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ON THE A-INTEGRABILITY OF SINGULAR INTEGRAL TRANSFORMS

by Shobha MADAN

1. Introduction.

In this paper we shall generalize a theorem of Alexandrov on the A-Integrability of Riesz transforms [1].

Let $L^{1,\infty}(\mathbf{R}^n)$ denote the weak-L^1 space consisting of measurable functions f on $\mathbf{R}^n$ for which $\sup\limits_{\alpha>0} \alpha m\{x \in \mathbf{R}^n : |f(x)| > \alpha\} = K < \infty$, where m denotes the Lebesgue measure on $\mathbf{R}^n$; let $L_0^{1,\infty}(\mathbf{R}^n)$ (resp. $L_{00}^{1,\infty}(\mathbf{R}^n)$) be the subspace of $L^{1,\infty}(\mathbf{R}^n)$ consisting of functions which satisfy $\lim\limits_{\alpha\to\infty} \alpha m\{x : |f(x)|>\alpha\} = 0$ (resp. the subspace of $L_0^{1,\infty}(\mathbf{R}^n)$ of functions satisfying $\lim\limits_{\alpha\to 0+} \alpha m\{|f(x)|>\alpha\} = 0$). For brevity we shall write $L^1_{(0,(0))}(\mathbf{R}^n)$ to mean the space « $L^{1,\infty}(\mathbf{R}^n)$ (resp. $L_0^{1,\infty}$, resp. $L_{00}^{1,\infty}$) ». A similar notation will be used for the weak Hardy spaces defined below. For a function f, we write $\lambda_f(\alpha)$ for its distribution function, i.e. $\lambda_f(\alpha) = m\{x \in \mathbf{R}^n : |f(x)|>\alpha\}$, $\alpha > 0$. In the following C, C', K will denote several different constants.

Let $u(x,y)$, $x \in \mathbf{R}^n$, $y > 0$ be a harmonic function on the upper half plane $\mathbf{R}_+^{n+1}$, and for $x \in \mathbf{R}^n$, $\Gamma_a(x) = \{(x',y) \in \mathbf{R}_+^{n+1} : |x'-x|<ay\}$ is the cone of aperture a at x. When $a = 1$, we shall simply write $\Gamma(x)$. The non tangential maximal function of u is the function $u^\star(x) = \sup\limits_{\Gamma(x)} |u(x',y)|$.

We define $H^{1,\infty}_{(0,(0))} = \{u(x,y) : u$ a harmonic function on $\mathbf{R}_+^{n+1}$ such that $u^* \in L^{1,\infty}_{(0,(0))}(\mathbf{R}^n)\}$. These are the spaces considered by Alexandrov in [1], where he proves an A-Integrability result for the system of conjugate functions of u.

Let (X,μ) be a measure space and f a measurable function on X. Then f is said to be A-integrable if

(i) $\alpha\mu\{x \in X : |f(x)| > \alpha\} = o(1), \quad \alpha \to +\infty, \quad \alpha \to 0_+$

(ii) $\lim\limits_{\substack{\varepsilon \to 0_+ \\ \alpha \to +\infty}} \int_X [f]_{\varepsilon,\alpha}(x)\, d\mu(x)$ exists

where
$$[f]_{\varepsilon,\alpha}(x) = f(x) \text{ if } \varepsilon < |f(x)| \leqslant \alpha$$
$$= 0 \text{ if not.}$$

The limit in (ii) is called the A-integral of f and is denoted by (A) $\int f\, d\mu$ [2].

THEOREM (Alexandrov). — *Let* $u_0 \in H^{1,\infty}_{00}$ *and let* $u_1, \ldots, u_n$ *be the system of conjugate harmonic functions of* u_0. *If* $f_0, f_1 \ldots f_n$ *denote the non-tangential boundary functions of* $u_0, u_1 \ldots u_n$ *and* $g_0, g_1 \ldots g_n$ *is another such system of boundary functions such that* $g_k \in L^2 \cap L^\infty(\mathbf{R}^n)$, $k = 0, 1 \ldots n$, *then*

$$(A) \int (f_k g_0 + f_0 g_k)\, dx = 0, \quad k = 1, 2 \ldots n.$$

In section 3, we shall prove a similar result for singular integral transforms, using real variable methods, and the fact that a certain set of transforms forms a conjugate system does not play any essential role. Our result then contains the above result of Alexandrov.

2.

The $H^{1,\infty}_{(0,(0))}$ spaces have been defined above by means of a non-tangential maximal function with respect to a cone of aperture 1. But this is in fact not a restriction, and we have

PROPOSITION 1. — *Let* $u(x, y)$ *be any continuous function on* $\mathbf{R}^{n+1}_+$. *Then the following are equivalent :*

1) $u^\star(x) = \sup\limits_{\Gamma(x)} |u(x',y)| \in L^{1,\infty}_{(0,(0))}(\mathbf{R}^n)$

2) $u_{\mathrm{N}}^{\star}(x) = \sup\limits_{\Gamma_{\mathrm{N}}(x)} |u(x',y)| \in \mathrm{L}^{1,\infty}_{(0,(0))}(\mathbf{R}^n)$

3) $u^{\star\star}(x) = \sup\limits_{(x',y)\in \mathbf{R}^{n+1}_{+}} |u(x',y)| \left(\dfrac{y}{|x-x'|+y}\right)^{\mathrm{M}} \in \mathrm{L}^{1,\infty}_{(0,(0))}(\mathbf{R}^n)$

where $\mathrm{M} > n$.

The proof of this proposition is only a slight modification of the proof of lemma 1 of [3], where the equivalence of $\mathrm{L}^p(\mathbf{R}^n)$ $(0<p<\infty)$ norms of these functions has been proved.

Further, these spaces can also be characterized using the area function,

$$\mathrm{S}_a(u)(x) = \left(\int_{\Gamma_a(x)}\int |\nabla(x',y)|^2 y^{1-n}\, dx\, dy\right)^{1/2}$$

as a consequence of the following inequality [3]

$$\lambda_{\mathrm{S}(u)}(\alpha) < \mathrm{C}\left\{\lambda_{u^\star}(\alpha) + \frac{1}{\alpha^2}\int_0^\alpha \beta\lambda_{u^\star}(\beta)\, d\beta\right\}$$

and a corresponding inequality with the roles of $\mathrm{S}(u)$ and $u^\star$ interchanged. These inequalities have been proved in [3] for harmonic functions $u(x,y)$ which are Poisson Integrals of L^2-functions, a restriction which can easily be removed. Also the restriction on the cones can be removed using Proposition 1. A similar characterization also holds for the radial maximal function $u^+(x) = \sup\limits_{y>0} |u(x,y)|$ and for the g-function

$$g(u)(x) = \left(\int_0^\infty |\nabla u(x,y)|^2 y\, dy\right)^{1/2}$$

(see [5] for details). We summarize these results in

PROPOSITION 2. — *Let* $u(x,y)$ *be a harmonic function on* $\mathbf{R}^{n+1}_{+}$. *Then the following are equivalent :*

1) $u^\star \in \mathrm{L}^{1,\infty}_{(0,(0))}(\mathbf{R}^n)$

2) $u^+ \in \mathrm{L}^{1,\infty}_{(0,(0))}(\mathbf{R}^n)$

3) $\mathrm{S}(u) \in \mathrm{L}^{1,\infty}_{(0,(0))}(\mathbf{R}^n)$

4) $g(u) \in \mathrm{L}^{1,\infty}_{(0,(0))}(\mathbf{R}^n)$.

It is well-known that if $u(x,y)$ is the Poisson integral of a bounded measure $\left(\text{i.e. } u(x,y) = P_y{\star}\mu(x) = C_n \int_{\mathbf{R}^n} \frac{y}{(|x-t|^2+y^2)^{\frac{n+1}{2}}}\, d\mu(t)\right)$ then $u \in H^{1,\infty}$ [6] and μ is absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}^n$ if and only if $u \in H_0^{1,\infty}$ [4]. It is not difficult to see that not every function of $H^{1,\infty}$(resp. $H_0^{1,\infty}$) can be obtained in this way. In the following proposition we characterize those bounded measures on $\mathbf{R}^n$ whose Poisson integrals are in $H_{00}^{1,\infty}$.

PROPOSITION 3. — *Let* μ *be a bounded measure on* $\mathbf{R}^n$ *and let* $u(x,y) = P_y{\star}\mu(x)$ *be its harmonic extension to* $\mathbf{R}_+^{n+1}$.

Then $\lim_{\delta\to 0_+} \delta m\{u^\star > \delta\} = 0$ *if and only if* $\int_{\mathbf{R}^n} d\mu(x) = 0$.

Proof. — It is well-known that

$$\int_{\mathbf{R}^n} d\mu(x) = \lim_{y\to\infty} C_n y^n u(0,y).$$

From this it follows immediately that for δ small enough

$$\delta m\{u^\star > \delta\} \geqslant C\left|\int_{\mathbf{R}^n} d\mu(x)\right|.$$

Conversely, let $\int d\mu(x) = 0$. By an easy reduction we may assume that μ has compact support and that μ is supported on the unit cube Q_0 in $\mathbf{R}^n$.

$$\begin{aligned} u(x,y) &= C_n \int_{\mathbf{R}^n} \frac{y}{(|x-t|^2+y^2)^{\frac{n+1}{2}}}\, d\mu(t) \\ &= \int_{\mathbf{R}^n} [P_y(x-t) - P_y(x)]\, d\mu(t). \end{aligned}$$

Hence $|u(x,y)| < C_n\|\mu\| \sup_{t\in Q_0} |P_y(x-t)-P_y(x)|$.

If $|x|$ is large, then the supremum on the right hand side of the above inequality $\sim \frac{y|x|^n}{|x|^{2(n+1)}}$. Also since $u^\star \in H^{1,\infty}$, for $(x,y) \in \mathbf{R}_+^{n+1}$ fixed, the ball in $\mathbf{R}^n$ with center x and radius y is contained in the set $\{u^\star > |u(x,y)|\}$. Therefore

$$K \geqslant |u(x,y)| m\{u^\star > |u(x,y)|\} \geqslant C|u(x,y)|y^n$$
$$\text{i.e. } |u(x,y)| \leqslant C/y^n.$$

Consequently,

$$\{(x,y) \in \mathbf{R}_+^{n+1} . |u(x,y)| > \delta\} \subseteq \{(x,y) : |x| \leqslant 1/\delta^{1/n(n+2)},\ y \leqslant C/\delta^{1/n}\}.$$

Hence

$$\delta m\{u^+(x) > \delta\} \leqslant C\|\mu\| \, \delta^{\frac{n+1}{n+2}} = o(1) \quad \text{as} \quad \delta \to 0.$$

This with Proposition 2 completes the proof.

COROLLARY. – $H_{00}^{1,\infty} \cap \{P_y \star \mu(x);\ \mu \text{ a bounded measure}\}$

$$= \{P_y \star f(x) : f \in L^1(\mathbf{R}^n), \int f(x)\, dx = 0\}.$$

In the next proposition, we prove that if $u \in H^{1,\infty}$ then $u(\cdot, y)$ converges in the sense of tempered distributions as $y \to 0$. The proof of the corresponding result for the H^p spaces [3] does not directly apply since in this case the fact that $u^\star \in L^{1,\infty}(\mathbf{R}^n)$ does not necessarily imply that for $y > 0$, $u(\cdot, y) \in L^1(\mathbf{R}^n)$.

PROPOSITION. – *Let* $u \in H^{1,\infty}$. *Then* $\lim_{y \to 0} u(\cdot, y) = f$ *exists in the sense of tempered distribution.*

Proof. – We have seen above that $u^\star \in L^{1,\infty}$ implies that $|u(x,y)| \leqslant C/y^n$. Hence for every $y > 0$, the function $u_y(x) = u(x,y) \in L^2(\mathbf{R}^n)$ and

$$\|u_y\|_2^2 = \int_{\mathbf{R}^n} |u(x,y)|^2\, dx$$
$$= \int_{\{|u_y| \leqslant Cy^{-n}\}} |u(x,y)|^2\, dx \leqslant \int_0^{Cy^{-n}} \beta \lambda_{u_y}(\beta)\, d\beta = C/y^n.$$

Now for $\delta > 0$ fixed we define a function almost everywhere by

$$\hat{u}_0(\xi) = \hat{u}(\xi, \delta) e^{2\bar{\lambda}|\xi|\delta},$$

$\xi \in \mathbf{R}^n$ where $\hat{u}(\xi,\delta)$ is the Plancherel transform of $u_\delta(x)$. Since $u(x,y)$ is a harmonic function, we have $\hat{u}(\cdot,\delta') = \hat{u}(\cdot,\delta)e^{2\bar{\lambda}|\cdot|(\delta'-\delta)}$, $\delta, \delta' > 0$; hence the definition of $\hat{u}_0$ does not depend on the choice of δ. It is clear that $\hat{u}_0$ defines a distribution, denoted by $T_{\hat{u}_0}$. To show that this distribution is in fact tempered, it is enough to prove that for every rapidly decreasing C^∞ function $\psi(h)$ on $\mathbf{R}^n$, the distributions $\psi(h)\tau_h T_{\hat{u}_0}$ are bounded in the space of distributions (here τ_h is the translation by h). Let φ be a C^∞ function with compact support (say Q), then

$$|\langle \psi(h)\tau_h T_{\hat{u}_0}, \varphi\rangle| \leqslant |\psi(h)| \int_Q |\hat{u}(\xi,\delta)| e^{2\bar{\lambda}|\xi|\delta} |\varphi(\xi+h)| \, d\xi .$$

Choose $\delta = 1/K(1+|h|)$ where K is a suitable constant depending on the support of φ then

$$\begin{aligned} |\langle \psi(h)\tau_h T_{\hat{u}_0}, \varphi\rangle| &\leqslant C'|\psi(h)| \; \|\hat{u}_\delta\|_2 \; \|\varphi\|_2 \\ &\leqslant C|\psi(h)|(1+|h|)^{n/2}\|\varphi\|_2 \leqslant C\|\varphi\|_2 . \end{aligned}$$

This proves that $T_{\hat{u}_0}$ is a tempered distribution. Let $f = \mathscr{F}^{-1}(\hat{u}_0)$ (the inverse Fourier transform of $T_{\hat{u}_0}$). Then, if φ is in the Schwarz class $\mathscr{S}$,

$$\begin{aligned} \int u(x,y)\overline{\varphi(x)} \, dx &= \int \hat{u}(\xi,y)\hat{\varphi}(\xi) \, d\xi \\ &= \int \hat{u}_0(\xi) e^{-2\bar{\lambda}|\xi|y} \hat{\varphi}(\xi) \, d\xi \\ &\xrightarrow[y \to 0]{\mathscr{S}'} \langle T_{\hat{u}_0}, \hat{\varphi}\rangle = \langle f, \varphi\rangle \end{aligned}$$

so that $u(\cdot,y) \to f$ as $y \to 0$ in the sense of tempered distributions.

We shall not go into the details, but with the estimates proved in [3] for H^p spaces $(0<p<\infty)$ it can be shown that the $H^{1,\infty}_{(0,(0))}$ spaces can be realized as certain spaces of tempered distributions:

Let: $\varphi \in \mathscr{S}$, $\int_{\mathbf{R}^n} \varphi(x)\, dx = 1$ and $\varphi_t(x) = t^{-n}\varphi(x/t)$. Then if $H^{1,\infty}_{(0,(0))}$ is identified with the space of boundary distributions (Proposition 3), we have

$$H^{1,\infty}_{(0,(0))} = \{ f \in \mathscr{S}' : \sup_{\Gamma(x)} |\varphi_t \star f(x')| \in L^{1,\infty}_{(0,(0))}(\mathbf{R}^n) \}$$

(for details, see theorem 11 in [3]).

3. The A-integral.

Let K be a tempered distribution on $\mathbf{R}^n$, which is C^1 away from the origin and

(i) $|\hat{K}(\xi)| \leqslant B < \infty$

(ii) $|\nabla K(x)| \leqslant C|x|^{-n-1}$.

For $f \in L^1(\mathbf{R}^n)$, $Tf = K \star f$ (which exists as a limit) is a tempered distribution and belongs to $H_0^{1,\infty}$ i.e. it arises as the boundary distribution of a harmonic function $v(x,y)$ such that $v^\star \in L_0^1(\mathbf{R}^n)$. We let Tf also denote the non-tangential boundary function of $v(x,y)$. Further, if $\int_{\mathbf{R}^n} f(x)\,dx = 0$ (i.e. the associated harmonic function is in $H_{00}^{1,\infty}$) then $Tf \in L_{00}^{1,\infty}(\mathbf{R}^n)$.

THEOREM. — *Let* $f \in L^1(\mathbf{R}^n)$, $\int f(x)\,dx = 0$, *and let* Tf *be as defined above. If* $\psi \in L^2 \cap L^\infty(\mathbf{R}^n)$ *is such that* $T\psi \in L^2 \cap L^\infty(\mathbf{R}^n)$, *then*

$$\text{(A)} \quad \int_{\mathbf{R}^n} Tf(x)\psi(x) = -\int_{\mathbf{R}^n} f(x)T\psi(x)\,dx.$$

Proof. — Let $M = \max(\|\psi\|_2, \|\psi\|_\infty, \|T\psi\|_2, \|T\psi\|_\infty)$ and suppose $\varepsilon > 0$ is small and $\alpha > 0$ is large

$$\text{(1)} \quad \int_{\mathbf{R}^n} [Tf]_{\varepsilon,\alpha}(x)\,dx = \int_{\{\varepsilon < u^\star \leqslant \alpha\}} [Tf\psi]_{\varepsilon,\alpha}\,dx + \int_{\{u^\star \leqslant \varepsilon\}} [Tf\psi]_{\varepsilon,\alpha}\,dx + \int_{\{u^\star > \alpha\}} [Tf\psi]_{\varepsilon,\alpha}\,dx = I_1 + I_2 + I_3.$$

Clearly

$$\text{(2)} \quad |I_3| \leqslant \alpha m\{u^\star > \alpha\} = o(1) \text{ as } \alpha \to \infty, \text{ uniformely in } \varepsilon.$$

To estimate I_1 and I_2 we do a Calderon Zygmund decomposition at the level α. Then f can be written as $f(x) = g(x) + b(x)$, where

$|g(x)| \leqslant C\alpha$ and $\|g\|_1 \leqslant \|f\|_1$ (hence $\|g\|_2^2 \leqslant C\alpha\|f\|_1$), and the function b satisfies

$$\int b(x)\, dx = 0$$

$$\|b\|_1 \leqslant \int_{\{u^\star > \alpha\}} |f(x)|\, dx + C\alpha m\{u^\star > \alpha\}$$

$$\int_{\{u^\star \leqslant \alpha\}} |\mathrm{T}b(x)| \leqslant C\alpha m\{u^\star > \alpha\}. \tag{3}$$

Consider the integral

$$\begin{aligned} \mathrm{I}_1 &= \int_{\mathrm{F}_{\varepsilon,\alpha}} [\mathrm{T}f\psi]_{\varepsilon,\alpha}\, dx, \quad \text{where} \quad \mathrm{F}_{\varepsilon,\alpha} = \{x : \varepsilon < u^\star(x) \leqslant \alpha\} \\ &= \int_{\mathrm{F}_{\varepsilon,\alpha}} \mathrm{T}f\psi\, dx - \int_{\mathrm{F}_{\varepsilon,\alpha}\cap\{|\mathrm{T}f\psi| \leqslant \varepsilon\}} \mathrm{T}f\psi\, dx - \int_{\mathrm{F}_{\varepsilon,\alpha}\cap\{|\mathrm{T}f\psi| > \alpha\}} \mathrm{T}f\psi\, dx \\ &= \int_{\mathrm{F}_{\varepsilon,\alpha}} \mathrm{T}f\psi\, dx - \mathrm{J}_1 - \mathrm{J}_2. \end{aligned}$$

We have $|\mathrm{J}_1| < \varepsilon m\{u^\star > \varepsilon\} = o(1)$ as $\varepsilon \to 0$, uniformely in α and

$$\begin{aligned} |\mathrm{J}_2| &\leqslant \int_{\mathrm{F}_{\varepsilon,\alpha}\cap\{|\mathrm{T}f\psi| > \alpha\}} |\mathrm{T}g\,\psi|\, dx + \int_{\{u^\star \leqslant \alpha\}} |\mathrm{T}b\,\psi|\, dx \\ &\leqslant C\|\mathrm{T}g\|_2\, \|\psi\chi_{\{|\mathrm{T}f\psi| > \alpha\}}\|_2 + C\alpha m\{u^\star > \alpha\} \end{aligned}$$

using Holder's inequality and (3). But since g is in L^2 and T is a bounded operator on L^2,

$$\begin{aligned} |\mathrm{J}_2| &\leqslant C\|g\|_2 \mathrm{M}(m\{|\mathrm{T}f\psi| > \alpha^2\})^{1/2} + C\alpha m\{u^\star > \alpha\} \\ &\leqslant C\mathrm{M}\|f\|_1(\alpha m\{|\mathrm{T}f\psi| > \alpha\})^{1/2} + C\alpha m\{u^\star > \alpha\} \\ &= o(1) \quad \text{as} \quad \alpha \to \infty \qquad \text{uniformely in } \varepsilon. \end{aligned}$$

Hence we get

$$\begin{aligned} \mathrm{I}_1 &= \int_{\mathrm{F}_{\varepsilon,\alpha}} \mathrm{T}f(x)\psi(x)\, dx + o(1), \quad \alpha \to \infty, \quad \varepsilon \to 0_+ \\ &= \int_{\mathrm{F}_{\varepsilon,\alpha}} \mathrm{T}g(x)\psi(x)\, dx + o(1), \quad \alpha \to \infty, \quad \varepsilon \to 0_+. \end{aligned} \tag{4}$$

It remains to evaluate I_2. Let $F_\varepsilon = \{u^\star \leqslant \varepsilon\}$

$$
\begin{aligned}
I_2 &= \int_{F_\varepsilon} [Tf\psi]_{\varepsilon,\alpha}(x)\,dx \\
&= \int_{F_\varepsilon} Tf\psi\,dx - \int_{F_\varepsilon \cap \{|Tf\psi| \leqslant \varepsilon\}} Tf\psi\,dx - \int_{F_\varepsilon \cap \{|Tf\psi| > \alpha\}} Tf\psi\,dx \\
&= \int_{F_\varepsilon} Tf\psi\,dx - K_1 - K_2 .
\end{aligned}
$$

K_2 can be estimated in the same way as J_2 and we get $|K_2| = o(1)$ as $\alpha \to \infty$ uniformely in ε.

Note that K_1 is independent of α; to estimate we do a Calderon-Zymund decomposition of f at a level α_0 chosen large enough depending on ε. Write $f = g_0 + b_0$ with g_0 and b_0 as above with respect to α_0. Then

$$
\begin{aligned}
|K_1| &\leqslant \int_{\{|Tf\psi| \leqslant \varepsilon,\, |Tg_0\psi| > \varepsilon\}} |Tf\psi|\,dx + \int_{\{|Tf\psi| < \varepsilon,\, |Tg_0\psi| \leqslant \varepsilon\} \cap F_\varepsilon} |Tf\psi|\,dx \\
&\leqslant \varepsilon m\{|Tg_0\psi| > \varepsilon\} + \int_{\{|Tg_0\psi| \leqslant \varepsilon\}} |Tg_0\psi|\,dx + \int_{\{u^\star \leqslant \varepsilon\}} |Tb_0\psi|\,dx \\
&= o(1) \quad \text{as} \quad \varepsilon \to 0 .
\end{aligned}
$$

Hence

$$
(5) \qquad |I_2| = \int_{\{u^\star \leqslant \varepsilon\}} Tg\psi\,dx + o(1), \quad \alpha \to \infty, \quad \varepsilon \to 0 .
$$

Combining (2), (4) and (5),

$$
\begin{aligned}
\int_{\mathbf{R}^n} [Tf\psi]_{\varepsilon,\alpha}\,dx &= \int_{\{u^\star \leqslant \alpha\}} Tg\psi\,dx + o(1) \\
&= \int_{\mathbf{R}^n} Tg(x)\psi(x)\,dx + o(1) \\
&= -\int g(x)T\psi(x)\,dx + o(1) \\
&= -\int f(x)T\psi(x)\,dx + o(1), \quad \alpha \to \infty,\ \varepsilon \to 0 .
\end{aligned}
$$

In the last step we have used the estimate

$$\|b\|_1 \leqslant \int_{\{u^\star > \alpha\}} |f(x)|\, dx + C\alpha m\{u^\star > \alpha\} = o(1) \quad \text{as} \quad \alpha \to \infty .$$

This completes the proof of the theorem.

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