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ON THE A-INTEGRABILITY
OF SINGULAR INTEGRAL TRANSFORMS

by Shobha MADAN

1. Introduction.

In this paper we shall generalize a theorem of Alexandrov on the A-Integrability of Riesz transforms [1].

Let \( L^{1,\infty}(\mathbb{R}^n) \) denote the weak-\( L^1 \) space consisting of measurable functions \( f \) on \( \mathbb{R}^n \) for which \( \sup_{a>0} \alpha m\{x \in \mathbb{R}^n : |f(x)| > \alpha\} = K < \infty \), where \( m \) denotes the Lebesgue measure on \( \mathbb{R}^n \); let \( L^{1,\infty}_0(\mathbb{R}^n) \) (resp. \( L^{1,\infty}_{00}(\mathbb{R}^n) \)) be the subspace of \( L^{1,\infty}(\mathbb{R}^n) \) consisting of functions which satisfy \( \lim_{a \to \infty} \alpha m\{x : |f(x)| > \alpha\} = 0 \) (resp. the subspace of \( L^{1,\infty}_0(\mathbb{R}^n) \) of functions satisfying \( \lim_{a \to 0^+} \alpha m\{|f(x)| > \alpha\} = 0 \)). For brevity we shall write \( L^{1,\infty}_{0,(0,0)}(\mathbb{R}^n) \) to mean the space \( \langle L^{1,\infty}(\mathbb{R}^n) \rangle \) (resp. \( L^{1,\infty}_0(\mathbb{R}^n) \) resp. \( L^{1,\infty}_{00}(\mathbb{R}^n) \)).

A similar notation will be used for the weak Hardy spaces defined below.

For a function \( f \), we write \( \lambda_f(\alpha) \) for its distribution function, i.e. \( \lambda_f(\alpha) = \{x \in \mathbb{R}^n : |f(x)| > \alpha\} \), \( \alpha > 0 \). In the following \( C, C', K \) will denote several different constants.

Let \( u(x,y), x \in \mathbb{R}^n, y > 0 \) be a harmonic function on the upper half plane \( \mathbb{R}^{n+1}_+ \), and for \( x \in \mathbb{R}^n, \Gamma_a(x) = \{(x',y) \in \mathbb{R}^{n+1}_+ : |x' - x| < ay\} \) is the cone of aperture \( a \) at \( x \). When \( a = 1 \), we shall simply write \( \Gamma(x) \). The non tangential maximal function of \( u \) is the function \( u^*(x) = \sup_{\Gamma(x)} |u(x',y)| \).

We define \( H_{0,(0,0)}^{1,\infty} = \{u(x,y) : u \) a harmonic function on \( \mathbb{R}^{n+1}_+ \) such that \( u^* \in L^{1,\infty}_{0,(0,0)}(\mathbb{R}^n) \} \). These are the spaces considered by Alexandrov in [1], where he proves an A-Integrability result for the system of conjugate functions of \( u \).
Let \((X, \mu)\) be a measure space and \(f\) a measurable function on \(X\). Then \(f\) is said to be \(A\)-integrable if

(i) \(\alpha \mu \{ x \in X : |f(x)| > \alpha \} = o(1), \ \alpha \to + \infty, \ \alpha \to 0^+\)

(ii) \(\lim_{\varepsilon \to 0^+} \int_X \left[ f_{\varepsilon, \alpha}(x) \right] d\mu(x) \) exists

where \(f_{\varepsilon, \alpha}(x) = f(x)\) if \(\varepsilon < |f(x)| \leq \alpha\)

\(= 0\) if not.

The limit in (ii) is called the \(A\)-integral of \(f\) and is denoted by \((A) \int f \, d\mu\) [2].

**Theorem (Alexandrov).** — Let \(u_0 \in H_0^1, \infty\) and let \(u_1, \ldots, u_n\) be the system of conjugate harmonic functions of \(u_0\). If \(f_0, f_1 \ldots f_n\) denote the non-tangential boundary functions of \(u_0, u_1 \ldots u_n\) and \(g_0, g_1 \ldots g_n\) is another such system of boundary functions such that \(g_k \in L^2 \cap L^\infty(R^n), k = 0, 1 \ldots n\), then

\[(A) \int (f_k g_0 + f_0 g_k) \, dx = 0, \ \ k = 1, 2 \ldots n.\]

In section 3, we shall prove a similar result for singular integral transforms, using real variable methods, and the fact that a certain set of transforms forms a conjugate system does not play any essential role. Our result then contains the above result of Alexandrov.

2.

The \(H_{(0, \infty)}^{1, \infty}\) spaces have been defined above by means of a non-tangential maximal function with respect to a cone of aperture 1. But this is in fact not a restriction, and we have

**Proposition 1.** — Let \(u(x, y)\) be any continuous function on \(R_+^{n+1}\). Then the following are equivalent:

1) \(u^*(x) = \sup_{\Gamma(x)} |u(x', y)| \in L_0^{1, \infty}(R^n)\)
2) \( u^*_N(x) = \sup_{\gamma_N(x)} |u(x',y)| \in L^{1,\infty}_{(0,0)}(\mathbb{R}^n) \)

3) \( u^{**}(x) = \sup_{(x,y) \in \mathbb{R}^{n+1}} |u(x',y)| \left( \frac{y}{|x-x'| + y} \right)^M \in L^{1,\infty}_{(0,0)}(\mathbb{R}^n) \)

where \( M > n \).

The proof of this proposition is only a slight modification of the proof of lemma 1 of [3], where the equivalence of \( L^p(\mathbb{R}^n) \) \((0 < p < \infty)\) norms of these functions has been proved.

Further, these spaces can also be characterized using the area function,

\[
S_a(u)(x) = \left( \int_{\gamma_d(x)} \int |\nabla u(x',y)|^2 y^{1-n} \, dx \, dy \right)^{1/2}
\]

as a consequence of the following inequality [3]

\[
\lambda_{S(u)}(\alpha) < C \left\{ \lambda_{u^*}^{\alpha}(\alpha) + \frac{1}{\alpha^2} \int_0^\alpha \beta \lambda_{u^*}^{\beta}(\beta) \, d\beta \right\}
\]

and a corresponding inequality with the roles of \( S(u) \) and \( u^* \) interchanged. These inequalities have been proved in [3] for harmonic functions \( u(x,y) \) which are Poisson Integrals of \( L^2 \)-functions, a restriction which can easily be removed. Also the restriction on the cones can be removed using Proposition 1. A similar characterization also holds for the radial maximal function \( u^+(x) = \sup_{y>0} |u(x,y)| \) and for the \( g \)-function

\[
g(u)(x) = \left( \int_0^\infty |\nabla u(x,y)|^2 y \, dy \right)^{1/2}
\]

(see [5] for details). We summarize these results in

**Proposition 2.** — Let \( u(x,y) \) be a harmonic function on \( \mathbb{R}^{n+1}_+ \). Then the following are equivalent :

1) \( u^* \in L^{1,\infty}_{(0,0)}(\mathbb{R}^n) \)
2) \( u^+ \in L^{1,\infty}_{(0,0)}(\mathbb{R}^n) \)
3) \( S(u) \in L^{1,\infty}_{(0,0)}(\mathbb{R}^n) \)
4) \( g(u) \in L^{1,\infty}_{(0,0)}(\mathbb{R}^n) \).
It is well-known that if \( u(x,y) \) is the Poisson integral of a bounded measure \( \mu \) (i.e. \( u(x,y) = P_y \mu(x) = \int_{\mathbb{R}^n} \frac{y}{(|x-t|^2 + y^2)^{\frac{n+1}{2}}} \, d\mu(t) \)) then \( u \in H^{1,\infty} [6] \) and \( \mu \) is absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R}^n \) if and only if \( u \in H_0^{1,\infty} [4] \). It is not difficult to see that not every function of \( H^{1,\infty} (\text{resp. } H_0^{1,\infty}) \) can be obtained in this way. In the following proposition we characterize those bounded measures on \( \mathbb{R}^n \) whose Poisson integrals are in \( H_0^{1,\infty} \).

**Proposition 3.** — Let \( \mu \) be a bounded measure on \( \mathbb{R}^n \) and let \( u(x,y) = P_y \mu(x) \) be its harmonic extension to \( \mathbb{R}_+^{n+1} \).

Then \( \lim_{\delta \to 0^+} \delta m\{u^* > \delta\} = 0 \) if and only if \( \int_{\mathbb{R}^n} d\mu(x) = 0 \).

**Proof.** — It is well-known that

\[
\int_{\mathbb{R}^n} d\mu(x) = \lim_{t \to \infty} C_n y^* u(0,y).
\]

From this it follows immediately that for \( \delta \) small enough

\[
\delta m\{u^* > \delta\} \geq C|\int_{\mathbb{R}^n} d\mu(x)|.
\]

Conversely, let \( \int d\mu(x) = 0 \). By an easy reduction we may assume that \( \mu \) has compact support and that \( \mu \) is supported on the unit cube \( Q_0 \) in \( \mathbb{R}^n \).

\[
u(x,y) = C_n \int_{\mathbb{R}^n} \frac{y}{(|x-t|^2 + y^2)^{\frac{n+1}{2}}} \, d\mu(t)
= \int_{\mathbb{R}^n} [P_y(x-t) - P_y(x)] \, d\mu(t).
\]

Hence \( |u(x,y)| < C_n \|\mu\| \sup_{t \in Q_0} |P_y(x-t) - P_y(x)| \).

If \( |x| \) is large, then the supremum on the right hand side of the above inequality \( \sim \frac{y|\mu|^n}{|x|^{2(n+1)}} \). Also since \( u^* \in H^{1,\infty} \), for \( (x,y) \in \mathbb{R}_+^{n+1} \) fixed, the
ball in \( \mathbb{R}^n \) with center \( x \) and radius \( y \) is contained in the set \( \{ u^* > |u(x,y)| \} \). Therefore

\[
K \geq |u(x,y)|m\{u^* > |u(x,y)|\} \geq C|u(x,y)|y^n
\]

i.e. \( |u(x,y)| \leq C/y^n \).

Consequently,

\[
\{(x,y) \in \mathbb{R}^{n+1}_+ : |u(x,y)| > \delta\} \subseteq \{(x,y) : |x| \leq 1/\delta^{1/(n+2)}, \ y \leq C/\delta^{1/n}\}.
\]

Hence

\[
\delta m\{u^+(x) > \delta\} \leq C||\mu|| \delta^{n+1/2} = o(1) \quad \text{as} \quad \delta \to 0.
\]

This with Proposition 2 completes the proof.

**Corollary.** — \( H^{1,\infty}_{00} \cap \{P_x \mu(x) ; \ \mu \ \text{a bounded measure}\} \)

\[
= \{P_x f(x) : f \in L^1(\mathbb{R}^n), \int f(x) \, dx = 0\}.
\]

In the next proposition, we prove that if \( u \in H^{1,\infty} \) then \( u(\cdot , y) \) converges in the sense of tempered distributions as \( y \to 0 \). The proof of the corresponding result for the \( H^p \) spaces [3] does not directly apply since in this case the fact that \( u^* \in L^{1,\infty}(\mathbb{R}^n) \) does not necessarily imply that for \( y > 0 \), \( u(\cdot , y) \in L^1(\mathbb{R}^n) \).

**Proposition.** — Let \( u \in H^{1,\infty} \). Then \( \lim_{y \to 0} u(\cdot , y) = f \) exists in the sense of tempered distribution.

**Proof.** — We have seen above that \( u^* \in L^{1,\infty} \) implies that \( |u(x,y)| \leq C/y^n \). Hence for every \( y > 0 \), the function \( u_y(x) = u(x,y) \in L^2(\mathbb{R}^n) \) and

\[
||u_y||_2^2 = \int_{\mathbb{R}^n} |u(x,y)|^2 \, dx
\]

\[
= \int_{|w| < Cy^{-n}} |u(x,y)|^2 \, dx \leq \int_0^{Cy^{-n}} \beta \lambda_{u_y}(\beta) \, d\beta = C/y^n.
\]

Now for \( \delta > 0 \) fixed we define a function almost everywhere by

\[
\hat{u}_0(\xi) = \hat{u}(\xi,\delta)e^{2\pi i \xi \delta},
\]
ξ ∈ ℝⁿ where \( \hat{u}(ξ, δ) \) is the Plancherel transform of \( u_δ(x) \). Since \( u(x, y) \) is a harmonic function, we have \( \hat{u}(\cdot, δ') = \hat{u}(\cdot, δ)e^{2\pi i \frac{(δ - δ')}{δ}} \), \( δ, δ' > 0 \); hence the definition of \( \hat{u}_δ \) does not depend on the choice of \( δ \). It is clear that \( \hat{u}_δ \) defines a distribution, denoted by \( T_{\hat{u}_δ} \). To show that this distribution is in fact tempered, it is enough to prove that for every rapidly decreasing \( C^∞ \) function \( ϕ(h) \) on \( ℝⁿ \), the distributions \( ϕ(h)τ_δT_{\hat{u}_δ} \) are bounded in the space of distributions (here \( τ_δ \) is the translation by \( h \)). Let \( ϕ \) be a \( C^∞ \) function with compact support (say \( Q \)), then

\[
|\langle ϕ(h)τ_δT_{\hat{u}_δ}, ϕ(x)⟩| ≤ C(1 + |h|)^{n/2} \|ϕ\|_2 \leq C\|ϕ\|_2.
\]

This proves that \( T_{\hat{u}_δ} \) is a tempered distribution. Let \( f = \mathcal{F}^{-1}(\hat{u}) \) (the inverse Fourier transform of \( T_{\hat{u}_δ} \)). Then, if \( ϕ \) is in the Schwarz class \( \mathcal{S} \),

\[
\int u(x, y)ϕ(x) \, dx = \int \hat{u}(ξ, y)ϕ(ξ) \, dξ
\]

\[
= \int \hat{u}_δ(ξ)e^{-2\pi i ξ y}\hat{ϕ}(ξ) \, dξ
\]

\[
\mathcal{F}^{\prime}
\]

\[
y \to 0 \quad \langle T_{\hat{u}_δ}, ϕ⟩ = \langle f, ϕ⟩
\]

so that \( u(\cdot, y) \to f \) as \( y \to 0 \) in the sense of tempered distributions.

We shall not go into the details, but with the estimates proved in [3] for \( H^p \) spaces \( 0 < p < \infty \) it can be shown that the \( H^{1,∞}_{(0,∞)} \) spaces can be realized as certain spaces of tempered distributions:

Let \( ϕ ∈ \mathcal{S} \), \( ∫ ϕ(x) \, dx = 1 \) and \( ϕ_t(x) = t^{-n}ϕ(x/t) \). Then if \( H^{1,∞}_{(0,∞)} \) is identified with the space of boundary distributions (Proposition 3), we have

\[
H^{1,∞}_{(0,∞)} = \{ f ∈ \mathcal{S}′ : \sup_{γ(x)} |ϕ_t ∗ f(x′)| ∈ L^{1,∞}_{(0,∞)}(ℝ^n) \}
\]

(for details, see theorem 11 in [3]).
3. The A-integral.

Let $K$ be a tempered distribution on $\mathbb{R}^n$, which is $C^1$ away from the origin and

(i) $|\hat{K}(\xi)| \leq B < \infty$

(ii) $|\nabla K(x)| \leq C|x|^{-n-1}$.

For $f \in L^1(\mathbb{R}^n)$, $Tf = K * f$ (which exists as a limit) is a tempered distribution and belongs to $H_0^{1,\infty}$ i.e. it arises as the boundary distribution of a harmonic function $v(x,y)$ such that $v^* \in L^1_0(\mathbb{R}^n)$. We let $Tf$ also denote the non-tangential boundary function of $v(x,y)$. Further, if $\int_{\mathbb{R}^n} f(x) \, dx = 0$ (i.e. the associated harmonic function is in $H_0^{1,\infty}$) then $Tf \in L_{00}^{1,\infty}(\mathbb{R}^n)$.

**THEOREM.** Let $f \in L^1(\mathbb{R}^n)$, $\int f(x) \, dx = 0$, and let $Tf$ be as defined above. If $\psi \in L^2 \cap L^\infty(\mathbb{R}^n)$ is such that $T\psi \in L^2 \cap L^\infty(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} Tf(x) \psi(x) \, dx = \int_{\mathbb{R}^n} f(x) T\psi(x) \, dx. \quad \text{(A)}$$

**Proof.** Let $M = \max(||\psi||_2, ||\psi||_\infty, ||T\psi||_2, ||T\psi||_\infty)$ and suppose $\varepsilon > 0$ is small and $\alpha > 0$ is large

$$\int_{\mathbb{R}^n} Tf_{\varepsilon,\alpha}(x) \, dx = \int_{[\varepsilon < u^* \leq \alpha]} [Tf\psi]_{\varepsilon,\alpha} \, dx$$

$$+ \int_{[u^* \leq \varepsilon]} [Tf\psi]_{\varepsilon,\alpha} \, dx + \int_{[u^* > \alpha]} [Tf\psi]_{\varepsilon,\alpha} \, dx$$

$$= I_1 + I_2 + I_3.$$

Clearly

$$|I_3| \leq \alpha m\{u^* > \alpha\} = o(1) \text{ as } \alpha \to \infty, \text{ uniformly in } \varepsilon.$$

To estimate $I_1$ and $I_2$ we do a Calderon Zygmund decomposition at the level $\alpha$. Then $f$ can be written as $f(x) = g(x) + b(x)$, where
\[ |g(x)| \leq C\alpha \quad \text{and} \quad \|g\|_1 \leq \|f\|_1 \quad \text{(hence} \quad \|g\|_2^2 \leq C\alpha \|f\|_1), \quad \text{and the function} \quad b \quad \text{satisfies} \]
\[ \int b(x) \, dx = 0 \]
\[ \|b\|_1 \leq \int_{\{u^* > \alpha\}} |f(x)| \, dx + C\alpha m\{u^* > \alpha\} \]

(3) \[ \int_{\{u^* \leq \alpha\}} |Tb(x)| \leq C\alpha m\{u^* > \alpha\}. \]

Consider the integral
\[ I_1 = \int_{F_{e,\alpha}} [Tf \psi]_{e,\alpha} \, dx, \quad \text{where} \quad F_{e,\alpha} = \{x : \varepsilon < u^*(x) \leq \alpha\} \]
\[ = \int_{F_{e,\alpha}} Tf \psi \, dx - \int_{F_{e,\alpha} \cap \{|Tf \psi| \leq \varepsilon\}} Tf \psi \, dx - \int_{F_{e,\alpha} \setminus \{|Tf \psi| > \varepsilon\}} Tf \psi \, dx \]
\[ = \int_{F_{e,\alpha}} Tf \psi \, dx - J_1 - J_2. \]

We have \[ |J_1| < \varepsilon m\{u^* > \varepsilon\} = o(1) \quad \text{as} \quad \varepsilon \to 0, \quad \text{uniformly in} \quad \alpha \quad \text{and} \]
\[ |J_2| \leq \int_{F_{e,\alpha} \cap \{|Tf \psi| > \varepsilon\}} |Tg \psi| \, dx + \int_{\{u^* \leq \alpha\}} |Tb \psi| \, dx \]
\[ \leq C\|Tg\|_2 \|\psi\|_{L^2} \|Tf \psi\|_{L^2} + C\alpha m\{u^* > \alpha\} \]

using Holder’s inequality and (3). But since \( g \) is in \( L^2 \) and \( T \) is a bounded operator on \( L^2 \),
\[ |J_2| \leq C\|g\|_2 M(m\{|Tf \psi| > \alpha^2\})^{1/2} + C\alpha m\{u^* > \alpha\} \]
\[ \leq CM\|f\|_1 (\alpha m\{|Tf \psi| > \alpha\})^{1/2} + C\alpha m\{u^* > \alpha\} \]
\[ = o(1) \quad \text{as} \quad \alpha \to \infty \quad \text{uniformly in} \quad \varepsilon. \]

Hence we get
\[ (4) \quad I_1 = \int_{F_{e,\alpha}} Tf(x) \psi(x) \, dx + o(1), \quad \alpha \to \infty, \quad \varepsilon \to 0_+. \]
\[ = \int_{F_{e,\alpha}} Tg(x) \psi(x) \, dx + o(1), \quad \alpha \to \infty, \quad \varepsilon \to 0_+. \]
It remains to evaluate $I_2$. Let $F_\varepsilon = \{u^* \leq \varepsilon\}$.

$$I_2 = \int_{F_\varepsilon} [Tf\psi]_{k,\alpha}(x) \, dx$$

$$= \int_{F_\varepsilon} Tf\psi(x) \, dx - \int_{F_\varepsilon \cap \{\|T\psi\| \leq \varepsilon\}} Tf\psi(x) \, dx - \int_{F_\varepsilon \cap \{\|T\psi\| > \varepsilon\}} Tf\psi(x) \, dx$$

$$= \int_{F_\varepsilon} Tf\psi(x) \, dx - K_1 - K_2.$$

$K_2$ can be estimated in the same way as $J_2$ and we get $|K_2| = o(1)$ as $\alpha \to \infty$ uniformly in $\varepsilon$

Note that $K_1$ is independent of $\alpha$; to estimate we do a Calderon-Zymund decomposition of $f$ at a level $\alpha_0$ chosen large enough depending on $\varepsilon$. Write $f = g_0 + b_0$ with $g_0$ and $b_0$ as above with respect to $\alpha_0$. Then

$$|K_1| \leq \int_{\{\|T\psi\| \leq \varepsilon, \|Tg_0\| > \varepsilon\}} |Tf\psi| \, dx + \int_{\{\|T\psi\| \leq \varepsilon, \|Tg_0\| \leq \varepsilon\} \cap F_\varepsilon} |Tf\psi| \, dx$$

$$\leq \varepsilon m \{\|Tg_0\| > \varepsilon\} + \int_{\{\|Tg_0\| \leq \varepsilon\}} |Tg_0\psi| \, dx + \int_{\{u^* \leq \varepsilon\}} |Tg_0\psi| \, dx$$

$$= o(1) \quad \text{as} \quad \varepsilon \to 0.$$

Hence

$$|I_2| = \int_{\{u^* \leq \varepsilon\}} Tg\psi \, dx + o(1), \quad \alpha \to \infty, \quad \varepsilon \to 0.$$  

Combining (2), (4) and (5),

$$\int_{\mathbb{R}^n} [Tf\psi]_{k,\alpha} \, dx = \int_{\{u^* \leq \alpha\}} Tg\psi \, dx + o(1)$$

$$= \int_{\mathbb{R}^n} Tg(x)\psi(x) \, dx + o(1)$$

$$= - \int g(x)T\psi(x) \, dx + o(1)$$

$$= - \int f(x)T\psi(x) \, dx + o(1), \quad \alpha \to \infty, \quad \varepsilon \to 0.$$
In the last step we have used the estimate

$$\|b\|_1 \leq \int_{\{u^* > \alpha\}} |f(x)| \, dx + C\alpha n\{u^* > \alpha\} = o(1) \quad \text{as} \quad \alpha \to \infty.$$ 

This completes the proof of the theorem.

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