=1}^{t-1} \left(\eta + \frac{d_{\kappa_\rho} - d_k}{d_{\kappa_\rho}}\right)^{-1} = \mathrm{R}_\rho^{(1)}(\eta)$$

which is the form of $\mathrm{R}_\rho(\eta)$ appearing in the formula for $\mu(f_1)$. Thus, from (4.2) we obtain

$$\mu(\mathrm{F}) - (\ell - 1)\mu(f)$$
$$= \ell\left(\sum_{\rho=1}^{r} \frac{d_1 \ldots d_{t-1}}{w_1 \ldots w_s} d_{\kappa_\rho}^{s-(t-1)} \,\mathrm{res}_{\eta=0}\,(\mathrm{R}_\rho^{(1)}(\eta))\right) = \ell\mu(f_1).$$

Remark. — If d_{κ_ρ} does not occur among $d_1, \ldots, d_{t-1}$, then $R_\rho^{(1)}(\eta)$ is holomorphic at $\eta = 0$ so there is no contribution from this term in this case.

□

The importance of knowing μ is that by a result of Greuel [12, III, 3.1], if f is weighted homogeneous of finite singularity type,

$$\mu(f) = \begin{cases} \dim_k \tilde{N}(f) & s > t \\ \delta(f) - 1 & s = t \end{cases} \tag{4.3}$$

where $\delta(f) = \dim_k Q(f)$ with $Q(f) = \mathscr{C}_x / f^* m_y . \mathscr{C}_x$.

Remark. — If $s = t$, then how much $\delta(f)$ deviates from $\dim_k \tilde{N}(f)$ is an important deformation property of f. If we write $\dim_k \tilde{N}(f) = \delta(f) + \lambda$, then $\lambda + 1$ is the maximum length of a string of successive flat deformations of f which can be made [7]; or using the methods of [8, § 5], it can also be shown to be a relative codimension of the orbit of $f^* m_y . \mathscr{C}_x$ in an appropriate Hilbert Scheme.

A special case of interest to us is

Corollary 4.4. — *Let f and* F *be as above but with $t = 2$ and f_1 a non-singular quadric. Then*

$$\dim_k \tilde{N}(F) = \begin{cases} (\ell - 1) \dim_k \tilde{N}(f) + \ell & s > 2 \\ (\ell - 1)\, \delta(f) + 1 & s = 2 \end{cases}$$

Knowing $\mu(F)$, we turn to the question of explicitly determining $\tilde{N}(F)$ and $\mathscr{J}(F)$. This is given by

Proposition 4.5. — *Consider the germ $F(x,z) = (f_1(x), f_2(x) + z^\ell)$ obtained by adjoining z^ℓ to the germ $f : k^s, 0 \to k^t, 0$ (where f need not be weighted homogeneous), with both f and f_1 of finite singularity type. Let $\mathscr{C}_z / m_z^r$ be denoted by A_r. Then, there are sequences which are right exact :*

$$\text{(4.6}a\text{)}\quad 0 \to \tilde{N}(f_1) \tilde{\otimes} A_\ell \xrightarrow{\nu} \tilde{N}(F) \xrightarrow{\rho} \tilde{N}(f) \otimes A_{\ell-1} \to 0$$

$$\text{(4.6}b\text{)}\quad 0 \to \mathscr{J}(f_1) \tilde{\otimes} A_\ell \xrightarrow{\nu'} \mathscr{J}(F) \xrightarrow{\rho'} \mathscr{J}(f) \otimes A_{\ell-1} \to 0 .$$

If f is weighted homogeneous and $s > t$ then the sequences are exact. Also, (4.6 *a*) *can be viewed as a sequence of* $\mathcal{J}$(F)*-modules, while for* (4.6 *b*) ρ' *is an algebra homomorphism and* ν' *is a homomorphism of* $\mathcal{J}$(F)*-modules.*

Remarks. — All tensor products are taken over k. The $\tilde{\otimes}$ indicates that a twisted $\mathcal{J}$(F)-module structure is defined on the tensor product. An alternate way of representing $(\ )\tilde{\otimes} A_\ell$ as a $\mathcal{J}$(F)-module would be to replace it in the proposition by

$$(\quad)\otimes_{\mathscr{C}_x}\tilde{A} \text{ where } \tilde{A}=\mathscr{C}_x[z]/(z^\ell+f_2(x)).$$

In this paper we will only use tensor products over k.

Proof. — First consider (4.6 *a*). The map ρ is induced by the quotient map $\tilde{N}(F)\to\tilde{N}(F)/m_z^{\ell-1}\tilde{N}(F)$. This is because

$$\tilde{N}(F)/m_z^{\ell-1}\tilde{N}(F)\xrightarrow{\simeq}\mathscr{C}_{x,z}\{\varepsilon_i\}/T\mathscr{K}_eF+m_z^{\ell-1}\mathscr{C}_{x,z}\{\varepsilon_i\}$$

and in

$$\mathscr{C}_{x,z}\{\varepsilon_i\}/m_z^{\ell-1}\mathscr{C}_{x,z}\{\varepsilon_i\}\xrightarrow{\simeq}\mathscr{C}_x\{z^j\varepsilon_i;\,1\leqslant i\leqslant t,0\leqslant j\leqslant \ell-2\}$$

the image of $T\mathscr{K}_e.F$ is $(\mathscr{C}_z/m_z^{\ell-1}).(T\mathscr{K}_e.f)$. Hence, taking quotients we have

$$\tilde{N}(F)/m_z^{\ell-1}\tilde{N}(F)\xrightarrow{\simeq}\tilde{N}(f)\otimes A_{\ell-1}. \tag{4.7}$$

Then, ρ is the composition of the isomorphism (4.7) and the quotient map.

Consider the endomorphism of $\tilde{N}(F)$ given by multiplication by $z^{\ell-1}$. It has image $=\ker(\rho)$ and it also annihilates $\mathcal{J}(F).\varepsilon_t$ (in $\tilde{N}(F)$). Thus, there is an induced map

$$\tilde{\nu}:\ \tilde{N}(F)/\mathcal{J}(F).\varepsilon_t\to\tilde{N}(F)$$

with $\operatorname{Im}(\tilde{\nu})=\ker(\rho)$. We first construct a $\mathcal{J}(f_1)$-module isomorphism between $\tilde{N}(F)/\mathcal{J}(F).\varepsilon_t$ and $\tilde{N}(f_1)\otimes A_\ell$. Note that $\tilde{N}(F)/\mathcal{J}(F).\varepsilon_t$ can be identified with the image of $\tilde{N}(F)$ under the projection onto the first $t-1$ factors. Thus,

$$\tilde{N}(F)/\mathcal{J}(F).\varepsilon_t\xrightarrow{\simeq}\mathscr{C}_{x,u}\{\varepsilon_i\}_{i=1}^{t-1}/(\overline{T\mathscr{K}_e.f_1}+(f_2+z^\ell).\mathscr{C}_{x,u}\{\varepsilon_i\}_1^{t-1}) \tag{4.9}$$

.

where $\overline{\mathrm{T}\mathscr{K}_e.f_1}$ denotes the $\mathscr{C}_{x,u}$-module generated by the generators of $\mathrm{T}\mathscr{K}_e.f_1$. Via (4.9) there is a natural map of $\mathscr{J}(f_1)$-modules

$$\psi : \tilde{\mathrm{N}}(f_1) \otimes \mathrm{A}_\ell \to \tilde{\mathrm{N}}(\mathrm{F})/\mathscr{J}(\mathrm{F}).\varepsilon_t$$

sending $\varphi \otimes z^j \to z^j\varphi$. By examining fixed powers of z^j we see that this map is injective. Furthermore, by the division theorem, for any $g(x,z).\varepsilon_i$, $i < t$, we may divide in $\mathscr{C}_{x,z}$

$$g = \mathrm{G}.(z^\ell + f_2) + \sum_{i=0}^{\ell-1} g_i(x).z^i. \tag{4.10}$$

Thus, (4.10) implies (via (4.9)) that $\overline{g.\varepsilon_i}$ (the image in $\tilde{\mathrm{N}}(\mathrm{F})/\mathscr{J}(\mathrm{F}).\varepsilon_t$) belongs to $\mathrm{Im}(\psi)$. Hence, ψ is also onto. Hence, ψ is an isomorphism of $\mathscr{J}(f_1)$ modules. Thus, via ψ^{-1} we obtain a $\mathscr{J}(\mathrm{F})$-module structure on $\tilde{\mathrm{N}}(f_1) \otimes \mathrm{A}_\ell$ extending the $\mathscr{J}(f_1)$-module structure. To describe this « twisted » $\mathscr{J}(\mathrm{F})$-module structure, it is sufficient to describe multiplication by z. It is given by

$$z.(\varphi \otimes z^i) = \begin{cases} \varphi \otimes z^{i+1} & i < \ell - 1 \\ -f_2\varphi \otimes 1 & i = \ell - 1. \end{cases}$$

Then, $\nu = \tilde{\nu} \circ \psi$.

In the special case when f is weighted homogeneous and $s > t$, we use (4.3) to obtain from (4.6 *a*)

$$\mu(\mathrm{F}) = \dim_k(\mathrm{Im}\,\nu) + (\ell-1)\mu(f).$$

Together with (4.1), this implies $\dim_k(\mathrm{Im}\,\nu) = \ell.\mu(f_1)$ so that ν is injective.

The proof for (4.6 *b*) is similar. An explicit examination of the generators of $\mathscr{J}(\mathrm{F})$ yields

$$\mathscr{J}(\mathrm{F})/m_z^{\ell-1}.\mathscr{J}(\mathrm{F}) \xrightarrow{\simeq} \mathscr{C}_{x,z}/(\mathrm{J}(f).\mathscr{C}_{x,z} + m_z^{\ell-1}\mathscr{C}_{x,z})$$
$$\xrightarrow{\simeq} \mathscr{J}(f) \otimes \mathrm{A}_{\ell-1}.$$

As before, multiplication by $z^{\ell-1}$ induces a $\mathscr{J}(\mathrm{F})$-module endomorphism of $\mathscr{J}(\mathrm{F})$ with image $= \ker(\rho')$ and with kernel containing $\mathrm{J}(f_1).\mathscr{J}(\mathrm{F})$. Thus, it induces a homomorphism

$$\tilde{\nu}' : \mathscr{J}(\mathrm{F})/\mathrm{J}(f_1).\mathscr{J}(\mathrm{F}) \to \mathscr{J}(\mathrm{F}).$$

As before we construct a $\mathscr{J}(f_1)$-module isomorphism

$$\psi' : \mathscr{J}(f_1) \otimes A_\ell \xrightarrow{\simeq} \mathscr{J}(F)/J(f_1).\mathscr{J}(F).$$

Then, $v' = \tilde{v}' \circ \psi'$ and we use ψ' to extend the $\mathscr{J}(f_1)$-module structure on $\mathscr{J}(f_1) \otimes A_\ell$ to a « twisted » $\mathscr{J}(F)$-module structure. Again by a result of Greuel [13, cor. 5.8] if f is weighted homogeneous and $s > t$, $\dim_k \mathscr{J}(f) = \mu(f)$. Then, we proceed as before to obtain exactness.

□

On consequence of proposition 4.5 is a formula for $\tilde{N}(F)_{\geqslant 0}$.

Corollary 4.11. — *With the preceding notation with f weighted homogeneous, the following sequence is right exact (and exact if $s>t$)*

$$(4.12) \quad 0 \to (\tilde{N}(f_1) \tilde{\otimes} A_\ell)_{\geqslant(-(\ell-1).wt(z))} \xrightarrow{v} \tilde{N}(F)_{\geqslant 0} \xrightarrow{\rho} (\tilde{N}(f) \otimes A_{\ell-1})_{\geqslant 0} \to 0.$$

From this we can see how quickly the number of moduli increases when powers are adjoined. For example, consider $f : k^s, 0 \to k^2, 0$ with $d_1 = wt(f_1) < d_2 = wt(f_2)$. Let $F(x,z) = (f_1(x), f_2(x)+z^\ell)$. For F to be simple we must have $\tilde{N}(F)_{\geqslant 0} = 0$. If $s > 2$, then F is not simple for $z^{2\ell-2}\varepsilon_1$ is a non-zero element of $N(F)_{\geqslant}$. Similarly, for F to be unimodal (i.e. $\dim_k \tilde{N}(F)_{\geqslant 0} = 1$) with $s > 2$, we must have $\mu(f_1) = 1$ and $\ell = 2$ since

$$v(\tilde{N}(f_1) \tilde{\otimes} m_\ell) \subset \tilde{N}(F)_{\geqslant 0} \quad (wt(z^\ell.\varepsilon_1) > 0).$$

Nonetheless, we shall see, for example, that most unimodal surface singularities in k^4 arise from adjoining powers.

5. Gorenstein Properties of the Jacobian Algebra.

As before, let $f = (f_1, f_2) : k^s, 0 \to k^t, 0$ with $s > t$ and $f_1 : k^s, 0 \to k^{t-1}, 0$ and $f_2 : k^s, 0 \to k, 0$. Also, suppose f is weighted homogeneous with both f and f_1 of finite singularity type. Thus, we can adjoin z^ℓ to obtain $F(x,z) = (f_1(x), f_2(x)+z^\ell)$. Then, $\tilde{N}(F)^*$ being principal is related to properties of $\tilde{N}(f_1)^*$ and $\mathscr{J}(f)$ via

Theorem 5.1. — *With the preceding notation, $\tilde{N}(F)^*$ is a principal $\mathscr{J}(F)$-module iff $\tilde{N}(f_1)^*$ is principal and $\mathscr{J}(f)$ is Gorenstein.*

For the proof of the theorem, we make use of one additional exact sequence involving $\tilde{N}(F)$.

LEMMA 5.2. — *There is an exact sequence of* $\mathscr{J}(F)$*-modules*

$$0 \to \mathscr{J}(f) \otimes A_{\ell-1} \xrightarrow{\lambda} \tilde{N}(F) \xrightarrow{\pi} \tilde{N}(f_1) \tilde{\otimes} A_\ell \to 0$$

(where the $\mathscr{J}(F)$*-module structure on* $\mathscr{J}(f) \otimes A_{\ell-1}$ *is via* ρ' *of* (4.6 *b*) *and on* $\tilde{N}(F_1) \tilde{\otimes} A_\ell$ *it is the twisted structure defined in the same proposition. Also* $\lambda(g(x) \otimes z^i) = z^i g(x)\varepsilon_t$).

First, we use Lemma 5.2 to prove the theorem. We need the additional observation that by proposition 4.5 $v'(\mathscr{J}(f_1) \tilde{\otimes} A_\ell)$ annihilates $\mathscr{J}(f) \otimes A_{\ell-1}$. Thus, if $p \in \tilde{N}(F)^*$ and $g \in v'(\mathscr{J}(f_1) \tilde{\otimes} A_\ell)$ then using lemma 5.2 $g.p$ vanishes on $\lambda(\mathscr{J}(f) \otimes A_{\ell-1})$ and hence induces an element of $(\tilde{N}(f_1) \tilde{\otimes} A_\ell)^*$. Thus,

$$(5.3) \qquad v'(\mathscr{J}(f_1) \tilde{\otimes} A_\ell).p \subset (\tilde{N}(f_1) \tilde{\otimes} A_\ell)^*.$$

If now $\tilde{N}(F)^*$ is principal, then by Remark 2.1, it has a generator p which vanishes on non-maximal weights. By the first observation, the $\mathscr{J}(F)$-module structure on $(\mathscr{J}(f) \otimes A_{\ell-1})^*$ factors to make $(\mathscr{J}(f) \otimes A_{\ell-1})^*$ a $\mathscr{J}(f) \otimes A_{\ell-1}$ module via its algebra structure. If $p_1 = p \circ \lambda$, then

$$(5.4) \qquad \mathscr{J}(F).(p_1) = (\mathscr{J}(f) \otimes A_{\ell-1}).p_1 \subset (\mathscr{J}(f) \otimes A_{\ell-1})^*.$$

Then, by dimension count (since $\dim_k \mathscr{J}(F) = \dim_k \tilde{N}(F)$), if $\tilde{N}(F)^*$ is generated by p then we must have equality in both (5.3) and (5.4).

Then, for example, equality in (5.4) is equivalent to $\mathscr{J}(f) \otimes A_{\ell-1}$ being Gorenstein. Since the algebra structure is that of tensor product

$$\text{socle}\,(\mathscr{J}(f) \otimes A_{\ell-1}) = \text{socle}\,(\mathscr{J}(f)) \otimes \text{socle}\,(A_{\ell-1}).$$

Thus, being Gorenstein implies $\dim_k \text{socle}\,(\mathscr{J}(f) \otimes A_{\ell-1}) = 1$ so $\dim_k \text{socle}\,(\mathscr{J}(f)) = 1$ and hence $\mathscr{J}(f)$ is Gorenstein. Similarly, suppose we have equality in (5.3). Note that

$$v'(\mathscr{J}(f_1) \tilde{\otimes} A_\ell).p = (\mathscr{J}(f_1) \tilde{\otimes} A_\ell).z^{\ell-1}.p.$$

However, $g.z^{\ell-1}.p$ on $\tilde{N}(f_1) \tilde{\otimes} A_\ell$ is just $g.p$ on $v(\tilde{N}(f_1) \tilde{\otimes} A_\ell)$ where v is given in proposition 4.5 as multiplication by $z^{\ell-1}$. Thus,

$\nu(\tilde{N}(f_1)\tilde{\otimes} A_\ell)^*$ is a principal $\mathscr{J}(f_1)\ \tilde{\otimes}\ A_\ell$-module generated by p. Now, an argument similar to that given for $\mathscr{J}(f)\otimes A_{\ell-1}$ shows that $\tilde{N}(f_1)^*$ is a principal $\mathscr{J}(f_1)$-module.

For the converse, suppose $\tilde{N}(f_1)$ principal and $\mathscr{J}(f)$ Gorenstein. Let $\tilde{\omega}$ generate the socle of $\mathscr{J}(f)$ and ω' generate the subspace of maximal weight of $\tilde{N}(f_1)$. Then, $\tilde{N}(f_1)\ \tilde{\otimes}\ A_\ell$ is a principal $\mathscr{J}(f_1)\ \tilde{\otimes}\ A_\ell$ module so there is a $p\in\tilde{N}(F)^*$, with $p(z^{2\ell-2}\omega')=1$ (here $z^{2\ell-2}\omega'=\nu(\omega'\otimes z^{\ell-1})$) and vanishing on non-maximal weights in $\nu(\tilde{N}(f_1)\tilde{\otimes} A_\ell)$ so that $\nu(\tilde{N}(f_1)\tilde{\otimes} A_\ell)^*$ is generated as a $\mathscr{J}(f_1)\ \tilde{\otimes}\ A_\ell$-module by p. Then, we have equality in (5.3) (for the p we have just defined). Similarly, if p_2 is defined so $p_2(\tilde{\omega}\otimes z^{\ell-2})=1$ and vanishes on lower weights then $(\mathscr{J}(f)\otimes A_{\ell-1})^*$ is generated as a $\mathscr{J}(f)\otimes A_{\ell-1}$ module by p_2. Thus, to prove $\tilde{N}(F)^*$ is principal it is sufficient to show that

$$z^{2\ell-2}\omega'\equiv z^{\ell-2}.\tilde{\omega}.\varepsilon_t \text{ in } \tilde{N}(F). \tag{5.5}$$

This is because (5.3) will be an equality by the choice of p while $\lambda^*p(\omega'\otimes z^{\ell-2})=1$ and vanishes on lower weights. Thus, $\lambda^*p=p_2$ so (5.4) is also an equality and p generates $\tilde{N}(F)^*$. For (5.5) it is sufficient to know

$$z^{\ell}.\omega'\equiv\tilde{\omega}.\varepsilon_t \text{ in } \tilde{N}(F),$$

or

$$-f_2.\omega'\equiv\tilde{\omega}.\varepsilon_t \text{ in } \tilde{N}(F).$$

Now, by the assumption of maximal weight of ω', $f_2.\omega'\equiv 0$ in $\tilde{N}(f_1)$. Thus, there is a vector field $\xi\in\theta_s$ so that

$$f_2\omega'\equiv\xi(f_1)\ \mathrm{mod}\, f_1^*m_{y_1}.\theta(f_1).$$

Define $\omega=\xi(f_2)$. Then, by construction $f_2\omega'\equiv-\omega\varepsilon_t$ *in* $\tilde{N}(F)$. The theorem is then completed by the following lemma.

LEMMA 5.6. — *If $\tilde{N}(f_1)^*$ is principal and $\mathscr{J}(f)$ is Gorenstein then ω (defined above) generates the socle of $\mathscr{J}(f)$.*

It remains to prove the lemmas.

Proof of lemma 5.2. — Consider $\alpha_1:\mathscr{J}(F)\to\tilde{N}(F)$ sending $g\mapsto g.\varepsilon_t$. Then,

$$\mathrm{coker}\,(\alpha_1)\xrightarrow{\simeq}\tilde{N}(F)/\mathscr{J}(F).\varepsilon_t\xrightarrow{\simeq}\tilde{N}(f_1)\ \tilde{\otimes}\ A_\ell$$

where the last isomorphism follows by the proof of proposition 4.5. Also, as $z^{\ell-1}$ annihilates ε_t in $\tilde{\mathrm{N}}(\mathrm{F})$, by (4.6 *b*) $\nu'(\mathscr{J}(f_1)\tilde{\otimes}\mathrm{A}_\ell) \subset \ker(\alpha_1)$. Since ν' is injective, a dimension count shows we have equality; thus 4.6 *b* implies $\mathscr{J}(\mathrm{F})/\ker(\alpha_1) \xrightarrow{\simeq} \mathscr{J}(f) \otimes \mathrm{A}_{\ell-1}$.

□

Proof of lemma 5.6. — As $\mathscr{J}(f)$ is assumed Gorenstein, it is only necessary to show that ω is a non-zero element of socle $(\mathscr{J}(f))$. Define a homomorphism $\alpha_2 : \mathscr{J}(f) \to \tilde{\mathrm{N}}(f)$ sending $g \to g.\varepsilon_t$, then if $\mathrm{K} = \ker(\alpha_2)$, it is a non-zero ideal and hence its non-zero elements of maximal weight belong to socle $(\mathscr{J}(f))$, thus socle $(\mathscr{J}(f)) \subset \mathrm{K}$. It is sufficient to show $\omega \in \mathrm{K}$ and has maximum weight in K.

By an earlier argument we have

$$\operatorname{coker}(\alpha_2) \xrightarrow{\simeq} \tilde{\mathrm{N}}(f)/\mathscr{J}(f).\varepsilon_t \xrightarrow{\simeq} \tilde{\mathrm{N}}(f_1)/f_2.\tilde{\mathrm{N}}(f_1).$$

As $\dim_k \mathscr{J}(f) = \dim_k \tilde{\mathrm{N}}(f)$, we have

$$\dim_k \ker(\alpha_2) = \dim_k \operatorname{coker}(\alpha_2),$$

or $\dim_k \mathrm{K} = \dim_k \tilde{\mathrm{N}}(f_1)/f_2.\tilde{\mathrm{N}}(f_1)$. Similarly, multiplication by f_2 in $\tilde{\mathrm{N}}(f_1)$ induces a homomorphism with $\operatorname{coker} = \tilde{\mathrm{N}}(f_1)/f_2.\tilde{\mathrm{N}}(f_1)$ and

$$\ker = \mathrm{B} = \{\varphi \in \tilde{\mathrm{N}}(f_1); f_2.\varphi = 0 \text{ in } \tilde{\mathrm{N}}(f_1)\}.$$

Again $\dim_k \mathrm{B} = \dim_k \tilde{\mathrm{N}}(f_1)/f_2.\tilde{\mathrm{N}}(f_1)$. Hence, $\dim_k \mathrm{K} = \dim_k \mathrm{B}$. By assumption ω' generates the subspace of maximal weight in B. We shall show the operation by which we constructed ω from ω' extends to give an isomorphism $\mathrm{B} \xrightarrow{\simeq} \mathrm{K}$ which changes weights by a fixed amount. Then, ω has the desired property.

To define a map $\beta : \mathrm{B} \to \mathrm{K}$, let $\varphi \in \theta(f_1)$ project to an element $\bar{\varphi}$ of B. Then there is a $\xi \in \theta_s$ so that

$$f_2.\varphi \equiv \xi(f_1) \bmod f_1^* m_{y_1}\theta(f_1).$$

We define $\beta(\bar{\varphi}) = \xi(f_2)$ viewed as an element of $\mathscr{J}(f)$. By its definition

$$\beta(\bar{\varphi}).\varepsilon_t \equiv \xi(f) - f_2.\varphi \bmod f_1^* m_{y_1}\theta(f_1).$$

Then $\beta(\bar{\varphi}).\varepsilon_t \in T\mathscr{K}_e.f$; so in $\tilde{N}(f)$, $\beta(\bar{\varphi}).\varepsilon_t = 0$ or $\beta(\bar{\varphi}) \in K$. A priori it is not even clear that β is well-defined.

PROPOSITION 5.7. — *For f, f_1 weighted homogeneous of finite singularity type and $s > t$ (but without assumptions of Gorenstein, etc.) $\beta : B \to K$ is a (well-defined) isomorphism of $\mathscr{J}(f_1)$-modules.*

Proof. — First suppose $\varphi \in \theta(f_1)$ projects to a non-zero element $\bar{\varphi}$ of B. We claim $\beta(\bar{\varphi}) \neq 0$. Otherwise consider F obtained by adjoining z^2. By assumption, $\beta(\bar{\varphi}) \in J(f)$. Then, $\beta(\bar{\varphi}).\varepsilon_t \in J(f).\varepsilon_t$. By considering the generators of $J(f)$, we see $J(f).\varepsilon_t \subset T\mathscr{K}_e.F$. Also,

$$f_2.\varphi + \beta(\bar{\varphi}).\varepsilon_t = \xi(f) \bmod T\mathscr{K}_e.F.$$

As $\xi \in \theta_s$, $\xi(f) = \xi(F)$, so $f_2.\varphi \in T\mathscr{K}_e.F$. Now in $\tilde{N}(F)$, $f_2.\bar{\varphi} \equiv -z^2\bar{\varphi} = -\nu(\bar{\varphi} \otimes z)$ (ν as in proposition 4.5). Thus, $\nu(\bar{\varphi} \otimes z) = 0$ but $\bar{\varphi} \otimes z \neq 0$ in $\tilde{N}(f_1) \tilde{\otimes} A_2$, contradicting the injectivity of ν.

Claim 2. — If $\{\bar{\varphi}_i\}$ are linearly independent in B then $\{\beta(\bar{\varphi}_i)\}$ are linearly independent (for any choice of $\beta(\bar{\varphi}_i)$ since β is still not necessarily well-defined). If $\Sigma c_i\beta(\bar{\varphi}_i) = 0$ and some $c_i \neq 0$ then $\Sigma c_i\bar{\varphi}_i \neq 0$. However, $\Sigma c_i\beta(\bar{\varphi}_i)$ is at least one value for $\beta(\Sigma c_i\bar{\varphi}_i)$. This contradicts claim 1.

Claim 3. — β is a well-defined isomorphism of vector spaces. If it is well-defined then it is an isomorphism by claim 2 since K and B have the same dimension. Suppose there are two distinct possible values for $\beta(\bar{\varphi}), h_1, h'_1$. Extend $\bar{\varphi}$ to a basis $\{\bar{\varphi}_i\}$ for B so $\bar{\varphi}_1 = \bar{\varphi}$ and pick values $h_i = \beta(\bar{\varphi}_i)$ so $\beta(\bar{\varphi}_1) = h_1$. The $\{h_i\}$ are linearly independent by claim 2 and so form a basis for K. Let $h'_1 = \Sigma c_ih_i$ with some $c_i \neq 0, i > 1$. Then, $\bar{\varphi}' = \bar{\varphi} - \Sigma c_i\bar{\varphi}_i \neq 0$, but has a value $= 0$ for $\beta(\bar{\varphi}')$, contradicting claim 1.

Lastly, β is clearly $\mathscr{C}_x$-linear. Since B is a $\mathscr{J}(f_1)$-module and β is well-defined, K is also a $\mathscr{J}(f_1)$-module and β is a $\mathscr{J}(f_1)$-module homomorphism.

□

Remark. — The preceding proposition is also a type of duality result in that it establishes an isomorphism between the $\mathscr{J}(f_1)$-submodule of elements of $\tilde{N}(f_1)$ *annihilated* by f_2 *and the ideal of* $\mathcal{J}(f)$ which *annihilates* the element ε_t in $\tilde{N}(f)$ (and this latter ideal is not even a priori a $\mathcal{J}(f_1)$-module).

As an important special case we consider $f: k^s, 0 \to k^2, 0$ with $s > 2$ and f_1 a non-singular quadric. By a weighted homogeneous change of coordinates we may write f_1 in the form

$$f_1 = \sum_{i=1}^{\ell} x_i x_{2\ell-i+1} + \sum_{i=2\ell+1}^{s} \delta_i x_i^2 \quad (\delta_i = \pm 1).$$

Then, $B = \tilde{N}(f_1) = \mathcal{C}_x/m_x$; thus, K has dimension $= 1$ and is generated by $\omega = \beta(1)$, which in this case is given by the formula

$$(d_1 d_2^{-1})\omega = \sum_{i=1}^{\ell} \frac{\partial f_2}{\partial x_i} \cdot \frac{\partial f_2}{\partial x_{2\ell-i+1}} + \frac{1}{4} \cdot \sum_{i=2\ell+1}^{s} \delta_i \left(\frac{\partial f_2}{\partial x_i}\right)^2 .$$

COROLLARY 5.8. — *With f as above:*

i) *if $\mathcal{J}(f)$ is Gorenstein, then $\tilde{N}(F)^*$ is a principal $\mathcal{J}(F)$-module.*

ii) *if instead $\mathcal{J}(f)/(\omega)$ is Gorenstein then $N(f)^*$ is a principal $\mathcal{J}(f)$-module.*

Proof. — i) is a direct consequence of Theorem 5.1. For ii), by proposition 5.7, $\mathcal{J}(f)/(\omega) \xrightarrow{\simeq} \mathrm{Im}(\alpha_2)$. While

$$\operatorname{coker}(\alpha_2) \xrightarrow{\simeq} \tilde{N}(f_1) = \mathcal{C}_x/m_x .$$

Thus $m_x . \mathcal{J}(f)/(\omega) \xrightarrow{\simeq} N(f)$, and $\mathcal{J}(f)/(\omega)$ Gorenstein implies $(m_x . \mathcal{J}(f)/(\omega))^*$ is principal.

□

6. Sufficient conditions for topological triviality in versal unfoldings.

In preparation for the next two sections we must recall the conditions that must be verified to apply [4, I, II] to obtain topological triviality in a versal unfolding. Let $f_0 : k^s, 0 \to k^t, 0$ be a weighted homogeneous germ of finite singularity type (and of rank $=0$). Suppose f_0 is unimaximal. Using a basis $\{\varphi_j\}$ for $N(f_0)_{<\max wt}$, there is defined a map τ_0 which has image

$$\mathrm{Im}(\tau_0) = \mathscr{C}_x\left\{\frac{\partial f_0}{\partial x_i}\right\} + \mathscr{C}_y\{\varepsilon_i, \varphi_j\}.$$

We *first require* that $m_y . N(f_0)_{\max} \subset \mathrm{Im}(\tau_0)$. If this is satisfied then there are defined maps

$$\tau_1 : \ker(\tau_0) \rightarrow (m_u/m_u^2) . N(f_0)_{\max}$$

or τ_1^+ if m_u is replaced by m_{u_+} ($\underline{u}$ denotes unfolding parameters for $f(x,u) = (f_0(x) + \Sigma u_i \varphi_i, \underline{u})$ and u_+ denotes the parameters of positive weight; f is said to be *versal in non-maximal weight*). The surjectivity of either τ_1 or τ_1^+ is sufficient to conclude that the versal unfolding of f_0 is topologically trivial along the parameter subspace corresponding to $N(f_0)_{\max}$ [4, II]. In turn, if $N(f_0)^*$ is principal then τ_1 and τ_1^+ will be surjective in weights $\neq d_i + \max wt$. The *remaining condition requires* a verification that τ_1 (or τ_1^+) is surjective in weights $= d_i + \max wt$.

The first condition can be disposed of in most of our cases by the following simple lemma:

LEMMA 6.1. — *Let $f_0 : k^s, 0 \rightarrow k^2, 0$ be a weighted homogeneous unimaximal germ with $d_1 < d_2$ and $d_1 \nmid d_2$. If for $i = 1, 2$ there are generators of $N(f_0)_{\max}$ of the form $h_i \varepsilon_i$, then $m_y N(f_0)_{\max} \subset \mathrm{Im}(\tau_0)$.*

Proof. — Let ψ generate $N(f_0)_{\max}$. Then, by the preparation theorem it follows that

$$\mathrm{Im}(\tau_0) + \mathscr{C}_y\{\psi\} = \mathscr{C}_x\{\varepsilon_i\}.$$

Elements of $\mathscr{C}_y\{\psi\}$ have weights of the form $\max wt + nd_1 + md_2; n, m \geqslant 0$. Hence, τ_0 is surjective in weights not of this form. For $i = 1, 2$

$$\theta(f_0)_{\max wt} = \mathrm{Im}(\tau_0)_{\max wt} + \langle h_i \varepsilon_i \rangle.$$

Thus, it is sufficient to show $y_1 h_1 \varepsilon_1, y_2 h_2 \varepsilon_2 \in \mathrm{Im}(\tau_0)$. For example, the Euler relation implies

$$h_1 y_1 . \varepsilon_1 \equiv - d_2 d_1^{-1} h_1 y_2 \varepsilon_2 \bmod \mathrm{Im}(\tau_0).$$

As $wt(h_1 \varepsilon_1) = \max wt$, $wt(h_1 \varepsilon_2) = \max wt + d_1 - d_2$. Thus, $h_1 \varepsilon_2$ and then $h_1 y_2 \varepsilon_2 \in \mathrm{Im}(\tau_0)$ unless $d_1 - d_2 = nd_1 + md_2$, an impossibility as $d_1 < d_2$. For $i = 2$, if we similarly obtained $d_2 - d_1 = nd_1 + md_2$ then $m = 0$ and hence $d_1 | d_2$, a contradiction.

□

Secondly, we must deal with the case of $wt = d_i + \max wt$. The reason this difficulty arises is that in the construction of [4, II, § 6], if there is an element $g \in \mathrm{Q}(f_0)$ such that $g.\varepsilon_i \in \mathrm{Im}(\tau_0)$, then there is no guarantee that the element of $(m_u/m_u^2).\mathrm{N}(f_0)_{\max}$ corresponding to $g.p$ can be constructed (p a generator of $\mathrm{N}(f_0)^*$). We consider the situation of ii) in Corollary 5.8.

LEMMA 6.2. — *If $f: k^s, 0 \to k^2, 0$ satisfies the hypotheses of Corollary 5.8 with $\mathscr{J}(f_0)$ Gorenstein and $m_y.\mathrm{N}(f_0)_{\max} \subset \mathrm{Im}(\tau_0)$. Then, for* F *obtained by adjoining a power to f, τ_1^+ is surjective.*

Proof. — It is sufficient to show τ_1^+ is surjective in $wt = \max wt + d_1$ as $d_1 \leqslant d_2$.

By the proof of theorem 5.1 in this case $z^{2\ell-2}.p \equiv 0$ on $\mathrm{N}(\mathrm{F})$, and $z^{2\ell-2}\varepsilon_1$ generates $\mathrm{N}(\mathrm{F})_{\max}$. Given $b \in \mathscr{J}(\mathrm{F})_{\max wt + d_1}$, there is a constant c_b so that $(b - c_b z^{2\ell-2}).\varepsilon_1 = 0$ in $\mathrm{N}(\mathrm{F})$, but $(b - c_b.z^{2\ell-2})p = b.p$ on $\mathrm{N}(\mathrm{F})$. Thus, for any b with $b.\varepsilon_1 \neq 0$ in $\mathrm{N}(\mathrm{F})_{\max}$, there is a b_1 with $b_1.p = b.p$ on $\mathrm{N}(\mathrm{F})$ and $b_1.\varepsilon_1 = 0$ in $\mathrm{N}(\mathrm{F})_{\max}$. This observation together with the construction in [4, II, § 6] implies τ_1 or τ_1^+ is surjective in $wt = \max wt + d_1$.

□

7. Curve Singularities in k^3.

In this section we give a result for curves which has consequences in the next section for surface singularities. We also give several counterexamples to « expected behavior » for curve singularities which contrasts with the behavior of surface singularities in the next section.

PROPOSITION 7.1. — *Let $f: k^3, 0 \to k^2, 0$ be a weighted homogeneous germ defining an isolated curve singularity, then $\mathscr{J}(f)$ is Gorenstein.*

Proof. — If $k = \mathbf{R}$, then the results for f or its complexification are equivalent. Thus, we may assume $k = \mathbf{C}$. Then, $\mathrm{J}(f)$ only vanishes at 0 which has codimension 3 in $\mathbf{C}^3$. Thus, we may use the Buchsbaum-

Eisenbud criteria [3 a] and show that $J(f)$ is generated by the pfaffians of a 5×5 skew-symmetric matrix. Consider the following skew-symmetric matrix.

$$\begin{pmatrix} 0 & a_1x & -a_3z & f_{1y} & f_{2y} \\ -a_1x & 0 & a_2y & f_{1z} & f_{2z} \\ \cdot & \cdot & 0 & f_{1x} & f_{2x} \\ \cdot & & \cdot & 0 & 0 \\ \cdot & & & \cdot & 0 \end{pmatrix}$$

Here $f_{ix} = \frac{\partial f_i}{\partial x}$, etc, where $f = (f_1,f_2)$, and $a_1 = wt(x)$, etc. A straightforward computation of the pfaffians shows they consist of the 2×2 minors of df, and $a_1xf_{ix} + a_2yf_{iy} + a_3zf_{iz}$, $i = 1, 2$. By the weighted homogeneity of f, these are the generators of $J(f)$.

□

Observe that the principal consequence of proposition 7.1 is for surface singularities. We next see via several examples of uni-modal curve singularities that the situation is less than might be expected for versal unfoldings of curve singularities.

Example 7.2. — The curve singularities

$$f_0(x,y,z) = (xy, x^a+y^b+z^c), \quad 2 \leqslant a \leqslant b, \quad 2 \leqslant c$$

are uni-modal singularities exactly for $(a,b,c) = (4,4,2), (3,6,2)$, and $(3,3,3)$. The versal unfoldings for these singularities are *not* topologically trivial along the weight zero direction. For the first two examples, $f_t(x,y,z) = (xy+tz, x^a+y^b+z^c)$ is the weight zero deformation. However, for $t = 0$, f_t is a Σ_3-singularity while for $t \neq 0$, it is a Σ_2-singularity. Hence, by results of [6], the versal unfolding of f_0 changes topological type at $t = 0$. A more involved argument using results of [6] also show that the versal unfolding of $f_t(x,y,z) = (xy+tz^2, x^3+y^3+z^3)$ changes topological type at $t = 0$.

Note. — Adjoining powers to the first two examples give surface singularities for which the same phenomena holds. However, these surface singularities are not uni-modal. Also, Wirthmuller [29] has shown this phenomena can also occur for zero-dimensional uni-modal singularities.

Example 7.3. — For versal unfoldings viewed as germs, the $\mathscr{A}$-orbit is open in the $\mathscr{K}$-orbit. We can weaken versality in the simplest way by considering negative versal unfoldings of uni-modal singularities which are still finitely $\mathscr{A}$-determined (so they are topologically versal). The condition of openess of the $\mathscr{A}$-orbit is equivalent to the surjectivity of τ_1. This has been found to hold for all other uni-modal singularities. However, surprisingly this condition fails for a uni-modal curve singularity.

PROPOSITION 7.3. — *The germ* $f_0(x,y,z) = (yz-x^2, xy+\varepsilon xz^3+tz^5)$, $4+27\varepsilon t^2 \neq 0$, $t \neq 0$, $\varepsilon = \pm 1$ *defines a curve singularity for which* t *is a simple modulus and for which the negative versal unfolding is finitely* $\mathscr{A}$-*determined. However,* τ_1 *is not surjective for any value of* t.

Proof. — A calculation shows that $\ker(\tau_0)$ in weight $= 1$ has dimension $= 1$ while $(m_u/m_u^2).N(f_0)_{\max}$ in weight $= 1$ has dimension $= 2$. Thus, τ_1 cannot be surjective. The fact that the modulus is simple [4, I] implies that for most values of t the negative versal unfolding is finitely $\mathscr{A}$-determined. However, to determine that it is finitely $\mathscr{A}$-determined for all values of t except those for which finite singularity type fails requires an extensive calculation to show that eventually τ_5 is surjective! □

8. Families of surface singularities in $\mathbf{C}^4$ beginning at the uni-modals.

Uni-modal surface singularities in $\mathbf{C}^4$ with $\mathbf{C}^*$-action consist of the simple elliptic singularity of Saito [25] and the exceptional uni-modal quotient singularities of Dolgachev [9, 10] which are complete intersections but not hypersurfaces. These singularities are listed in Table 1 (in a slightly modified form as that given by Pinkham [23]). The equations also define real singularities in $\mathbf{R}^4$. These singularities arise either by adjoining a power to a curve singularity or as the intersection of a non-singular quadric and a hypersurface. As such they fit into infinite families of surface singularities arising by similar construction. These are given in Table 2. Then, for these families we have the main result of this paper.

THEOREM 8.1. — i) *The infinite families (in Table 2) beginning at the exceptional uni-modal and simple elliptic singularities are uni-maximal germs and have unfoldings versal in non-maximal weight which are infinitesimally stable off the subspace of non positive weight. For the simple elliptic and exceptional uni-modal singularities these unfoldings are actually finitely* $\mathscr{A}$-*determined as germs.*

ii) *Hence, these unfoldings are topologically versal (and over* $\mathbf{R}$

topologically stable), so that the versal unfoldings are topologically trivial along the parameter corresponding to the term of maximal weight.

Proof. – First, we verify that for the germs f_0 defining the surface singularities, $N(f_0)^*$ is principal. For those obtained by adjoining powers to curve singularities, it is guaranteed by proposition 7.1 and Corollary 5.8. For the last three families which are intersections of hypersurfaces with non-singular quadrics we can verify by direct calculation that $\mathscr{J}(f)/(\omega)$ is Gorenstein, so again Corollary 5.8 applies.

Next, we verify the conditions described in § 6. For $m_y . N(f_0)_{\max} \subset \operatorname{Im}(\tau_0)$: we note that except for those special integral parameter values which yield germs obtained by adjoining powers to curves defined by a pair of quadrics, $d_1 \nmid d_2$. The remaining condition for lemma 6.1 follows from (5.5) for adjoining powers and by direct verification for the last three families. For the special integral values the check must be made directly.

Lastly, we consider the surjectivity of τ_1 or τ_1^+ on $wt = \max wt + d_1$. By lemma 6.2, this is taken care of for all but the last three families. For those families we directly observe that $\omega\varepsilon_1 \neq 0$ in $N(f_0)$, and by examining weights, we see it generates $N(f_0)_{\max}$. Also, $\omega . p = 0$ on $N(f_0)$; thus, the proof of proposition 6.2 shows that τ_1^+ is surjective in $wt = \max wt + d_1$. For the simple elliptic and exceptional uni-modal singularities, this implies that τ_1 is surjective, and yields the finite $\mathscr{A}$-determinacy. □

9. A geometric condition and pairs of generic quadrics.

All of our efforts have been concentrated on establishing algebraic criteria for unfoldings to be finitely $\mathscr{A}$-determined or infinitesimally stable off the subspace of non-positive weight. This is because generally it is extremely difficult to establish infinitesimal stability geometrically. In this section we present an interesting exception to this rule.

Generically, pencils of quadrics in $\mathbf{C}^n$ contain non-singular quadrics, and we may choose coordinates so a pair of such quadrics is given by

$$(x_1^2 + \cdots + x_n^2, a_1 x_1^2 + \cdots + a_n x_n^2).$$

If $a_i \neq a_j$, $i \neq j$, then the germ $f_0 : k^n, 0 \to k^2, 0$ (we may choose $k = \mathbf{R}$ if $a_i \in \mathbf{R}$, all i) is a complete intersection defining an isolated singularity.

For such germs there are $n - 3$ moduli obtained by allowing $n - 3$ of the a_i to vary; and if $n = 4$, we obtain the simple elliptic surface singularity in k^4.

The versal unfoldings of such generic pairs of quadrics has been studied by Knörrer [16] and [17]. In his work, among other things he succeeded in showing that the discriminant of the versal unfolding is topologically a product of the discriminant of the negative versal unfolding with the subspace of moduli parameters. However, he was not able to obtain the topological triviality of the versal unfolding along the subspace of moduli parameters. We show how to obtain this result using results which Knorrer used for analyzing the discriminant together with a result from [4, I].

We consider an unfolding $F : \mathbf{C}^{s+q}, 0 \to \mathbf{C}^{t+q}, 0$ of the germ $f_0 : \mathbf{C}^s, 0 \to \mathbf{C}^t, 0$, with $F(x,u) = (\bar{F}(x,u),u)$, such that: i) F is infinitesimally stable, and ii) there is a neighborhood U of 0 in $\mathbf{C}^{s+q}$ and a representative F_1 of F, $F_1 : U \to \mathbf{C}^{t+q}$, such that F_1 is infinitesimally stable and there are only a finite number of germs $F_{(x,u)}$ (up to analytic equivalence) for $(x,u) \in U - (\{0\} \times \mathbf{C}^q)$. Then, we can stratify the discriminant of F, denoted by C(F), by the analytic types of the germs

$$S(y,u) = \{F_{(x,u)} : (x,u) \in \Sigma(F) \cap F^{-1}(y,u)\}$$

for $(y,u) \in C(F) - (\{0\} \times \mathbf{C}^q)$ (recall $\Sigma(F)$ denotes the critical set of F). By the multi-transversality of F (see [19, V]), the set of (y,u) with $S(y,u)$ constant (up to analytic equivalence) is a smooth submanifold. By the above assumption ii) on F, there are only a finite number of strata $\{S_i\}$. Lastly, let $\pi : \mathbf{C}^{t+q} \to \mathbf{C}^q$ denote the projection along $\mathbf{C}^t \times \{0\}$. Then, we have the following geometric condition for finite $\mathscr{A}$-determinacy.

PROPOSITION 9.1. — *For the unfolding* F *above, suppose there is a neighborhood* V *of* 0 *in* $\mathbf{C}^{t+q}$ *such that* π *is a submersion on each* $S_i \cap V$. *Then, there is a neighborhood* U_1 *of* 0 *in* $\mathbf{C}^q$ *such that the germs* $F_1(.,u) : \mathbf{C}^s, 0 \to \mathbf{C}^t, 0$ *are finitely* $\mathscr{A}$*-determined for* $u \in U$. *In particular,* f_0 *is finitely* $\mathscr{A}$*-determined.*

Proof. — The proof is analogous to that of proposition 4.2 of [4, I]. We choose neighborhoods U_1 of 0 in $\mathbf{C}^q$ and U_2 of 0 in $\mathbf{C}^s$ so that $U_2 \times U_1 \subset U$ and $F_1(U_2 \times U_1) \subset V$. We define a parametrized family of mappings $i_u : \mathbf{C}^t \to \mathbf{C}^{t+q}$ by $i_u(y) = (y,u)$, for $u \in U$. For a given $u \in U_1$, there is a neighborhood V_u of 0 in $\mathbf{C}^t$ so that $i_u(V_u) \subset V$. By the assumption on $\pi|(S_i \cap V)$, $i_u|V_u$ is transverse to each S_i. Then, i_u is also transverse to $\{0\} \times \mathbf{C}^q$; hence i_u is transverse to F_1. Also, if $i_u^* F_1$ denotes the pull-back map then $i_u^* F_1 = F_1(.,u) | F_1^{-1}(V_u \times \{u\})$. Lastly, by the same argument used in proposition 4.2 of [4, I], i_u transverse to the $\{S_i\}$ implies that $i_u^* F_1$ is multi-transverse to contact orbits off of $F_1^{-1}(0,u) \cap \Sigma(F_1)$, which is a finite set in a compact neighborhood of $(0,u)$. Thus, on a small enough punctured neighborhood of $(0,u)$, $F_1(.,u)$ is infinitesimally stable by [19, V]. Thus, the germ $F(.,u)$ is finitely $\mathscr{A}$-determined by the geometric characterization of Mather and Gaffney referred to in [4, I]. □

Remark. – For a real analytic germ $F : \mathbf{R}^{s+q}, 0 \to \mathbf{R}^{t+q}, 0$, if its complexification satisfies the hypothesis of proposition 9.1, then the same conclusion holds for F by the geometric characterization.

Now by Theorem 3.23 of Knörrer's thesis (also see remark 2 after theorem 3.1 in [17]) the versal unfolding of a pair of generic quadrics, viewed as an unfolding by the moduli parameters of the negative versal unfolding f of f_0, satisfies the conditions of proposition 9.1 (the proof of this uses simultaneous uniformization of Riemann surfaces). Thus, by proposition 9.1 we can conclude.

THEOREM 9.2. – *The negative versal unfolding of a germ f_0 defined by a pair of generic quadrics is finitely $\mathscr{A}$-determined. Hence, it is C^0-versal and the versal unfolding is a topologically trivial unfolding of the negative versal unfolding.*

Remark. – In this theorem, the C^0-versality follows from corollary 4 of [4, I] rather than Theorem 4 of [4, II].

TABLE 1

germs defining unimodal surface singularities	weights of functions (d_1, d_2)	weights of variables (x, y, z, w)	Milnor number	Simple Curve
$(yz-x^2, xy+z^3+w^2)$	(16, 18)	(8, 10, 6, 9)	9	$i\delta$
$(yz-x^2, xy+xz^2+w^2)$	(12, 14)	(6, 8, 4, 7)	10	$i\delta$
$(yz-x^2, xy+z^4+w^2)$	(10, 12)	(5, 6, 3, 6)	11	$i\delta$
$(yz-x^2, y^2+z^3+w^2)$	(10, 12)	(5, 6, 4, 6)	10	$i\varepsilon 1$
$(yz-x^2, y^2+xz^2+w^2)$	(8, 10)	(4, 5, 3, 5)	11	$i\varepsilon 1$
$(yz-x^2+w^2, xy+z^3)$	(8, 9)	(4, 5, 3, 4)	11	$i\delta$
$(xw+yz, z^2-yw+x^3)$	(11, 12)	(4, 5, 6, 7)	10	
$(xw+yz, z^2-yw+yx^2)$	(9, 10)	(3, 4, 5, 6)	11	

Here the first six singularities are obtained from curve singularities by adjoining w^2. The notation for the curve singularities is that of Mather [19, VI] or also [22]. These curves also appear in the list of simple singularities of Giusti [11] using different notation.

TABLE 2

Functions	Weights of functions (d_1, d_2)	Weights of variables (x, y, z, w)	Milnor number	Curve singularity
$(xy-2^2, x^a+y^b+w^c)$	$(2c(a+b), 2abc)$	$(2bc, 2ac, c(a+b), 2ab)$	$(c-1)\{ab+(a-1)(b-1)\}+c$	$(xy-z^2, x^a+y^b)$
$(yz-x^2, xy+z^\ell+w^k)$	$(k(2\ell+2), 3\ell k)$	$(k(\ell+1), k(2\ell-1), 3k, 3\ell)$	$(k-1)(2\ell+1)+k$	$i\delta$
$(yz-x^2, xy+xz^{\ell-1}+w^k)$	$(2k\ell, k(3\ell-2)$	$(k\ell, 2k(\ell-1), 2k, 3\ell-2)$	$(k-1)(3\ell-1)+k$	$i\delta$
$(yz-x^2, y^2+xz^{\ell-1}+w^k)$	$(2k(\ell+1), 2k(2\ell-1))$	$(k(\ell+1), k(2\ell-1), 3k, 2(2\ell-1))$	$(k-1)(4\ell-3)+k$	$i\varepsilon 1$
$(yz-x^2+w^2, xy+z^\ell)$	$(2\ell+2, 3\ell)$	$(\ell+1, 2\ell-1, 3, \ell+1)$	$4\ell-1$	$i\delta$
$(xw+yz, z^2-yw+x^\ell)$	$(3\ell+2, 4\ell)$	$(4, \ell+2, 2\ell, 3\ell-2)$	$3\ell+1$	
$(xw+yz, z^2-yw+yx^{\ell-1})$	$(3\ell, 4\ell-2)$	$(3, \ell+1, 2\ell-1, 3\ell-3)$	$4\ell-1$	

(For all examples $\ell \geqslant 3$, $k \geqslant 2$, and for the first family $2 \leqslant a \leqslant b$, $c \geqslant 2$. Also, it is to be understood that for certain values of the integral parameters, it may be necessary to divide the weights by their g.c.d.).

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