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A CONTINUOUS HELSON SURFACE IN \mathbb{R}^3

by Detlef MÜLLER

1.

Let G be a locally compact abelian group, and let $A(G)$ denote the Fourier algebra on G and $B(G)$ the Fourier-Stieltjes algebra on G . If $E \subset G$ is a compact subset of G , then $A(E)$ will denote the quotient Banach algebra $A(G)/I(E)$, where $I(E)$ is the ideal of all functions in $A(G)$ which vanish on E . E is a *Helson set* if $A(E) = C(E)$ (see [6] as a general reference). Let $M(G)$ denote the algebra of bounded Radon measures on G , $M(E)$ the subspace of all measures with support contained in E , and let $PM(G)$ be the dual space of $A(G)$. Then E is a Helson set if and only if its Helson constant

$$\begin{aligned} \alpha(E) &= \sup \{ \|f\|_{A(E)} : f \in A(E) \text{ and } \|f\|_{C(E)} \leq 1 \} \\ &= \sup \{ \|\mu\| : \mu \in M(E) \text{ and } \|\mu\|_{PM} \leq 1 \} \end{aligned}$$

is finite.

A comprehensive study of the question when a continuous submanifold of \mathbb{R}^n is a Helson set has been carried out in [5] by O. C. McGehee and G. S. Woodward. They proved among other results that there exists a Helson curve in \mathbb{R}^2 which is the graph of a Lip(1) function, and that there is a continuous Helson k -manifold in $\mathbb{R}^{\ell k}$ whenever $\ell \geq k + 1$. The former result had essentially already been obtained by J. P. Kahane in [3] in connection with studies on Lusin's problem, but the proof in [5] gives a concrete construction instead of Baire category arguments which were used by Kahane. A variant of the proof in [5] did already appear in [4]. Two years after Kahane's result N. Th. Varopoulos proved that continuous Sidon manifolds of dimension $n - 1$ are abundant in \mathbb{R}^n [8], but it was not clear whether at least some of these Sidon manifolds were Helson sets.

In this paper we will construct a Helson surface in \mathbf{R}^3 which is the graph of a Lip (1) function. In addition to this our methods also offer the possibility of a proof by induction over n that every \mathbf{R}^n contains a Helson manifold of dimension $n - 1$. But, to avoid technical complications, we will restrict ourselves to the case $n = 3$. The proof will be based on the result (Theorem 1) that there even exists a sequence $\{\Gamma_k\}_k$ of Helson curves in \mathbf{R}^2 such that $\cup \Gamma_k$ is dense in some open part of \mathbf{R}^2 and such that $\alpha\left(\bigcup_{k \leq m} \Gamma_k\right)$ is uniformly bounded for all m .

We would like to thank Professor McGehee for helpful conversations and suggestions.

2.

We will now introduce some notations. G will in general denote a locally compact abelian group. Let W be a symmetric neighborhood of the neutral element in G , let D be a subset of \mathbf{C} and let E be a compact subset of G . Then $C_{\sigma, w}(E, D)$ will denote the set of all continuous functions f on E with values in D , such that $|f(x) - f(y)| < \sigma$ whenever $x, y \in E$ and $x - y \in W$.

By T we will denote the subset $T = \{\zeta \in \mathbf{C} : |\zeta| = 1\}$ of \mathbf{C} .

If $G = \mathbf{R}^n$ for some n , then for any $\delta > 0$, $U(\delta)$ will denote the open ball with radius δ and center 0 in \mathbf{R}^n .

If f is a Lip (1) function on some subset Q of \mathbf{R}^n , then we write

$$L(f) = \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|} : x, y \in Q, x \neq y \right\}.$$

Finally the graph of a function f will be denoted by $G(f)$.

3.

In this section we will prove a result which is related to the deep separation results that emerged with the solution to the union problem for Helson sets (see [9], [2], and [1] as a general reference).

LEMMA 1. — *Let E be a compact Helson set in the locally compact abelian group G . Let $\sigma > 0$, and let W be a symmetric neighborhood of*

the neutral element in G . Then there exists a neighborhood $V = V(E, \sigma, W)$ such that for any function $f \in C_{\sigma/4, W}(E, T)$ there exists some $g \in A(G)$ with

- (i) $|f(x) - g(x+z)| < \sigma$ for $x \in E$ and $z \in V$,
- (ii) $\|g\|_A \leq \alpha(E)$.

Proof. — Assume E, σ and W are given as above. Choose a symmetric neighborhood W_0 of the neutral element in G whose closure is compact, such that $W_0 + W_0 + W_0 \subset W$.

We claim :

- (1) There exist finitely many functions $\tilde{g}_1, \dots, \tilde{g}_m$ in $C_{\sigma/2, W_0}(E, C)$ with $\|\tilde{g}_i\|_{C(E)} = 1$ such that for every $f \in C_{\sigma/8, W}(E, T)$ there exists a \tilde{g}_j with $\|f - \tilde{g}_j\|_{C(E)} < \sigma/3$.

To prove (1), fix $\kappa > 0$ such that $3\left(\frac{1}{8} + 2\kappa\right) < \frac{1}{2}$ and $\frac{1}{4} + 3\kappa < \frac{1}{3}$, and choose a finite subset $D \subset T$ such that each point of T lies within distance $\kappa\sigma$ from D .

Let $E_0 = \{x_1, x_2, \dots, x_n\} \subset E$ such that $E \subset \bigcup_{i=1}^n (x_i + W_0)$ and $x_j \notin x_i + W_0$ for $i \neq j$. Let $\sigma' = \left(\frac{1}{8} + 2\kappa\right)\sigma$. Then $C_{\sigma', W}(E_0, D)$ is a finite set. We will show that every function $h \in C_{\sigma', W}(E_0, D)$ can be extended to a function $\tilde{h} \in C_{3\sigma', W_0}(E, C)$ with $\|\tilde{h}\|_{C(E)} = 1$.

In fact, choose a finite partition of unity $\{\varphi_i\}_i$ of continuous functions φ_i on E such that $\text{supp } \varphi_i \subset (x_i + W_0)$, $0 \leq \varphi_i \leq 1$ and $\varphi_i(x_i) = 1$ for $i = 1, \dots, n$, and let $\tilde{h} = \sum h(x_i)\varphi_i$. Then \tilde{h} of course extends h , $\|\tilde{h}\|_{C(E)} \leq \|h\|_{C(E_0)} = 1$, and an easy estimate shows that $h \in C_{\sigma', W}(E_0, D)$ implies $\tilde{h} \in C_{3\sigma', W_0}(E, C)$.

Now let $f \in C_{\sigma/8, W}(E, T)$, and choose $h: E_0 \rightarrow D$ such that $\|h - f\|_{C(E_0)} < \kappa\sigma$. Then it follows easily that $h \in C_{\sigma', W}(E_0, D)$, hence $\tilde{h} \in C_{3\sigma', W_0}(E, C) \subset C_{\sigma/2, W_0}(E, C)$ and $\|\tilde{h}\|_{C(E)} = 1$. Moreover, if $x \in E$, then

$$(2) \quad |f(x) - \tilde{h}(x)| \leq |f(x) - f(x_i)| + |f(x_i) - h(x_i)| + |\tilde{h}(x_i) - \tilde{h}(x)| \\ \leq \sigma/8 + \kappa\sigma + \sigma' \leq \sigma/3,$$

if $x_i \in E_0$ is chosen such that $x \in x_i + W_0$.

So (1) holds with $\{\tilde{g}_1, \dots, \tilde{g}_m\} = \{\tilde{h} : h \in C_{\sigma', W}(E_0, D)\}$.

Now choose $\beta > \alpha(E)$. There exist functions $g_1, \dots, g_m \in A(G)$ such that $g_i|_E = \tilde{g}_i$ and $\|g_i\|_A < \beta$. Choose a neighborhood V of the neutral element in G such that for $i = 1, \dots, m$,

$$(3) \quad |g_i(x) - g_i(x+z)| < \sigma/12 \quad \text{for } x \in E \quad \text{and } z \in V.$$

If then $f \in C_{\sigma/8, W}(E, T)$, if \tilde{g}_i is chosen according to (1) for f , and if g_i denotes the above extension of \tilde{g}_i , then (1) and (3) yield

$$|f(x) - g_i(x+z)| < \frac{5}{12} \sigma \quad \text{for } x \in E \quad \text{and } z \in V.$$

Assuming that $\beta > \alpha(E)$ had been chosen close enough to $\alpha(E)$, we may take g to be a multiple (at most slightly different from one) of g_i . Replacing finally σ by 2σ , the lemma is proved.

PROPOSITION 1. — *Let E be a compact Helson set in the locally compact abelian group G . Let $0 < \varepsilon < 1$ and $\sigma > 0$, and let W be a symmetric neighborhood of the neutral element in G . Then there exist neighborhoods $V = V(E, \sigma, W)$ and $U = U(E, \varepsilon, \sigma, W)$ of the neutral element in G such that for any function $f \in C_{\sigma/8, W}(E, T)$ there exists some $g \in A(G)$ with*

- (i) $|f(x) - g(x+z)| < \sigma$ for $x \in E$ and $z \in U$;
- (ii) $\|g(x+z)|f(x) - g(x+z)| < \alpha(E)^4 \varepsilon^{-1/2} \sigma$ for $x \in E$ and $z \in V$;
- (iii) $|g(y)| \leq \alpha(E)^5 \varepsilon$ for $y \notin E + V$;
- (iv) $\|g\|_A \leq \alpha(E)^5 \varepsilon^{-1/2}$.

Proof. — Let E, ε, σ and W be given as above. Fix $\beta > \alpha(E)$. Following the proof of Lemma 1, there exist functions $g_1, \dots, g_m \in A(G)$ with $\|g_i\|_A < \beta$ and a neighborhood $V = V(E, \sigma, W)$ of the neutral element in G such that for any $f \in C_{\sigma/8, W}(E, T)$

$$(4) \quad |f(x) - g_i(x+z)| < \frac{5}{12} \sigma \quad \text{for } x \in E \quad \text{and } z \in V$$

for some suitable g_i .

Moreover, after the separation-theorem 2.1.3 in [1] there exists a function $\chi_1 \in A(G)$ such that $\chi_1 = 1$ on E , $|\chi_1(y)| \leq \beta^2 \varepsilon^{1/2}$ for $y \notin E + V$ and $\|\chi_1\|_A \leq \beta^2 \varepsilon^{-1/4}$. Let $\chi = |\chi_1|^2$. Then $0 \leq \chi \leq \beta^4 \varepsilon^{-1/2}$, $\chi = 1$ on E , $|\chi(y)| \leq \beta^4 \varepsilon$ for $y \notin E + V$ and $\|\chi\|_A \leq \beta^4 \varepsilon^{-1/2}$.

Finally choose a neighborhood $U \subset V$ of the neutral element in G such that

$$(5) \quad |1 - \chi(x+z)| < \sigma/3\beta \quad \text{for } x \in E \text{ and } z \in U.$$

Let $f \in C_{\sigma, \beta, w}(E, T)$, choose g_i as in (4) and set $g = \chi g_i$. Then

$$|f(x) - g(x+z)| < \frac{5}{12} \sigma + \frac{\sigma}{3\beta} \beta < \frac{3}{4} \sigma, \quad \text{if } x \in E, z \in U,$$

and

$$\begin{aligned} \|g(x+z)|f(x) - g(x+z)| \\ \leq \chi(x+z) \{ |f(x) - g_i(x+z)| + |1 - |\tilde{g}_i(x)|| \cdot |f(x)| \\ + \|g_i(x) - |g_i(x+z)|| |f(x)| \}, \end{aligned}$$

where \tilde{g}_i is chosen as in the proof of Lemma 1, hence

$$\|g(x+z)|f(x) - g(x+z)| \leq \beta^4 \varepsilon^{-1/2} \left(\frac{5}{12} \sigma + \frac{1}{3} \sigma + \frac{1}{12} \sigma \right) = \frac{5}{6} \beta^4 \varepsilon^{-1/2} \sigma$$

for $x \in E$ and $z \in V$ (compare with (1), (3) and (4)). Since $|g(y)| \leq \beta^4 \varepsilon \beta = \beta^5 \varepsilon$ for $y \notin E + V$ and $\|g\|_\Lambda \leq \beta^4 \varepsilon^{-1/2} \beta = \beta^5 \varepsilon^{-1/2}$, again we see that if β has been chosen close enough to $\alpha(E)$ we may replace g by a suitable multiple of itself to obtain (i) to (iv) of Proposition 1.

Remark. — The final remark in [7] would even allow us to replace (ii) in Proposition 1 by

$$(ii)' \quad \|g(x+z)|f(x) - g(x+z)| < \alpha(E)^4 \sigma \quad \text{for } x \in E \text{ and } z \in V$$

(if $\varepsilon < \frac{1}{4}$), but we do not need this in the following.

4.

The next proposition is a simple extension of Theorem 3.2 in [5] and is proved by the same method. We will nevertheless include a proof, because in combination with the other results of this paper it will indicate the possibility for an inductive proof for the existence of a Helson hypersurface in any \mathbb{R}^n .

PROPOSITION 2. — Assume that real numbers $a_1 < a_2 < \dots < a_n$ and $d > 0$ are given. There exist non-decreasing functions f_1, \dots, f_n in $\text{Lip}(1)([0, 1])$ such that

- (i) $\|f_j - a_j\|_{C([0, 1])} \leq d$ for $j = 1, \dots, n$,
- (ii) $L(f_j) \leq d$ for $j = 1, \dots, n$, and
- (iii) $\alpha(\Gamma) \leq 3^{3/2}$, where $\Gamma = \cup G(f_j)$.

Proof. — Let

$$D = \{d_1 < d_2 < \dots < d_m\} \text{ and } E = \{e_1 < e_2 < \dots < e_m\}$$

be two subsets of \mathbf{R} which are independent over \mathbf{Q} .

Let $\tau = (1, 0) \in \mathbf{R}^2$. If $\eta = (\eta_1, \eta_2) \in \mathbf{R}^2$ is a second unit vector with $\eta_i > 0$, then let $P(D, E; \eta)$ denote the polygonal path in \mathbf{R}^2 whose $2m - 1$ vertices, in order, are $d_1\tau + e_1\eta$, $d_1\tau + e_2\eta$, $d_2\tau + e_2\eta$, $d_2\tau + e_3\eta$, \dots , $d_m\tau + e_m\eta$. As in [5], such a path P will be called an *I-polygonal path*. Let $s(P)$ denote the largest distance between two consecutive vertices of P .

Let η' and τ' be unit vectors perpendicular to η and τ , respectively.

In the following we will assume that all I-polygonal paths P which we will consider contain the graph of a function $f_P \in \text{Lip}(1)([0, 1])$, and further that

$$3d < \min_j (a_{j+1} - a_j).$$

We fix a unit vector $\eta = (\eta_1, \eta_2)$ such that $\eta_2/\eta_1 < d/2$, and denote $P(D, E; \eta)$ by $P(D, E)$. Note that then $L(f_P) \leq d/2$.

Fix $0 < \varepsilon \leq 1$. If $P^j = P(D^j, E^j)$, $j = 1, \dots, n$, are I-polygonal paths such that $D = \cup D^j$ and $E = \cup E^j$ are independent, then also $\tilde{D} = \{d\tau \cdot \eta' : d \in D\}$ and $\tilde{E} = \{e\eta \cdot \tau' : e \in E\}$ are independent. Thus, by Proposition 1, for every $\sigma > 0$ there exist

$$\delta = \delta(P^1, \dots, P^n, \varepsilon, \sigma) > 0, \quad \rho = \rho(P^1, \dots, P^n, \sigma) > 0$$

such that for any function $f: \tilde{D} \rightarrow \mathbf{T}$ there exists $g \in A(\mathbf{R})$ with

- (6) $|f(s) - g(s+t)| < \sigma$ for $s \in \tilde{D}$ and $t \in U(\delta)$;
- (7) $\|g(x+t)f(s) - g(s+t)\| < \varepsilon^{-1/2}\sigma$ for $s \in \tilde{D}$ and $t \in U(\rho)$;
- (8) $|g(s)| \leq \varepsilon$ for $s \notin \tilde{D} + U(\rho)$;
- (9) $\|g\|_\infty \leq \varepsilon^{-1/2}$,

and such that the analogue of (6) to (9) also holds for \tilde{E} instead of \tilde{D} . (Notice that \tilde{D} and \tilde{E} are Kronecker sets, hence $\alpha(\tilde{D}) = \alpha(\tilde{E}) = 1$.)

In order to construct functions f_1, \dots, f_n , divide for each $j = 1, \dots, n$ a sequence of I-polygonal paths $P_m^j = P(D_m^j, E_m^j)$ such that

$$(10) \quad D_m = \bigcup_j D_m^j \text{ and } E_m = \bigcup_j E_m^j \text{ are independent for each } m;$$

$$(11) \quad s_m = \max_j s(P_m^j) \downarrow 0 \text{ as } m \rightarrow \infty;$$

(12) every point of P_{m+1}^j lies within distance

$$\delta_m = 2^{-1} \delta(P_m^1, \dots, P_m^n, \varepsilon, m^{-1}) \text{ away from } P_m^j;$$

$$(13) \quad \|f_{P_m^j} - a_j\|_{C([0,1])} < d \text{ for all } j \text{ and } m.$$

Since $\delta_m \downarrow 0$ as $m \rightarrow \infty$, the functions $f_{P_m^j}$ converge for fixed j uniformly towards a Lip (1) functions f_j on $[0, 1]$, which clearly satisfies (i) and (ii) of Proposition 2.

In order to prove (iii), let $\mu \in M(\Gamma)$ be a measure of norm one. Fix $\sigma > 0$, let Q be a compact rectangle whose interior contains Γ , and choose a continuous function $h : Q \rightarrow \mathbb{T}$ such that $\|h\mu - |\mu|\| < \sigma$. Pick $\alpha > 0$ such that $h \in C_{\sigma, U(\alpha)}(Q, \mathbb{T})$, and choose m large enough such that

$$m^{-1} < \sigma, \quad s_m < \alpha/2 \quad \text{and} \quad \rho_m = \rho(P_m^1, \dots, P_m^n, \sigma) < \alpha/12.$$

For $d \in D_m$ and $e \in E_m$ let

$$R_d^1 = \{a\eta' + b\eta : b \in \mathbb{R}, |a - d\tau \cdot \eta'| < 2\delta_m\},$$

$$R_e^2 = \{a\tau + b\tau' : a \in \mathbb{R}, |b - e\eta \cdot \tau'| < 2\delta_m\},$$

$$S_d^1 = \{a\eta' + b\eta : b \in \mathbb{R}, |a - d\tau \cdot \eta'| < \rho_m\},$$

$$S_e^2 = \{a\tau + b\tau' : a \in \mathbb{R}, |b - e\eta \cdot \tau'| < \rho_m\},$$

and let

$$R^1 = \bigcup_{d \in D} R_d^1, \quad R^2 = \bigcup_{e \in E} R_e^2, \quad S^1 = \bigcup_{d \in D} S_d^1, \quad S^2 = \bigcup_{e \in E} S_e^2.$$

Because of the choice of η we may assume the following important property :

$$(14) \quad \text{If } d \in D_m^j, \text{ then } S_d^1 \cap G(f_{P_m^\ell}) = \emptyset \text{ for } \ell \neq j, \text{ and if}$$

$$e \in E_m^j, \quad \text{then } S_e^2 \cap G(f_{P_m^\ell}) = \emptyset \quad \text{for } \ell \neq j.$$

Since Γ lies within distance $2\delta_m$ from $\bigcup_j G(f_{P_m^j})$, $\Gamma \subset \mathbf{R}^1 \cup \mathbf{R}^2$. Therefore, either $|\mu|(\mathbf{R}^1) \geq \frac{1}{2}$ or $|\mu|(\mathbf{R}^2) \geq \frac{1}{2}$. We shall assume the former, the other case being equivalent to deal with.

For $d \in \mathbf{D}$, there exist exactly two vertices $d\tau + e_d\eta$ and $d\tau + e'_d\eta$ (with $e_d < e'_d$) of $\bigcup_j P_m^j$ which have d as τ -component. We define a function f on $\mathring{\mathbf{D}}$ by $f(d\tau \cdot \eta') = h(d\tau + e_d\eta)$. Choose $g \in \mathbf{A}(\mathbf{R}^2)$ corresponding to f with properties (6) to (9), and define g_1 on \mathbf{R}^2 by $g_1(t\eta' + s\eta) = g(t)$. Then $g_1 \in \mathbf{B}(\mathbf{R}^2)$ with $\|g_1\|_{\mathbf{B}} \leq \varepsilon^{-1/2}$, where $\mathbf{B}(\mathbf{R}^2)$ denotes the Banach algebra of Fourier-Stieltjes transforms of bounded Radon measures on \mathbf{R}^2 . Since $s_m < \alpha/2$ and $\rho_m < \alpha/12$, and since $\text{dist}(\Gamma \cap S_d^1, G(f_{P_m^j})) < \sigma/12$ for $d \in \mathbf{D}_m^j$, we conclude from (14) that

$$(15) \quad |x - y| < \alpha \text{ for any } d \in \mathbf{D} \text{ and } x, y \in \Gamma \cap S_d^1.$$

This together with (6) and (7) implies

$$(16) \quad |h(x) - g_1(x)| \leq 2\sigma \text{ for } x \in \mathbf{R}^1 \cap \Gamma,$$

and

$$(17) \quad |g_1(x)h(x) - g_1(x)| \leq 2\varepsilon^{-1/2}\sigma \text{ for } x \in S^1 \cap \Gamma.$$

Finally we have $|g_1(y)| < \varepsilon$ for $y \notin S^1$.

Since $\Gamma \setminus S^1 \subset \mathbf{R}^2$, all this together implies

$$\|\mu\|_{\text{PM}}\varepsilon^{-1/2} \geq \left| \int_{\Gamma} g_1 d\mu \right| \geq \left| \int_{S^1} g_1 d\mu \right| - \frac{1}{2}\varepsilon,$$

and

$$\begin{aligned} \int_{S^1} g_1 d\mu &= \int_{\mathbf{R}^1} h d\mu + \int_{\mathbf{R}^1} (g_1 - h) d\mu + \int_{S^1 \setminus \mathbf{R}^1} |g_1| d(h\mu - |\mu|) \\ &\quad + \int_{S^1 \setminus \mathbf{R}^1} |g_1| d|\mu| + \int_{S^1 \setminus \mathbf{R}^1} (g_1 - |g_1|h) d\mu, \end{aligned}$$

hence

$$\left| \int_{S^1} g_1 d\mu \right| > (|\mu|(\mathbf{R}^1) - \sigma) - 2\sigma - \varepsilon^{-1/2}\sigma - 2\varepsilon^{-1/2}\sigma,$$

i.e.

$$\|\mu\|_{\text{PM}}\varepsilon^{-1/2} \geq \frac{1}{2} - (3 + 3\varepsilon^{-1/2})\sigma - \frac{1}{2}\varepsilon.$$

Since $\sigma > 0$ was arbitrary, we get

$$\|\mu\|_{\text{PM}} \geq \frac{1}{2}(\varepsilon^{1/2} - \varepsilon^{3/2}),$$

which is at maximum $3^{-3/2}$ for $\varepsilon = 1/3$. This proves (iii).

LEMMA 2. — Let $\sigma > 0$, and let $\nu(n) = 2^{n-1} + 1$. There exists a double sequence $\{f_k^n\}_{n \geq 1, 1 \leq k \leq \nu(n)}$ of non-decreasing Lip(1) functions on $[0, 1]$ with the following properties :

(18) $\Gamma_k^n \cap \Gamma_\ell^n = \emptyset$ for $k \neq \ell$; where $\Gamma_k^n = G(f_k^n)$.

(19) $\alpha(\Gamma^n) \leq 3^6$ for every $n \geq 1$, where $\Gamma^n = \bigcup_k \Gamma_k^n$.

(20) If $k_1, k_2, \dots, k_{\nu(n)}$ are chosen such that

$$f_{k_1}^n < f_{k_2}^n < \dots < f_{k_{\nu(n)}}^n, \quad \text{and if} \quad h_j^n = \frac{1}{2}(f_{k_j}^n + f_{k_{j+1}}^n)$$

for $j = 1, \dots, \nu(n) - 1$, then

$$\|f_k^n - f_k^{n+1}\|_C < \delta_n \quad \text{for} \quad k = 1, \dots, \nu(n),$$

and

$$\|h_j^n - f_{\nu(n)+j}^{n+1}\|_C < \delta_n \quad \text{for} \quad j = 1, \dots, \nu(n) - 1,$$

where δ_n is determined as follows :

Let $\delta = \delta(\Gamma^n, \sigma, n^{-1})$ be chosen corresponding to Lemma 1 such that for any $f \in C_{\sigma/4, U(n-1)}(\Gamma^n, \mathbb{T})$ there is a $g \in A(\mathbb{R}^2)$ with $\|g\|_A \leq 3^6$ and $|f(x) - g(x+z)| < \sigma$ for $x \in \Gamma^n$ and $z \in U(\delta)$. Then $\delta_n > 0$ is chosen such that $2\delta_n < \delta$, $6\delta_n < \delta_{n-1}$ and

$$6\delta_n < \min \{|f_{k_{j+1}}^n(x) - f_{k_j}^n(x)| : x \in [0, 1], j = 1, \dots, \nu(n) - 1\}.$$

(21) $L(f_k^n) \leq 1$ for $n \geq 1$ and $1 \leq k \leq \nu(n)$.

Proof. — Fix $\sigma > 0$, and choose an increasing sequence $0 < d_1 < d_2 < \dots$ of real numbers $d_j < 1$. We will define $\{f_k^n\}$ by induction over n .

For $n = 1$ choose any two non-decreasing functions f_1^1 and f_2^1 on $[0, 1]$ with $L(f_k^1) \leq d_1$, $f_1^1 < f_2^1 < f_1^1 + 1$ and $\alpha(G(f_1^1) \cup G(f_2^1)) \leq 3^{3/2}$. This is possible by Proposition 2.

Assume that functions f_k^m for $m \leq n$ and $1 \leq k \leq v(m)$ have been defined which satisfy (18) to (20) and

$$(21)' \quad L(f_k^m) \leq d_m \text{ for } m \leq n$$

instead of (21).

Choose δ_n as in (20) of Lemma 2.

Similarly as in the proof of Proposition 2, let $\tau = (1, 0)$ and $\eta = (\eta_1, \eta_2)$ be unit vectors in \mathbf{R}^2 such that $\eta_i > 0$ and $\eta_2/\eta_1 = d_n$, and let τ' and η' be unit vectors perpendicular to τ and η , respectively. If we define the functions h_j^n as in (20), then $f_1^n, \dots, f_{v(n)}^n$ and $h_1^n, \dots, h_{v(n)-1}^n$ are non-decreasing functions on $[0, 1]$ with $L(f_k^n) \leq d_n$ and $L(h_j^n) \leq d_n$. It is easily seen that this allows us to find I-polygonal paths $P_k = P(D_k, E_k; \eta)$ for $k = 1, \dots, v(n+1)$ such that each path P_k contains the graph of a continuous function f_{P_k} on $[0, 1]$, and such that

$$(22) \quad \|f_k^n - f_{P_k}\|_C \leq \delta_n/2 \text{ for } k = 1, \dots, v(n)$$

and

$$\|h_j^n - f_{P_{v(n)+j}}\|_C \leq \delta_n/2 \text{ for } j = 1, \dots, v(n) - 1.$$

(In fact we even do not need that the sets D_k and E_k are independent.)

We will now replace the line segments in the paths P_k by pieces of Helson curves. Let $D = \cup D_k$ and $E = \cup E_k$, and choose $\alpha > 0$ such that $3\alpha < \min \{|d - d'| : d, d' \in D, d \neq d'\}$, $3\alpha < \min \{|e - e'| : e, e' \in E, e \neq e'\}$ and $\alpha < \delta_n/4$. By Proposition 2 there exist non-decreasing functions $g_e^2 \in \text{Lip}(1)([0, 1])$ such that

$$\|g_e^2 - e\eta \cdot \tau'\|_C \leq \alpha, \quad L(g_e^2) \leq d_{n+1} - d_n \text{ and } \alpha(\Gamma_E^2) \leq 3^{3/2},$$

where $\Gamma_E^2 = \bigcup_{e \in E} G(g_e^2)$. And similarly there exist non-decreasing functions $g_d^1 \in \text{Lip}(1)([0, 1])$ such that $\|g_d^1 - \ell_d\|_C \leq \alpha$, where ℓ_d denotes the affine linear function $\ell_d(x) = d + \eta_2 x / \eta_1$ whose graph is the line $\mathbf{R}\eta + d\tau \cdot \eta'$, and such that $L(g_d^1) \leq d_n + (d_{n+1} - d_n) = d_{n+1}$ and $\alpha(\Gamma_D^1) \leq 3^{3/2}$,

where $\Gamma_D^1 = \bigcup_{d \in D} G(g_d^1)$. By the union Theorem 2.1.2 for Helson sets in [1], the set $\Gamma_D^1 \cup \Gamma_E^2$ is a Helson set with $\alpha(\Gamma_D^1 \cup \Gamma_E^2) \leq 3^6$.

It is easy to see that $\Gamma_D^1 \cup \Gamma_E^2$ contains the graphs of $v(n+1)$ non-decreasing Lip(1) functions f_k^{n+1} on $[0, 1]$ with

$$(23) \quad \|f_k^{n+1} - f_{p_k}\|_C \leq \delta_n/2 \text{ for } k = 1, \dots, v(n+1),$$

which agree piecewise with the functions g_d^1 or g_e^2 . Of course we then have $L(f_k^{n+1}) \leq d_{n+1}$ for $k = 1, \dots, v(n+1)$, which implies (21)' for $n + 1$, and from (22) and (23) we get

$$\|f_k^{n+1} - f_k^n\|_C < \delta_n \text{ for } k = 1, \dots, v(n)$$

and $\|h_j^n - f_{v(n)+j}^{n+1}\|_C < \delta_n$ for $j = 1, \dots, v(n) - 1$. This, together with the choice of δ_n , guarantees (18) for $n + 1$, and thus also (20) holds for $n + 1$. Finally, (19) holds for Γ^{n+1} , since Γ^{n+1} is a closed subset of $\Gamma_D^1 \cup \Gamma_E^1$.

THEOREM 1. — *For any $\beta > 3^6$ there exists a sequence $\{f_k\}_{k \geq 1}$ of non-decreasing functions $f_k \in \text{Lip}(1)([0, 1])$ with $L(f_k) \leq 1$ such that :*

(i) $f_1 < f_k < f_2$ for all $k \geq 3$, and

$G(f_k) \cap G(f_\ell) = \emptyset$ for $\ell \neq k$.

(ii) For each $\varepsilon > 0$ and $k \neq \ell$ with $f_k < f_\ell$ there exist k_1, k_2, \dots, k_n such that

$$f_k = f_{k_1} < f_{k_2} < \dots < f_{k_n} = f_\ell \text{ and } \|f_{k_{j+1}} - f_{k_j}\|_C \leq \varepsilon.$$

(iii) $\alpha\left(\bigcup_{k=1}^n G(f_k)\right) \leq \beta$ for every $n \geq 1$.

Proof. — Fix $\beta > 3^6$. Choose $\sigma > 0$ such that $\beta(1 - 9\sigma/4) > 3^6$, and choose a double sequence $\{f_k^n\}_{n \geq 1, 1 \leq k \leq v(n)}$ of non-decreasing functions on $[0, 1]$ with the properties stated in Lemma 2. Assume in addition that $f_1^1 < f_2^1$.

Because of (20), for each $k \geq 1$ there exists an $f_k \in C([0, 1])$ such that

$$(24) \quad \|f_k - f_k^n\|_C \leq \frac{6}{5} \delta_n, \text{ if } v(n) \geq k.$$

Since $L(f_k^n) \leq 1$, this implies $L(f_k) \leq 1$, and of course f_k is non-decreasing. (24) also implies that $G(f_k) \cap G(f_\ell) = \emptyset$ if $\ell \neq k$, since for any n with $v(n) \geq \max(k, \ell)$ and any $x \in [0, 1]$

$$\begin{aligned} |f_k(x) - f_\ell(x)| &\geq |f_k^n(x) - f_\ell^n(x)| - \|f_k - f_k^n\|_C - \|f_\ell - f_\ell^n\|_C \\ &\geq 6\delta_n - \frac{6}{5}\delta_n - \frac{6}{5}\delta_n > 3\delta_n. \end{aligned}$$

Proceeding inductively we prove:

$$(25) \quad f_{k_j}^n + \delta_{n-1} < f_{k_{j+1}}^n + 4\left(\frac{2}{3}\right)^n.$$

For $n = 1$ this is true if we choose $\delta_0 < 1$ suitably. Assuming that (25) holds for some $n \geq 1$ we pick for instance a particular k_j . Then the smallest of the functions f_ℓ^{n+1} with $\ell \neq k_j$ and $f_{k_j}^{n+1} < f_\ell^{n+1}$ is $f_{v(n)+j}^{n+1}$, and the equalities

$$\begin{aligned} f_{v(n)+j}^{n+1} - f_{k_j}^{n+1} &= (h_j^n - f_{k_j}^n) + (f_{v(n)+j}^{n+1} - h_j^n) + (f_{k_j}^n - f_{k_j}^{n+1}) \\ &= \frac{1}{2}(f_{k_{j+1}}^n - f_{k_j}^n) + (f_{v(n)+j}^{n+1} - h_j^n) + (f_{k_j}^n - f_{k_j}^{n+1}) \end{aligned}$$

together with (20) and (25) imply

$$f_{k_j}^{n+1} + \delta_n < f_{v(n)+j}^{n+1} < f_{k_j}^{n+1} + 4\left(\frac{2}{3}\right)^{n+1}$$

Since (i) and (ii) of Theorem 1 are easy consequences of (24) and (25), we are left with the proof of (iii).

Fix $N \geq 1$, let $E = \bigcup_{k=1}^N G(f_k)$, and let $\mu \in M(E)$ be a measure of norm one. Let Q be a compact cube whose interior contains E , and choose a continuous function $h \in C(Q, \mathbf{T})$ such that $\|h\mu - |\mu|\| < \sigma$.

Pick $\alpha > 0$ such that $h \in C_{\sigma/4, U(\alpha)}(Q, \mathbf{T})$, choose n large enough so that $n^{-1} < \alpha$ and $v(n) \geq N$, and write $\delta = \delta(\Gamma^n, \sigma, n^{-1})$ as in Lemma 2.

Since $h|_{\Gamma^n} \in C_{\sigma/4, U(n^{-1})}(\Gamma^n, \mathbf{T})$, we can find, after (20), a function $g \in A(\mathbf{R}^2)$ with $\|g\|_\Lambda \leq 3^6$ and $|h(x) - g(x+z)| < \sigma$ for $x \in \Gamma^n$ and $z \in U(\delta)$. Moreover, (24) implies that

$$\text{dist}\left(E, \bigcup_{k=1}^N \Gamma_k^n\right) \leq \frac{6}{5}\delta_n \leq \min(\delta, n^{-1}) \quad \text{for } n \geq 2.$$

Hence for any $x \in E$ there exists $y \in \Gamma^n$ such that

$$|x - y| \leq n^{-1} \quad \text{and} \quad |x - y| < \delta,$$

which implies

$$\begin{aligned} |g(x) - h(x)| &\leq |g(x) - h(y)| + |h(y) - h(x)| \\ &\leq \sigma + \sigma/4 = 5\sigma/4. \end{aligned}$$

Thus we get

$$\|\mu\|_{\text{PM}} 3^6 \geq \left| \int_E g \, d\mu \right| \geq \left| \int_E h \, d\mu \right| - \left| \int_E (h - g) \, d\mu \right| \geq 1 - \sigma - \frac{5\sigma}{4},$$

or

$$\|\mu\| \leq \frac{3^6}{1 - 9\sigma/4} \|\mu\|_{\text{PM}} \leq \beta \|\mu\|_{\text{PM}}.$$

This proves Theorem 1.

THEOREM 2. — *For every $\gamma > 3^{9/2}$ there exists a surface $\Sigma \subset \mathbf{R}^3$ which is the graph of a Lip (1) function and such that $\alpha(\Sigma) \leq \gamma$.*

Proof. — The proof is similar to the proof of Proposition 2. Fix $\beta > 3^6$ such that $3^{3/2}\beta^{1/2} < \gamma$, and choose a sequence $\{f_k\}_{k \geq 1}$ of non-decreasing Lip (1) functions on $[0, 1]$ with the properties stated in Theorem 1. Let $\mathcal{L} = \{f_k : k \geq 1\}$. Let $\xi = (1, 0, 0)$, $\tau = (0, 1, 0)$ and $\eta = (0, \eta_2, \eta_3)$ be unit vectors, and assume $\eta_2 > 0$, $\eta_3 > 0$.

If $D = \{d_1 < d_2 < \dots < d_m\}$ and $E = \{e_1 < e_2 < \dots < e_m\}$ are finite subsets of \mathcal{L} , then let $Q(D, E)$ denote the surface in \mathbf{R}^3 whose trace in the plane $H_x = \{(x, y, z) \in \mathbf{R}^3 : y, z \in \mathbf{R}\}$ is the polygonal path $P_x = P(D_x, E_x, (\eta_2, \eta_3))$ for every $x \in [0, 1]$, where

$$D_x = \{d_1(x) < d_2(x) < \dots < d_m(x)\} \text{ and } E_x = \{e_1(x) < e_2(x) < \dots < e_m(x)\},$$

and where P_x is defined as in the proof of Proposition 2. Such surfaces $Q = Q(D, E)$ will be called \mathcal{L} -surfaces, and we will assume that all \mathcal{L} -surfaces Q considered in the following will contain the graph of a function $f_Q \in C([0, 1]^2)$. This can be achieved by applying, if necessary, a suitable affine linear transformation to \mathbf{R}^3 . Since $Q(D, E)$ is contained in the union of the surfaces

$$\Sigma_d^1 = \{x\xi + d(x)\tau + t\eta : x \in [0, 1], t \in \mathbf{R}\}$$

and

$$\Sigma_\varepsilon^2 = \{x\xi + e(x)\eta + t\tau : x \in [0, 1], t \in \mathbf{R}\}$$

for $d \in D$ and $e \in E$, the functions f_Q are Lip(1) functions with $L(f_Q) < \max(1, \eta_3/\eta_2)$. Let finally $s(Q) = \max_x s(P_x)$, where $s(P)$ is defined as in the proof of Proposition 2.

To construct Σ , fix $0 < \varepsilon \leq 1$ and $\sigma > 0$. If $Q = Q(D, E)$ is a \mathcal{L} -surface, then let $G(D) = \bigcup_{d \in D} G(d)$ and $G(E) = \bigcup_{e \in E} G(e)$. By Proposition 1 for any $\alpha > 0$ there exist $\delta = \delta(Q, \alpha, \varepsilon, \sigma) > 0$ and $\rho = \rho(Q, \alpha, \sigma) > 0$ such that for any function $f \in C_{\sigma/8, U(\alpha)}(G(D), \mathbf{T})$ there exists $g \in A(\mathbf{R}^2)$ with

$$(26) \quad |f(w) - g(w+z)| < \sigma \quad \text{for } w \in G(D) \text{ and } z \in U(\delta),$$

$$(27) \quad \|g(w+z) - f(w) - g(w+z)\| < \beta^4 \varepsilon^{-1/2} \sigma \quad \text{for } w \in G(D) \text{ and } z \in U(\rho),$$

$$(28) \quad |g(v)| \leq \beta^5 \varepsilon \quad \text{for } v \notin G(D) + U(\rho),$$

and

$$(29) \quad \|g\|_\Lambda \leq \beta^5 \varepsilon^{-1/2},$$

and such that the analogue of (26) to (29) also holds for $G(E)$ instead of $G(D)$.

Divide a sequence $Q_m = Q(D_m, E_m)$ of \mathcal{L} -surfaces such that

$$(30) \quad s(Q_m) \downarrow 0$$

and

$$(31) \quad \text{every point of } Q_{m+1} \text{ lies within distance}$$

$$\delta_m = 2^{-1} \eta_3 \delta(Q_m, m^{-1}, \varepsilon, \sigma) \quad \text{away from } Q_m.$$

This is possible because of (ii) of Theorem 1. Since $\delta_m \downarrow 0$, the surfaces $G(f_{Q_m}) \subset Q_m$ converge uniformly towards a surface Σ which is the graph of a Lip(1) function on $[0, 1]^2$.

To prove that Σ is a Helson surface, let $\mu \in M(\Sigma)$ be a measure of

norm one. Let Δ be a compact cube in \mathbb{R}^3 whose interior contains Σ , and let $h: \Delta \rightarrow \mathbb{T}$ be a continuous function such that $\|h\mu - |\mu|\| < \sigma$. Choose $\alpha > \sigma$ such that $h \in C_{\sigma/8, U(\alpha)}(Q, \mathbb{T})$, and choose m large enough so that $s(Q_m) < \alpha/2$, $2m^{-1} < \alpha$ and $\rho_m = \rho(Q_m, m^{-1}, \sigma) < \alpha/12$. For $Q = Q_m$ let Σ_d^1 and Σ_e^2 be defined as before, and write

$$\begin{aligned} \mathbb{R}^1 &= \bigcup_{d \in D} \Sigma_d^1 + U(2\delta_m), & \mathbb{S}^1 &= \bigcup_{d \in D} \Sigma_d^1 + U(\eta_3 \rho_m), \\ \mathbb{R}^2 &= \bigcup_{e \in E} \Sigma_e^2 + U(2\delta_m), & \mathbb{S}^2 &= \bigcup_{e \in E} \Sigma_e^2 + U(\eta_3 \rho_m). \end{aligned}$$

Since $\Sigma \subset \mathbb{R}^1 \cup \mathbb{R}^2$, either $|\mu|(\mathbb{R}^1) \geq \frac{1}{2}$ or $|\mu|(\mathbb{R}^2) \geq \frac{1}{2}$. We will assume the former. We define a function $f \in C_{\sigma/8, U(m^{-1})}(G(D), \mathbb{T})$ by

$$f(x, d_j(x)) = h(x\xi + d_j(x)\tau + e_j(x)\eta),$$

where we wrote

$$D = D_m = \{d_1 < \dots < d_k\} \text{ and } E = E_m = \{e_1 < \dots < e_k\}.$$

Choose $g \in A(\mathbb{R}^2)$ such that properties (26) to (29) hold for f and g with $\alpha = m^{-1}$, and define g_1 on \mathbb{R}^3 by $g_1(x\xi + y\tau + z\eta) = g(x, y)$. Then $g_1 \in B(\mathbb{R}^3)$,

$$\|g_1\|_B \leq \beta^5 \varepsilon^{-1/2} \quad \text{and} \quad |g_1(v)| \leq \beta^5 \varepsilon$$

for $v \notin \mathbb{S}^1$. And, by fixing the ξ -component of w , a similar argument as in the proof of Proposition 2 yields

$$(32) \quad |h(w) - g_1(w)| \leq 2\sigma \quad \text{for } w \in \mathbb{R}^1 \cap \Sigma,$$

and

$$(33) \quad \|g_1(w)\| |h(w) - g_1(w)| \leq 2\beta^4 \varepsilon^{-1/2} \sigma \quad \text{for } w \in \mathbb{S}^1 \cap \Sigma.$$

Now we can split up $\int_{\mathbb{S}^1} g_1 \, d\mu$ the same way as in Proposition 2 and obtain the estimate

$$\begin{aligned} \|\mu\|_{PM} \beta^5 \varepsilon^{-1/2} &\geq \left| \int_{\Sigma} g_1 \, d\mu \right| \\ &\geq (|\mu|(\mathbb{R}^1) - \sigma) - 2\sigma - \beta^5 \varepsilon^{-1/2} \sigma - 2\beta^4 \varepsilon^{-1/2} \sigma - \frac{1}{2} \beta^5 \varepsilon, \end{aligned}$$

or

$$\|\mu\|_{\text{PM}} \geq \beta^{-5} \frac{1}{2} (\varepsilon^{1/2} - \beta^5 \varepsilon^{3/2}) - (2\beta^{-5} \varepsilon^{1/2} + 1 + 2\beta^{-1}) \sigma.$$

For $\varepsilon = (3\beta^5)^{-1}$ the first term of the last sum is at maximum $(3\beta^5)^{-3/2}$. So, if we choose $\varepsilon = (3\beta^5)^{-1}$ and σ sufficiently small for the construction of Σ , then $\|\mu\|_{\text{PM}} \geq \gamma^{-1}$, which proves the theorem.

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