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On the $A$-integrability of singular integral transforms


<http://www.numdam.org/item?id=AIF_1984__34_2_53_0>
ON THE A-INTEGRABILITY OF SINGULAR INTEGRAL TRANSFORMS

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1. Introduction.

In this paper we shall generalize a theorem of Alexandrov on the A-Integrability of Riesz transforms [1].

Let $L^{1,\infty}(\mathbb{R}^n)$ denote the weak-$L^1$ space consisting of measurable functions $f$ on $\mathbb{R}^n$ for which $\sup_{a>0} \alpha m\{x \in \mathbb{R}^n : |f(x)| > \alpha\} = K < \infty$, where $m$ denotes the Lebesgue measure on $\mathbb{R}^n$; let $L^{1,\infty}_0(\mathbb{R}^n)$ (resp. $L^{1,\infty}_{00}(\mathbb{R}^n)$) be the subspace of $L^{1,\infty}(\mathbb{R}^n)$ consisting of functions which satisfy $\lim_{a \to \infty} \alpha m\{x : |f(x)| > \alpha\} = 0$ (resp. the subspace of $L^{1,\infty}_0(\mathbb{R}^n)$ of functions satisfying $\lim_{a \to 0^+} \alpha m\{|f(x)| > \alpha\} = 0$). For brevity we shall write $L^{1,\infty}_0(\mathbb{R}^n)$ (resp. $L^{1,\infty}_{00}(\mathbb{R}^n)$) to mean the space $L^{1,\infty}(\mathbb{R}^n)$ (resp. $L^{1,\infty}_0(\mathbb{R}^n)$, resp. $L^{1,\infty}_{00}(\mathbb{R}^n)$). A similar notation will be used for the weak Hardy spaces defined below.

For a function $f$, we write $\lambda_f(\alpha)$ for its distribution function, i.e. $\lambda_f(\alpha) = m\{x \in \mathbb{R}^n : |f(x)| > \alpha\}$, $\alpha > 0$. In the following $C$, $C'$, $K$ will denote several different constants.

Let $u(x,y)$, $x \in \mathbb{R}^n$, $y > 0$ be a harmonic function on the upper half plane $\mathbb{R}^{n+1}_+$, and for $x \in \mathbb{R}^n$, $\Gamma_a(x) = \{(x',y) \in \mathbb{R}^{n+1}_+ : |x' - x| < ay\}$ is the cone of aperture $a$ at $x$. When $a = 1$, we shall simply write $\Gamma(x)$. The non tangential maximal function of $u$ is the function $u^*(x) = \sup_{\Gamma(x)} |u(x',y)|$.

We define $H^{1,\infty}_{0,0}(\mathbb{R}^n) = \{u(x,y) : u$ a harmonic function on $\mathbb{R}^{n+1}_+$ such that $u^* \in L^{1,\infty}_{0,0}(\mathbb{R}^n)\}$. These are the spaces considered by Alexandrov in [1], where he proves an A-Integrability result for the system of conjugate functions of $u$. 
Let \((X, \mu)\) be a measure space and \(f\) a measurable function on \(X\). Then \(f\) is said to be \(A\)-integrable if

(i) \(\alpha \mu \{ x \in X : |f(x)| > \alpha \} = o(1), \ \alpha \to + \infty, \ \alpha \to 0^+\)

(ii) \(\lim_{\varepsilon \to 0^+} \int_X \left[ f_{\varepsilon, \alpha}(x) \right] d\mu(x) \) exists

where \(f_{\varepsilon, \alpha}(x) = f(x) \) if \(\varepsilon < |f(x)| \leq \alpha\)

= 0 if not.

The limit in (ii) is called the \(A\)-integral of \(f\) and is denoted by

\[ (A) \int f \, d\mu \ \text{[2].} \]

**Theorem (Alexandrov).** Let \(u_0 \in H_{00}^{1, \infty}\) and let \(u_1, \ldots, u_n\) be the system of conjugate harmonic functions of \(u_0\). If \(f_0, f_1 \ldots f_n\) denote the non-tangential boundary functions of \(u_0, u_1 \ldots u_n\) and \(g_0, g_1 \ldots g_n\) is another such system of boundary functions such that \(g_k \in L^2 \cap L^\infty(\mathbb{R}^n)\), \(k = 0, 1 \ldots n\), then

\[ (A) \int (f_k g_0 + f_0 g_k) \, dx = 0, \ \ k = 1, 2 \ldots n. \]

In section 3, we shall prove a similar result for singular integral transforms, using real variable methods, and the fact that a certain set of transforms forms a conjugate system does not play any essential role. Our result then contains the above result of Alexandrov.

2.

The \(H_{(0,0)}^{1, \infty}\) spaces have been defined above by means of a non-tangential maximal function with respect to a cone of aperture 1. But this is in fact not a restriction, and we have

**Proposition 1.** Let \(u(x, y)\) be any continuous function on \(\mathbb{R}^{n+1}\). Then the following are equivalent:

1) \(u^*(x) = \sup_{\Gamma(x)} |u(x', y)| \in L_{(0,0)}^{1, \infty}(\mathbb{R}^n)\)
2) \( u^*_N(x) = \sup_{\Gamma_N(x)} |u(x',y)| \in L^{1,\infty}_{(0,\infty)}(\mathbb{R}^n) \)

3) \( u^{**}(x) = \sup_{(x',y) \in \mathbb{R}^{n+1}} |u(x',y)| \left( \frac{y}{|x-x'|+y} \right)^M \in L^{1,\infty}_{(0,\infty)}(\mathbb{R}^n) \)

where \( M > n \).

The proof of this proposition is only a slight modification of the proof of lemma 1 of [3], where the equivalence of \( L^p(\mathbb{R}^n) \) \((0 < p < \infty)\) norms of these functions has been proved.

Further, these spaces can also be characterized using the area function,

\[
S_a(u)(x) = \left( \int_{\Gamma_d(x)} \int |\nabla (x',y)|^2 y^{1-n} \, dx \, dy \right)^{1/2}
\]
as a consequence of the following inequality [3]

\[
\lambda_{S(u)}(\alpha) < C \left\{ \lambda_{u^*}(\alpha) + \frac{1}{\alpha^2} \int_0^\alpha \beta \lambda_{u^*}(\beta) \, d\beta \right\}
\]
and a corresponding inequality with the roles of \( S(u) \) and \( u^* \) interchanged. These inequalities have been proved in [3] for harmonic functions \( u(x,y) \) which are Poisson Integrals of \( L^2 \)-functions, a restriction which can easily be removed. Also the restriction on the cones can be removed using Proposition 1. A similar characterization also holds for the radial maximal function \( u^+(x) = \sup_{y > 0} |u(x,y)| \) and for the \( g \)-function

\[
g(u)(x) = \left( \int_0^\infty |\nabla u(x,y)|^2 y \, dy \right)^{1/2}
\]
(see [5] for details). We summarize these results in

**Proposition 2.** — Let \( u(x,y) \) be a harmonic function on \( \mathbb{R}^{n+1}_+ \). Then the following are equivalent:

1) \( u^* \in L^{1,\infty}_{(0,\infty)}(\mathbb{R}^n) \)
2) \( u^+ \in L^{1,\infty}_{(0,\infty)}(\mathbb{R}^n) \)
3) \( S(u) \in L^{1,\infty}_{(0,\infty)}(\mathbb{R}^n) \)
4) \( g(u) \in L^{1,\infty}_{(0,\infty)}(\mathbb{R}^n) \).
It is well-known that if \( u(x,y) \) is the Poisson integral of a bounded measure \( \left( \text{i.e. } u(x,y) = P_y\mu(x) = C_n \int_{\mathbb{R}^n} \frac{y}{(|x-t|^2 + y^2)^{n+1}} \, d\mu(t) \right) \) then the supremum on the right hand side of the above inequality \( \lesssim \frac{y|y|^n}{|x|^{2(n+1)}} \). Also since \( u^* \in H^{1,\infty} \), for \( (x,y) \in \mathbb{R}^{n+1}_+ \) fixed, the

**Proposition 3.** Let \( \mu \) be a bounded measure on \( \mathbb{R}^n \) and let

\[
\lim_{\delta \to 0^+} \delta m\{u^* > \delta\} = 0 \text{ if and only if } \int_{\mathbb{R}^n} d\mu(x) = 0.
\]

**Proof.** It is well-known that

\[
\int_{\mathbb{R}^n} d\mu(x) = \lim_{y \to -\infty} C_n y^n u(0,y).
\]

From this it follows immediately that for \( \delta \) small enough

\[
\delta m\{u^* > \delta\} \geq C\int_{\mathbb{R}^n} d\mu(x).
\]

Conversely, let \( \int d\mu(x) = 0 \). By an easy reduction we may assume that \( \mu \) has compact support and that \( \mu \) is supported on the unit cube \( Q_0 \) in \( \mathbb{R}^n \).

\[
u(x,y) = C_n \int_{\mathbb{R}^n} \frac{y}{(|x-t|^2 + y^2)^{n+1}} \, d\mu(t)
= \int_{\mathbb{R}^n} [P_y(x-t) - P_y(x)] \, d\mu(t).
\]

Hence \( |u(x,y)| \lesssim C_n ||\mu|| \sup_{t \in Q_0} |P_y(x-t) - P_y(x)| \).

If \( |x| \) is large, then the supremum on the right hand side of the above inequality \( \sim \frac{y|x|^n}{|x|^{2(n+1)}} \). Also since \( u^* \in H^{1,\infty} \), for \( (x,y) \in \mathbb{R}^{n+1}_+ \) fixed, the
ball in \( \mathbb{R}^n \) with center \( x \) and radius \( y \) is contained in the set \( \{ u^* > |u(x,y)| \} \). Therefore

\[
K \geq |u(x,y)| m\{ u^* > |u(x,y)| \} \geq C|u(x,y)| y^n
\]

i.e. \( |u(x,y)| \leq C/y^n \).

Consequently,

\[
\{(x,y) \in \mathbb{R}^{n+1} : |u(x,y)| > \delta\} \subseteq \{ (x,y) : |x| \leq 1/\delta^{1/(n+2)}, y \leq C/\delta^{1/n} \}
\]

Hence

\[
d \delta m\{ u^* > \delta\} \leq C||\mu|| \delta^{n+1/2} = o(1) \quad \text{as} \quad \delta \to 0.
\]

This with Proposition 2 completes the proof.

**Corollary.** — \( H^{1,\infty}_{00} \cap \{ \mathcal{P}_x \mu(x) ; \mu \text{ a bounded measure} \} = \{ \mathcal{P}_x f(x) : f \in L^1(\mathbb{R}^n), \int f(x) \, dx = 0 \} \).

In the next proposition, we prove that if \( u \in H^{1,\infty} \) then \( u(\cdot,y) \) converges in the sense of tempered distributions as \( y \to 0 \). The proof of the corresponding result for the \( H^p \) spaces [3] does not directly apply since in this case the fact that \( u^* \in L^{1,\infty}(\mathbb{R}^n) \) does not necessarily imply that for \( y > 0 \), \( u(\cdot,y) \in L^1(\mathbb{R}^n) \).

**Proposition.** — Let \( u \in H^{1,\infty} \). Then \( \lim_{y \to 0} u(\cdot,y) = f \) exists in the sense of tempered distribution.

**Proof.** — We have seen above that \( u^* \in L^{1,\infty} \) implies that \( |u(x,y)| \leq C/y^n \). Hence for every \( y > 0 \), the function \( u_y(x) = u(x,y) \in L^2(\mathbb{R}^n) \) and

\[
\|u_y\|_2^2 = \int_{\mathbb{R}^n} |u(x,y)|^2 \, dx
\]

\[
= \int_{|u| \leq C_y^{-n}} |u(x,y)|^2 \, dx \leq \int_0^{C_y^{-n}} \beta \lambda_{u_y}(\beta) \, d\beta = C/y^n.
\]

Now for \( \delta > 0 \) fixed we define a function almost everywhere by

\[
\hat{u}_0(\xi) = \hat{u}(\xi,\delta)e^{2\pi i \xi \delta},
\]
$\xi \in \mathbb{R}^n$ where $\hat{u}(\xi, \delta)$ is the Plancherel transform of $u_\delta(x)$. Since $u(x, y)$ is a harmonic function, we have $\hat{u}(\cdot, \delta') = \hat{u}(\cdot, \delta)e^{2\pi i (\delta' - \delta)}$, $\delta, \delta' > 0$; hence the definition of $\hat{u}_0$ does not depend on the choice of $\delta$. It is clear that $\hat{u}_0$ defines a distribution, denoted by $T_{\hat{u}_0}$. To show that this distribution is in fact tempered, it is enough to prove that for every rapidly decreasing $C^\infty$ function $\psi(h)$ on $\mathbb{R}^n$, the distributions $\psi(h)\tau_h T_{\hat{u}_0}$ are bounded in the space of distributions (here $\tau_h$ is the translation by $h$). Let $\varphi$ be a $C^\infty$ function with compact support (say $Q$), then

$$\langle \psi(h)\tau_h T_{\hat{u}_0}, \varphi \rangle \leq |\psi(h)| \int_Q |\hat{u}(\xi, \delta)| e^{2\pi i \delta h} |\varphi(\xi + h)| d\xi.$$ 

Choose $\delta = 1/K (1 + |h|)$ where $K$ is a suitable constant depending on the support of $\varphi$ then

$$\langle \psi(h)\tau_h T_{\hat{u}_0}, \varphi \rangle \leq C' |\psi(h)| \|\hat{u}_0\|_2 \|\varphi\|_2 \leq C |\psi(h)|(1 + |h|)^{n/2} \|\varphi\|_2 \leq C \|\varphi\|_2.$$ 

This proves that $T_{\hat{u}_0}$ is a tempered distribution. Let $f = \mathcal{F}^{-1}(\hat{u}_0)$ (the inverse Fourier transform of $T_{\hat{u}_0}$). Then, if $\varphi$ is in the Schwarz class $\mathcal{S}$,

$$\int u(x, y)\overline{\varphi(x)} \, dx = \int \hat{u}(\xi, y)\overline{\varphi(\xi)} \, d\xi = \int \hat{u}_0(\xi)e^{-2\pi i \xi h} \overline{\varphi(\xi)} \, d\xi$$

$$\frac{\mathcal{F}'}{y \to 0} \langle T_{\hat{u}_0}, \varphi \rangle = \langle f, \varphi \rangle$$

so that $u(\cdot, y) \to f$ as $y \to 0$ in the sense of tempered distributions.

We shall not go into the details, but with the estimates proved in [3] for $H^p$ spaces ($0 < p < \infty$) it can be shown that the $H^1_{(0, \infty)}$ spaces can be realized as certain spaces of tempered distributions:

Let $\varphi \in \mathcal{S}$, $\int_{\mathbb{R}^n} \varphi(x) \, dx = 1$ and $\varphi_t(x) = t^{-n}\varphi(x/t)$. Then if $H^1_{(0, \infty)}$ is identified with the space of boundary distributions (Proposition 3), we have

$$H^1_{(0, \infty)} = \{ f \in \mathcal{S}' : \sup_{\Gamma(x)} |\varphi, \star f(x')| \in L^1_{(0, \infty)}(\mathbb{R}^n) \}$$

(for details, see theorem 11 in [3]).
3. The A-integral.

Let $K$ be a tempered distribution on $\mathbb{R}^n$, which is $C^1$ away from the origin and

(i) $|\hat{K}(\xi)| \leq B < \infty$

(ii) $|\nabla K(x)| \leq C|x|^{-n-1}.$

For $f \in L^1(\mathbb{R}^n)$, $Tf = K \star f$ (which exists as a limit) is a tempered distribution and belongs to $H^1_{0,\infty}$ i.e. it arises as the boundary distribution of a harmonic function $v(x,y)$ such that $v^* \in L^1_0(\mathbb{R}^n)$. We let $Tf$ also denote the non-tangential boundary function of $v(x,y)$. Further, if

$$\int_{\mathbb{R}^n} f(x) \, dx = 0 \quad \text{(i.e. the associated harmonic function is in $H^1_{0,\infty}$)}$$

then $Tf \in L^1_{0,\infty}(\mathbb{R}^n)$.

**Theorem.** Let $f \in L^1(\mathbb{R}^n)$, $\int f(x) \, dx = 0$, and let $Tf$ be as defined above. If $\psi \in L^2 \cap L^\infty(\mathbb{R}^n)$ is such that $T\psi \in L^2 \cap L^\infty(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} Tf(x)\psi(x) \, dx = -\int_{\mathbb{R}^n} f(x)T\psi(x) \, dx.$$

**Proof.** Let $M = \max(||\psi||_2, ||\psi||_\infty, ||T\psi||_2, ||T\psi||_\infty)$ and suppose $\varepsilon > 0$ is small and $\alpha > 0$ is large

$$\int_{\mathbb{R}^n} (Tf)_{k,\alpha}(x) \, dx = \int_{\{u^* < \alpha \}} [Tf\psi]_{k,\alpha} \, dx$$

$$+ \int_{\{u^* < \varepsilon \}} [Tf\psi]_{k,\alpha} \, dx + \int_{\{u^* > \alpha \}} [Tf\psi]_{k,\alpha} \, dx$$

$$= I_1 + I_2 + I_3.$$

Clearly

$$|I_3| \leq \alpha m\{u^* > \alpha\} = o(1) \quad \text{as} \quad \alpha \to \infty, \quad \text{uniformly in} \quad \varepsilon.$$

To estimate $I_1$ and $I_2$ we do a Calderon Zygmund decomposition at the level $\alpha$. Then $f$ can be written as $f(x) = g(x) + b(x)$, where
\[ |g(x)| \leq C\alpha \text{ and } \|g\|_1 \leq \|f\|_1 \text{ (hence } \|g\|_2 \leq C\alpha \|f\|_1), \text{ and the function } b \text{ satisfies} \]
\[
\int b(x) \, dx = 0
\]
\[
\|b\|_1 \leq \int_{[u^* > \alpha]} |f(x)| \, dx + C\alpha m\{u^* > \alpha\}
\]
(3)
\[
\int_{[u^* < \alpha]} |Tb(x)| \leq C\alpha m\{u^* > \alpha\}.
\]

Consider the integral

\[ I_1 = \int_{F_\varepsilon,\alpha} [Tf\psi]_{\varepsilon,\alpha} \, dx, \text{ where } F_{\varepsilon,\alpha} = \{x : \varepsilon < u^*(x) \leq \alpha\} \]
\[ = \int_{F_{\varepsilon,\alpha}} Tf\psi \, dx - \int_{F_{\varepsilon,\alpha}\cap \{|Tf\psi| < \varepsilon\}} Tf\psi \, dx - \int_{F_{\varepsilon,\alpha}\cap \{|Tf\psi| > \alpha\}} Tf\psi \, dx \]
\[ = \int_{F_{\varepsilon,\alpha}} Tf\psi \, dx - J_1 - J_2. \]

We have \( |J_1| < \varepsilon m\{u^* > \varepsilon\} = o(1) \) as \( \varepsilon \to 0 \), uniformly in \( \alpha \) and

\[ |J_2| \leq \int_{F_{\varepsilon,\alpha}\cap \{|Tf\psi| > \alpha\}} |Tg\psi| \, dx + \int_{[u^* < \alpha]} |Tb\psi| \, dx \]
\[ \leq C\|Tg\|_2 \|\psi\|_{L^\infty}(|Tf\psi| > \alpha) \|_{L^\infty} + C\alpha m\{u^* > \alpha\} \]

using Holder's inequality and (3). But since \( g \) is in \( L^2 \) and \( T \) is a bounded operator on \( L^2 \),

\[ |J_2| \leq C\|g\|_2 M\{m\{|Tf\psi| > \alpha^2\}\}^{1/2} + C\alpha m\{u^* > \alpha\} \]
\[ \leq CM\|f\|_1 (\alpha m\{|Tf\psi| > \alpha\})^{1/2} + C\alpha m\{u^* > \alpha\} \]
\[ = o(1) \quad \text{as } \alpha \to \infty \quad \text{uniformly in } \varepsilon. \]

Hence we get

(4) \[ I_1 = \int_{F_{\varepsilon,\alpha}} Tf(x)\psi(x) \, dx + o(1), \quad \alpha \to \infty, \quad \varepsilon \to 0_+ \]
\[ = \int_{F_{\varepsilon,\alpha}} Tg(x)\psi(x) \, dx + o(1), \quad \alpha \to \infty, \quad \varepsilon \to 0_+. \]
It remains to evaluate $I_2$. Let $F_\varepsilon = \{u^* \leq \varepsilon\}$

\[ I_2 = \int_{F_\varepsilon} [Tf \psi]_{u^*,\varepsilon}(x) \, dx \]
\[ = \int_{F_\varepsilon} Tf \psi \, dx - \int_{F_\varepsilon \cap \{Tf \psi \leq \varepsilon\}} Tf \psi \, dx - \int_{F_\varepsilon \cap \{Tf \psi > \varepsilon\}} Tf \psi \, dx \]
\[ = \int_{F_\varepsilon} Tf \psi \, dx - K_1 - K_2. \]

$K_2$ can be estimated in the same way as $J_2$ and we get $|K_2| = o(1)$ as $\alpha \to \infty$ uniformly in $\varepsilon$.

Note that $K_1$ is independent of $\alpha$; to estimate we do a Calderon-Zygmund decomposition of $f$ at a level $\alpha_0$ chosen large enough depending on $\varepsilon$.

Write $f = g_0 + b_0$ with $g_0$ and $b_0$ as above with respect to $\alpha_0$. Then

\[ |K_1| \leq \varepsilon m\{|Tg_0 \psi| > \varepsilon\} + \int_{\{Tg_0 \psi \leq \varepsilon\}} |Tg_0 \psi| \, dx + \int_{\{u^* \leq \varepsilon\}} |Tb_0 \psi| \, dx \]
\[ = o(1) \quad \text{as} \quad \varepsilon \to 0. \]

Hence

\[ (5) \quad |I_2| = \int_{\{u^* \leq \varepsilon\}} Tg \psi \, dx + o(1), \quad \alpha \to \infty, \quad \varepsilon \to 0. \]

Combining (2), (4) and (5),

\[ \int_{\mathbb{R}^n} [Tf \psi]_{u^*,\varepsilon} \, dx = \int_{\{u^* \leq \varepsilon\}} Tg \psi \, dx + o(1) \]
\[ = \int_{\mathbb{R}^n} Tg(x) \psi(x) \, dx + o(1) \]
\[ = -\int g(x) T\psi(x) \, dx + o(1) \]
\[ = -\int f(x) T\psi(x) \, dx + o(1), \quad \alpha \to \infty, \quad \varepsilon \to 0. \]
In the last step we have used the estimate
\[
\|b\|_1 \leq \int_{\{u^* > \alpha\}} |f(x)| \, dx + C\alpha n\{u^* > \alpha\} = o(1) \quad \text{as} \; \alpha \to \infty.
\]
This completes the proof of the theorem.

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Manuscrit reçu le 11 mars 1983.

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