ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 34, nº 3 (1984), p. 189-200

<http://www.numdam.org/item?id=AIF_1984__34_3_189_0>

Ann. Inst. Fourier, Grenoble
33, 3 (1984), 189-200

UNIQUE CONTINUATION FOR SCHRÖDINGER OPERATORS IN DIMENSION THREE OR LESS

by Eric T. SAWYER (1)

1. Introduction.

Suppose Ω is an open connected subset of $\mathbf{R}^n$ and $v(x)$ a nonnegative function on Ω. Let X denote a space of locally integrable functions on Ω and let L denote a partial differential operator. Following [1] we say that the differential inequality

$$|\mathrm{L}u(x)| \leqslant v(x)\,|u(x)| \quad \text{a.e. } x \text{ in } \Omega \tag{1.1}$$

has the unique continuation property, or u.c.p., relative to X if whenever u in X satisfies (1.1) (in the sense of distributions) and vanishes in (i.e. is zero a.e. in) a nonempty open subset of Ω, then u vanishes in Ω. The basic problem is to characterize in a useful way the operators L and the functions v for which (1.1) has the u.c.p. relative to a given space of functions X. In this paper we will be concerned with the case where L is the Laplacian Δ on $\mathbf{R}^n$ and X is the space $\mathrm{H}^{2,1}_{\mathrm{loc}}(\Omega)$ consisting of those functions u which, together with their distributional Laplacian Δu, are locally integrable on the open set $\Omega \subset \mathbf{R}^n$. This case arises in connection with unique continuation for Schrödinger operators, $\mathrm{H} = -\Delta + v$, which in turn has application to the question of the non-existence of positive eigenvalues for $\mathrm{H} = -\Delta + v$. We refer to [1], [2], [4], [10] and [13] and the references given there for details on this application.

In his article on Schrödinger semigroups ([13]), B. Simon essentially proposes (p. 519 of [13]) the problem of proving that if v satisfies the condition

$$(\mathrm{K}_n^{\mathrm{loc}}) \quad \lim_{r\to 0} \sup_{x\in \mathrm{K}} \mathrm{I}_2(\chi_{\mathrm{B}(x,r)}\, v)(x) = 0 \quad \text{for all compact } \mathrm{K} \subset \Omega\,,$$

(1) Research supported in part by NSERC grant A5149.

then the inequality

$$|\Delta u(x)| \leqslant v(x)\,|u(x)| \quad \text{a.e. } x \text{ in } \Omega \tag{1.2}$$

has the u.c.p. relative to $H^{2,1}_{\mathrm{loc}}(\Omega)$, Ω open connected. Here

$$I_2 f(y) = \phi_n * f(y) = \int \phi_n(y-z)f(z)\,dz$$

where $\qquad \phi_n(x) = c_n\,|x|^{2-n}\ (= c\log|x| \text{ if } n=2)$

is the fundamental solution of the Laplacian in $\mathbf{R}^n$. We refer the reader to [13] for a discussion of many illuminating characterizations of condition (K_n^{loc}). Results of W. Amrein, A. Berthier and V. Georgescu ([1]) and J. Saut and B. Scheurer ([11]) state that $v \in L^p_{\mathrm{loc}}(\Omega)$ is sufficient for (1.2) to have the u.c.p. relative to $H^{2,q}_{\mathrm{loc}}(\Omega)$ provided

$$p > \frac{n}{2},\ p \geqslant n-2,\ n \geqslant 2 \text{ and } q = \max\left\{1, \frac{2p}{p+2}\right\}$$

([1]) and relative to $H^{2,2}_{\mathrm{loc}}(\Omega)$ if $p > \dfrac{2n}{3}$ ([11]). See also [2], [3], [4], [5], [6], [10], [12] and [13] and references given there. More recently, C. Kenig and D. Jerison have obtained unique continuation for (1.2) relative to $H^{2,q}_{\mathrm{loc}}$, $q = \dfrac{2n}{n+2}$, when $v \in L^{n/2}_{\mathrm{loc}}$, $n \geqslant 3$ (private communication).

The main result of this paper is that condition (K_n^{loc}) is sufficient for (1.2) to have the u.c.p. relative to $H^{2,1}_{\mathrm{loc}}(\Omega)$ at least when $n \leqslant 3$, thereby establishing Simon's conjecture ([13], p. 519) for $n \leqslant 3$. We remark that $L^p_{\mathrm{loc}} \subset K_n^{\mathrm{loc}}$ for $p > \dfrac{n}{2}, n \geqslant 2$ ([13] ; see (A21)) while $L^{n/2}_{\mathrm{loc}}$ and K_n^{loc} are incomparable for $n \geqslant 3$. For $p \geqslant 1$, define

$$H^{2,p}_{\mathrm{loc}}(\Omega) = \{u \in L^1_{\mathrm{loc}}(\Omega);\ \Delta u \in L^p_{\mathrm{loc}}(\Omega)\}$$

and let $H^{2,p}_c(\Omega)$ denote those u in $H^{2,p}_{\mathrm{loc}}$ with compact support in Ω. Finally, for $w(x) \geqslant 0$ and $p \geqslant 1$, let

$$|||w|||_p = \sup\left\{\left(\int |I_2 f|^p\, w\right)^{1/p};\ \int |f|^p \leqslant 1,\ \operatorname{supp} f \subset \text{unit ball}\right\}.$$

THEOREM. – *Suppose Ω is a connected open subset of $\mathbf{R}^n$, $n = 2$ or 3, and that v is a nonnegative function on Ω. Let $p \geqslant 1$. Then inequality (1.2) has the u.c.p. relative to $H^{2,p}_{\mathrm{loc}}(\Omega)$ if*

$$\lim_{r\to 0} |||\chi_{B(x,r)}\, v^p|||_p = 0 \quad \textit{for all } x \textit{ in } \Omega. \tag{1.3}$$

Remarks. – (I) Since the u.c.p. is a local property, one need only require the vanishing of the limit in (1.3) outside a closed set E of measure zero such that $\Omega - E$ is connected.

(II) When $p = 1$, (1.3) is equivalent to (K_n^{loc}). Indeed, $|||w|||_1 = \sup_{|y| \leq 1} I_2\, w(y)$ and the maximum principle for harmonic functions shows that

$$\|I_2(\chi_{B(x,r)}\, v)\|_\infty = \sup_{y \in B(x,r)} I_2(\chi_{B(x,r)}\, v)(y)$$

$$\leq \sup_{y \in B(x,r)} I_2(\chi_{B(y,2r)}\, v)(y) \longrightarrow 0 \text{ as } r \longrightarrow 0 \quad \text{by } (K_n^{loc}).$$

A simple covering argument yields the converse.

(III) Characterizations of $|||\ |||_p$ for $p > 1$ can be obtained using the methods of [7], or [8] and [9]. Note however that if $v \in L^{3/2}_{loc}(\mathbf{R}^3)$, then Holder's inequality and the Sobolev theorem ([14] ; p. 119) show that (1.3) holds for $1 < p < \dfrac{3}{2}$.

(IV) In the simple case $n = 1$, (1.2) has the u.c.p. relative to $H^{2,1}_{loc}(\Omega)$ if and only if

$$\lim_{r \to 0} \int_{a-r}^{a+r} |x - a|\, v(x)\, dx = 0 \quad \text{for all } a \in \Omega. \tag{1.4}$$

The proof is left to the reader. Note that (1.4) is weaker than (K_1^{loc}).

2. Proof of the Theorem.

The theorem is proved using the approach of T. Carleman ([3] – see also [10]; p. 243) and the following estimate on Taylor polynomial approximations to $|\cdot - y|^{-\alpha}$.

Lemma. – *Let* $\psi_\alpha(x) = |x|^{-\alpha}$, $x \in \mathbf{R}^n$, $\alpha > 0$ *and* $\psi_0(x) = -\log|x|$. *Then for* $0 \leq \alpha \leq 1$, $N \geq 0$, *and* $x, y \in \mathbf{R}^n$

$$|\psi_\alpha(x-y) - \sum_{\ell=0}^{N} \frac{(x \cdot \nabla)^\ell}{\ell!} \psi_\alpha(-y)| \leq C_\alpha \left(\frac{|x|}{|y|}\right)^{N+1} \psi_\alpha(x-y) \tag{2.1}$$

where if $\alpha = 0$, *we restrict* x *and* y *to have modulus less than* $\dfrac{1}{4}$. *The constant* C_α *is independent of* N, n, x *and* y.

Remark. – (V) The Lemma fails for $\alpha > 1$. In fact, for fixed N and y, the left side of (2.1) behaves like $C|x|^N$ for large $|x|$ while the right side is $O(|x|^{N+1-\alpha})$ as $|x| \longrightarrow \infty$.

The Lemma will be proved in §3. Suppose now that $u \in C_c^\infty(\mathbf{R}^n - \{0\})$. Then $u(x) = \int \phi_n(x-y)\,\Delta u(y)\,dy$ and if we subtract the N^{th} degree Taylor polynomial of u at the origin (which is $\equiv 0$) and then interchange differentiation and integration, we obtain

$$u(x) = \int \left[\phi_n(x-y) - \sum_{\ell=0}^{N} \frac{(x\,.\,\nabla)^\ell}{\ell!}\,\phi_n(-y) \right] \Delta u(y)\,dy \tag{2.2}$$

for all x in $\mathbf{R}^n$. A standard limiting argument involving a C_c^∞ approximate identity now shows that if $u \in H_c^{2,1}(\mathbf{R}^n - \{0\})$, then (2.2) holds for a.e. x in $\mathbf{R}^n$.

We now prove the theorem. Suppose u and v satisfy (1.2) where $u \in H_{\text{loc}}^{2,p}(\Omega)$ and v satisfies (1.3). We may assume $v \geqslant 1$ since $v+1$ also satisfies (1.2) and (1.3). Suppose further that $0 \in \Omega$ and u vanishes in a neighbourhood of 0. Let $0 < r < \frac{1}{8}$ be such that $|||\chi_{B(0,r)}\, v^p|||_p < \frac{1}{2C_\alpha}$ where $\alpha = n-2$ and C_α is the constant appearing in (2.1). Choose $\eta \in C_c^\infty$ such that $\eta = 1$ on $B(0,r)$ and $\operatorname{supp}\eta$ is contained in $B\left(0, \frac{1}{4}\right)$. An easy computation shows that $\eta u \in H_c^{2,p}(\mathbf{R}^n - \{0\})$. Using (2.2) and the Lemma, we thus obtain for all $N \geqslant 1$,

$$\begin{aligned}
\int_{|x|<r} (v(x)\,|x|^{-N}\,|u(x)|)^p\,dx
&\leqslant C_\alpha^p \int_{|x|<r} v(x)^p\,[I_2(|y|^{-N}\,|\Delta(\eta u)(y)|)(x)]^p\,dx \\
&\leqslant C_\alpha^p\,|||\chi_{B(0,r)}\,v^p\,|||_p^p \int |y|^{-Np}\,|\Delta(\eta u)(y)|^p\,dy \\
&\leqslant \frac{1}{2^p}\int_{|y|<r} [v(y)\,|y|^{-N}\,|u(y)|]^p\,dy \\
&\quad + \frac{1}{2^p}\int_{|y|>r} |y|^{-Np}\,|\Delta(\eta u)(y)|^p\,dy
\end{aligned} \tag{2.3}$$

since $|\Delta(\eta u)| = |\Delta u| \leqslant v|u|$ in $B(0,r)$ by (1.2). The integrals in (2.3) are finite since $\Delta(\eta u)$ is in L^p and vanishes near the origin. Subtracting the first term on the right side of (2.3) from both sides and then letting $N \longrightarrow \infty$ shows that $v|u|$, and hence u, vanishes in $B(0,r)$. A standard connectedness argument now shows that u vanishes in Ω and this, given the lemma, completes the proof of the theorem.

3. Proof of Lemma.

We prove only the case $0 < \alpha \leqslant 1$, the case $\alpha = 0$ being similar but easier. Since ψ_α is radial and homogeneous, we may assume, after a dilation and a rotation, that $y = (1, 0, \dots, 0)$ and $x = (x_1, x_2, 0, \dots, 0)$. Thus we may as well suppose $n = 2$ and passing to polar co-ordinates $z = (x_1, x_2) = te^{i\theta}$, what we must prove is

$$||1 - te^{i\theta}|^{-\alpha} - P_N(t,\theta)| \leqslant C_\alpha t^{N+1} |1 - te^{i\theta}|^{-\alpha} \tag{3.1}$$

for all $t \geqslant 0$, $|\theta| \leqslant \pi$ and $N = 0, 1, 2, \dots$

where $P_N(t,\theta)$ denotes the Taylor polynomial of degree N at the origin for the real-analytic function $z = te^{i\theta} \longrightarrow |1 - z|^{-\alpha}$. In order to effectively compute $P_N(t,\theta)$, we write

$$|1 - te^{i\theta}|^{-\alpha} = (1 - te^{i\theta})^{-\alpha/2} (1 - te^{-i\theta})^{-\alpha/2}$$

and use the binomial expansion

$$(1 - z)^{-\alpha/2} = \sum_{k=0}^{\infty} (-1)^k \binom{-\alpha/2}{k} z^k = \sum_{k=0}^{\infty} \left[\begin{smallmatrix}\frac{\alpha}{2}-1\\ k\end{smallmatrix}\right] z^k, \quad |z| < 1$$

where the symbol $\left[\begin{smallmatrix}\gamma\\ k\end{smallmatrix}\right]$ denotes the product $\prod_{j=1}^{k} \left(1 + \frac{\gamma}{j}\right)$ for $k \geqslant 1$ and $\left[\begin{smallmatrix}\gamma\\ 0\end{smallmatrix}\right] = 1$. Thus with $\gamma = \frac{\alpha}{2} - 1$ we have for $t < 1$,

$$|1 - te^{i\theta}|^{-\alpha} = \left(\sum_{k=0}^{\infty} \left[\begin{smallmatrix}\gamma\\ k\end{smallmatrix}\right] (te^{i\theta})^k\right)\left(\sum_{\ell=0}^{\infty} \left[\begin{smallmatrix}\gamma\\ \ell\end{smallmatrix}\right] (te^{-i\theta})^\ell\right)$$
$$= \sum_{m=0}^{\infty} a_m(\theta)\, t^m \tag{3.2}$$

where $a_m(\theta) = \sum_{k+\ell=m} \genfrac{[}{]}{0pt}{}{\gamma}{k}\genfrac{[}{]}{0pt}{}{\gamma}{\ell} e^{i(k-\ell)\theta}$ $(= P_m^{\alpha/2}(\cos\ \theta)$ in terms of ultraspherical polynomials). Thus with $z = te^{i\theta}$

$$\begin{aligned} P_N(t,\theta) &= \sum_{k+\ell\leqslant N} \genfrac{[}{]}{0pt}{}{\gamma}{k}\genfrac{[}{]}{0pt}{}{\gamma}{\ell} (te^{i\theta})^k (te^{-i\theta})^\ell \\ &= \sum_{k+\ell\leqslant N} \genfrac{[}{]}{0pt}{}{\gamma}{k}\genfrac{[}{]}{0pt}{}{\gamma}{\ell} z^k \bar{z}^\ell = \sum_{m=0}^{N} a_m(\theta)\, t^m . \end{aligned} \tag{3.3}$$

We first dispose of the simple cases $t \leqslant \frac{1}{2}$ and $t \geqslant 2$. Since $(1-t)^{-\alpha} = \sum_{k=0}^{\infty} \genfrac{[}{]}{0pt}{}{\alpha-1}{k} t^k$ for $|t| < 1$ we conclude, on comparison with the case $\theta = 0$ of (3.2), that

$$|a_m(\theta)| \leqslant a_m(0) = \genfrac{[}{]}{0pt}{}{\alpha-1}{m},\ m \geqslant 0,\ |\theta| \leqslant \pi. \tag{3.4}$$

Now for $0 < \alpha \leqslant 1$, we have $\genfrac{[}{]}{0pt}{}{\alpha-1}{k} \leqslant C(k+1)^{\alpha-1} \leqslant C$ (see (3.6) below) and thus for $t \geqslant 2$ we have by (3.4) that

$$|P_N(t,\theta)| \leqslant \sum_{m=0}^{N} |a_m(\theta)|\, t^m \leqslant Ct^N \leqslant Ct^{N+1}\, |1-te^{i\theta}|^{-\alpha}$$

since $t^{-\alpha} \leqslant 2\,|1-te^{i\theta}|^{-\alpha}$ when $t \geqslant 2$. This proves (3.1) for $t \geqslant 2$. On the other hand, for $t \leqslant \frac{1}{2}$ we have by (3.2) and (3.4)

$$\big||1-te^{i\theta}|^{-\alpha} - P_N(t,\theta)\big| \leqslant \sum_{m=N+1}^{\infty} |a_m(\theta)|\, t^m \leqslant Ct^{N+1}\, |1-te^{i\theta}|^{-\alpha}$$

since $|1-te^{i\theta}| \geqslant \frac{1}{2}$ when $t \leqslant \frac{1}{2}$ and this yields (3.1) for $t \leqslant \frac{1}{2}$. We now consider the cases $1 \leqslant t \leqslant 2$ and $\frac{1}{2} \leqslant t < 1$ separately.

The case $1 \leqslant t \leqslant 2$

Let $\Delta\genfrac{[}{]}{0pt}{}{\gamma}{k} = \genfrac{[}{]}{0pt}{}{\gamma}{k} - \genfrac{[}{]}{0pt}{}{\gamma}{k+1}$ and $D_k(z) = \sum_{j=0}^{k} z^j = \dfrac{1-z^{k+1}}{1-z}$.

We continue to write $z = te^{i\theta}$. Summing by parts twice in the formula (3.3) for $P_N(t,\theta)$ we obtain

$$
\begin{aligned}
\mathrm{P_N}(t,\theta) &= \sum_{k=0}^{\mathrm{N}} \genfrac{[}{]}{0pt}{}{\gamma}{k} z^k \sum_{\ell=0}^{\mathrm{N}-k} \genfrac{[}{]}{0pt}{}{\gamma}{\ell} \bar{z}^{\ell} \\
&= \sum_{k=0}^{\mathrm{N}} \genfrac{[}{]}{0pt}{}{\gamma}{k} z^k \left\{ \sum_{\ell=0}^{\mathrm{N}-k-1} \Delta \genfrac{[}{]}{0pt}{}{\gamma}{\ell} \mathrm{D}_\ell(\bar{z}) + \genfrac{[}{]}{0pt}{}{\gamma}{\mathrm{N}-k} \mathrm{D}_{\mathrm{N}-k}(\bar{z}) \right\} \\
&= \sum_{\ell=0}^{\mathrm{N}-1} \Delta \genfrac{[}{]}{0pt}{}{\gamma}{\ell} \mathrm{D}_\ell(\bar{z}) \sum_{k=0}^{\mathrm{N}-\ell-1} \genfrac{[}{]}{0pt}{}{\gamma}{k} z^k + \sum_{k=0}^{\mathrm{N}} \genfrac{[}{]}{0pt}{}{\gamma}{k} \genfrac{[}{]}{0pt}{}{\gamma}{\mathrm{N}-k} z^k \mathrm{D}_{\mathrm{N}-k}(\bar{z}) \\
&= \sum_{\ell=0}^{\mathrm{N}-2} \Delta \genfrac{[}{]}{0pt}{}{\gamma}{\ell} \mathrm{D}_\ell(\bar{z}) \sum_{k=0}^{\mathrm{N}-\ell-2} \Delta \genfrac{[}{]}{0pt}{}{\gamma}{k} \mathrm{D}_k(z) \\
&\quad + \sum_{\ell=0}^{\mathrm{N}-1} \Delta \genfrac{[}{]}{0pt}{}{\gamma}{\ell} \genfrac{[}{]}{0pt}{}{\gamma}{\mathrm{N}-\ell-1} \mathrm{D}_\ell(\bar{z}) \mathrm{D}_{\mathrm{N}-\ell-1}(z) \\
&\qquad\qquad + \sum_{k=0}^{\mathrm{N}} \genfrac{[}{]}{0pt}{}{\gamma}{k} \genfrac{[}{]}{0pt}{}{\gamma}{\mathrm{N}-k} z^k \mathrm{D}_{\mathrm{N}-k}(\bar{z}) \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III}. \qquad (3.5)
\end{aligned}
$$

Let $d = |1-z|^{-1}$ and note that $d \geqslant \frac{1}{3}$ since $1 \leqslant t \leqslant 2$. We use the following estimates; recall that $\gamma = \frac{\alpha}{2} - 1$ satisfies $-1 < \gamma \leqslant -\frac{1}{2}$.

$$\genfrac{[}{]}{0pt}{}{\gamma}{k} \leqslant \mathrm{C}(k+1)^\gamma,\ \Delta \genfrac{[}{]}{0pt}{}{\gamma}{k} \leqslant \mathrm{C}(k+1)^{\gamma-1},\ -1<\gamma\leqslant 0,\ k \geqslant 0 \qquad (3.6)$$

$$|\mathrm{D}_k(z)| \leqslant \mathrm{C}t^{k+1} \min\{k+1, d\},\ t \geqslant 1,\ k \geqslant 0 \qquad (3.7)$$

$$\sum_{k=0}^{\infty} (k+1)^{\gamma-1} \min\{k+1, d\} \leqslant \mathrm{C}_\gamma d^{\gamma+1},\ -1<\gamma<0,\ d \geqslant \frac{1}{3} \qquad (3.8)$$

$$\sum_{k=0}^{\mathrm{N}} (k+1)^\gamma (\mathrm{N}-k+1)^\gamma \min\{\mathrm{N}-k+1, d\} \leqslant \mathrm{C}_\gamma d^{2\gamma+2}$$
$$-1<\gamma\leqslant -\frac{1}{2},\ d \geqslant \frac{1}{3}. \qquad (3.9)$$

For $k \geqslant 1$, $\gamma > -1$,

$$\genfrac{[}{]}{0pt}{}{\gamma}{k} = \prod_{j=1}^{k} \left(1 + \frac{\gamma}{j}\right) \leqslant \prod_{j=1}^{k} e^{\frac{\gamma}{j}} = e^{\gamma \sum_{j=1}^{k} \frac{1}{j}} \leqslant e^\gamma k^\gamma$$

which yields the first estimate in (3.6) and the second follows using $\Delta\left[{\gamma \atop k}\right] = -\dfrac{\gamma}{k+1}\left[{\gamma \atop k}\right]$. Since $\left|\sum_{j=0}^{k} z^j\right| \leqslant (k+1)\,t^k$ and $\left|\dfrac{1-z^{k+1}}{1-z}\right| \leqslant 2t^{k+1}\,d$ we have (3.7). Estimate (3.8) follows easily upon considering the sums $\sum_{k\leqslant d}$ and $\sum_{k>d}$ separately. Finally the left side of (3.9) is dominated by

$$\sum_{k=0}^{N} (k+1)^{\gamma}\,(N-k+1)^{\gamma+1} \leqslant (N+1)^{\gamma+1} \sum_{k=0}^{N} (k+1)^{\gamma} \leqslant CN^{2\gamma+2}$$

which yields (3.9) if $N \leqslant 2d$. If $N > 2d$, then the left side of (3.9) is dominated by

$$\sum_{k<\frac{N-d}{2}} (k+1)^{\gamma}\left(\frac{N}{2}\right)^{\gamma} d + \sum_{\frac{N-d}{2}\leqslant k\leqslant N-d} \left(\frac{N-d}{2}\right)^{\gamma} (N-k+1)^{\gamma}\, d$$
$$+ \sum_{N-d\leqslant k\leqslant N} (N-d)^{\gamma}\,(N-k+1)^{\gamma+1}$$
$$\leqslant C_{\gamma}\,[N^{2\gamma+1}\,d + N^{2\gamma+1}\,d + N^{\gamma}\,d^{\gamma+2}] \leqslant C_{\gamma}\,d^{2\gamma+2}$$

since both γ and $2\gamma+1$ are nonpositive.

We now return to (3.5) and show that the modulus of each of the terms I, II and III is dominated by

$$C_{\alpha}\,t^{N+1}\,d^{2\gamma+2} = C_{\alpha}\,t^{N+1}\,|1-te^{i\theta}|^{-\alpha}$$

as required in (3.1). We have

$$|I| \leqslant \sum_{k+\ell\leqslant N-2} \left|\Delta\left[{\gamma \atop k}\right] D_k(z)\right| \left|\Delta\left[{\gamma \atop \ell}\right] D_{\ell}(\bar{z})\right|$$

$$\leqslant Ct^{N} \sum_{k+\ell\leqslant N-2} (k+1)^{\gamma-1}\min\{k+1\,,d\}\;(\ell+1)^{\gamma-1}\min\{\ell+1\,,d\} \quad \text{by (3.6), (3.7)}$$

$$\leqslant Ct^{N}\left(\sum_{k=0}^{\infty} (k+1)^{\gamma-1}\min\{k+1\,,d\}\right)^2$$

and (3.8) now yields the desired result for term I. From (3.6) and (3.7) we have

$$|II| \leqslant Ct^{N+1} \sum_{\ell=0}^{N-1} (\ell+1)^{\gamma-1}\,(N-\ell)^{\gamma}\min\{\ell+1\,,d\}\min\{N-\ell\,,d\}$$

$$\leqslant Ct^{N+1} \sum_{\ell=0}^{N-1} (\ell+1)^{\gamma}\,(N-\ell)^{\gamma}\min\{N-\ell\,,d\} \leqslant C_{\alpha}\,t^{N+1}\,d^{2\gamma+2}$$

by (3.9). Finally $|\mathrm{III}| \leqslant C_\alpha t^{N+1} d^{2\gamma+2}$ follows immediately upon combining (3.6), (3.7) and (3.9) and this completes the proof of (3.1) for $1 \leqslant t \leqslant 2$.

The case $\frac{1}{2} \leqslant t < 1$

We continue to write $z = te^{i\theta}$ and $d = |1-z|^{-1}$. In place of (3.7) we have the estimate

$$|D_k(z)| \leqslant C \min\{k+1, d\} \quad k \geqslant 0, \ |z| \leqslant 1. \tag{3.10}$$

Let $T_{N+1}(z) = \sum_{k=N+1}^{\infty} [{}^{\gamma}_{k}] z^k = z^{N+1} \sum_{j=0}^{\infty} \Delta[{}^{\gamma}_{N+1+j}] D_j(z)$ for $|z| < 1$. By (3.6), (3.10) and (3.8),

$$|T_{N+1}(z)| \leqslant C|z|^{N+1} \Big(\sum_{j \leqslant d} (N+1+j)^{\gamma-1} (j+1) \tag{3.11}$$

$$+ \sum_{j > d} (N+1+j)^{\gamma-1} d \Big) \leqslant C_\alpha t^{N+1} d^{\gamma+1} = C_\alpha t^{N+1} |1-te^{i\theta}|^{-\alpha/2}.$$

From (3.2) and (3.3) we see that for $t < 1$,

$$|1-te^{i\theta}|^{-\alpha} - P_N(t,\theta) = \sum_{k+\ell \geqslant N+1} [{}^{\gamma}_{k}] z^k [{}^{\gamma}_{\ell}] \bar{z}^\ell \tag{3.12}$$

$$= T_{N+1}(z) T_0(\bar{z}) + T_0(z) T_{N+1}(\bar{z}) - T_{N+1}(z) T_{N+1}(\bar{z}) + R_N(t,\theta)$$

where $R_N(t,\theta) = \sum_{k=1}^{N} [{}^{\gamma}_{k}] z^k \sum_{\ell=N+1-k}^{N} [{}^{\gamma}_{\ell}] \bar{z}^\ell$. In view of (3.11) and (3.12), the estimate (3.1) for $\frac{1}{2} \leqslant t < 1$ will follow once we have shown

$$|R_N(t,\theta)| \leqslant C_\alpha t^{N+1} |1-te^{i\theta}|^{-\alpha}, \ \frac{1}{2} \leqslant t < 1, \ |\theta| \leqslant \pi. \tag{3.13}$$

To this end let $\tilde{\Delta}[{}^{\gamma}_{k}] = [{}^{\gamma}_{k}] - [{}^{\gamma}_{k-1}] = -\Delta[{}^{\gamma}_{k-1}]$ for $k \geqslant 1$. Two summations by part yield

$$R_N(t,\theta) = \sum_{k=1}^{N} [{}^{\gamma}_{k}] z^k \bar{z}^N \Big\{ \sum_{\ell=N+2-k}^{N} \tilde{\Delta}[{}^{\gamma}_{\ell}] D_{N-\ell}\Big(\frac{1}{\bar{z}}\Big)$$

$$+ [{}^{\gamma}_{N+1-k}] D_{k-1}\Big(\frac{1}{\bar{z}}\Big)\Big\}$$

$$= z^{N}\,\bar z^{N} \sum_{\ell=3}^{N} \tilde\Delta[{\gamma \atop \ell}]\, D_{N-\ell}\left(\frac{1}{\bar z}\right) \sum_{k=N+3-\ell}^{N} \tilde\Delta[{\gamma \atop k}]\, D_{N-k}\left(\frac{1}{z}\right)$$

$$+ z^{N}\,\bar z^{N} \sum_{\ell=2}^{N} \tilde\Delta[{\gamma \atop \ell}]\,[{\gamma \atop N+2-\ell}]\, D_{N-\ell}\left(\frac{1}{\bar z}\right) D_{\ell-2}\left(\frac{1}{z}\right)$$

$$+ \bar z^{N} \sum_{k=1}^{N} [{\gamma \atop k}]\, z^{k}\, [{\gamma \atop N+1-k}]\, D_{k-1}\left(\frac{1}{\bar z}\right)$$

$$= \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

Note that $td = |1 - \frac{1}{z}|^{-1} \geqslant \frac{1}{3}$ if $\frac{1}{2} \leqslant |z| < 1$. Since $\tilde\Delta[{\gamma \atop k}] = -\Delta[{\gamma \atop k-1}]$, we thus obtain from (3.6) and (3.7) that

$$[{\gamma \atop k}] \leqslant C k^{\gamma}, -1 < \gamma \leqslant 0,\ k \geqslant 1.$$

$$|\tilde\Delta[{\gamma \atop k}]| \leqslant C k^{\gamma-1}, -1 < \gamma \leqslant 0, k \geqslant 1. \tag{3.15}$$

$$|D_k\left(\frac{1}{z}\right)| \leqslant C t^{-k-1} \min\{k+1, td\},\ \frac{1}{2} \leqslant t < 1,\ k \geqslant 0.$$

We shall also need the following consequence of (3.9).

$$\sum_{\ell=3}^{N} \ell^{\gamma-1} \min\{N-\ell+1, td\} \sum_{k=N+3-\ell}^{N} k^{\gamma-1} \min\{N-k+1, td\}$$

$$\leqslant \sum_{\ell=3}^{N} \ell^{\gamma-1} \min\{N-\ell+1, td\}\, C_{\gamma} (N+3-\ell)^{\gamma}\, \ell \leqslant C_{\gamma} (td)^{2\gamma+2}$$

$$\text{by (3.9) for } -1 < \gamma \leqslant -\frac{1}{2},\ \ td \geqslant \frac{1}{3}. \tag{3.16}$$

From (3.15) and (3.16) we immediately deduce

$$|\mathrm{I}| \leqslant C_{\alpha}\, t^{N+1}\, |1 - te^{i\theta}|^{-\alpha}$$

for $\frac{1}{2} \leqslant t < 1$ and the corresponding estimates for II and III follow from (3.15) and (3.9) along lines similar to those used for their counterparts in (3.5). Thus (3.13) holds and this completes the proof of the Lemma.

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Manuscrit reçu le 25 août 1983
révisé le 30 novembre 1983.

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