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SHOBHA MADAN

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ON THE A-INTEGRABILITY OF SINGULAR INTEGRAL TRANSFORMS

by Shobha MADAN

1. Introduction.

In this paper we shall generalize a theorem of Alexandrov on the A-Integrability of Riesz transforms [1].

Let $L^{1,\infty}(\mathbf{R}^n)$ denote the weak-L^1 space consisting of measurable functions f on $\mathbf{R}^n$ for which $\sup_{\alpha>0} \alpha m\{x \in \mathbf{R}^n : |f(x)| > \alpha\} = K < \infty$, where m denotes the Lebesgue measure on $\mathbf{R}^n$; let $L_0^{1,\infty}(\mathbf{R}^n)$ (resp. $L_{00}^{1,\infty}(\mathbf{R}^n)$) be the subspace of $L^{1,\infty}(\mathbf{R}^n)$ consisting of functions which satisfy $\lim_{\alpha\to\infty} \alpha m\{x : |f(x)| > \alpha\} = 0$ (resp. the subspace of $L_0^{1,\infty}(\mathbf{R}^n)$ of functions satisfying $\lim_{\alpha\to 0+} \alpha m\{|f(x)| > \alpha\} = 0$). For brevity we shall write $L^1_{(0,(0))}(\mathbf{R}^n)$ to mean the space « $L^{1,\infty}(\mathbf{R}^n)$ (resp. $L_0^{1,\infty}$, resp. $L_{00}^{1,\infty}$) ». A similar notation will be used for the weak Hardy spaces defined below. For a function f, we write $\lambda_f(\alpha)$ for its distribution function, i.e. $\lambda_f(\alpha) = m\{x \in \mathbf{R}^n : |f(x)| > \alpha\}$, $\alpha > 0$. In the following C, C′, K will denote several different constants.

Let $u(x,y)$, $x \in \mathbf{R}^n$, $y > 0$ be a harmonic function on the upper half plane $\mathbf{R}_+^{n+1}$, and for $x \in \mathbf{R}^n$, $\Gamma_a(x) = \{(x',y) \in \mathbf{R}_+^{n+1} : |x'-x| < ay\}$ is the cone of aperture a at x. When $a = 1$, we shall simply write $\Gamma(x)$. The non tangential maximal function of u is the function $u^*(x) = \sup_{\Gamma(x)} |u(x',y)|$.

We define $H^{1,\infty}_{(0,(0))} = \{u(x,y) : u$ a harmonic function on $\mathbf{R}_+^{n+1}$ such that $u^* \in L^{1,\infty}_{(0,(0))}(\mathbf{R}^n)\}$. These are the spaces considered by Alexandrov in [1], where he proves an A-Integrability result for the system of conjugate functions of u.

Let (X,μ) be a measure space and f a measurable function on X. Then f is said to be A-integrable if

(i) $\alpha\mu\{x \in X : |f(x)| > \alpha\} = o(1), \quad \alpha \to +\infty, \quad \alpha \to 0_+$

(ii) $\lim\limits_{\substack{\varepsilon \to 0_+ \\ \alpha \to +\infty}} \int_X [f]_{\varepsilon,\alpha}(x)\, d\mu(x)$ exists

where $[f]_{\varepsilon,\alpha}(x) = f(x)$ if $\varepsilon < |f(x)| \leqslant \alpha$
$= 0$ if not.

The limit in (ii) is called the A-integral of f and is denoted by $(A)\int f\,d\mu$ [2].

THEOREM (Alexandrov). — *Let $u_0 \in H^{1,\infty}_{00}$ and let $u_1, \ldots, u_n$ be the system of conjugate harmonic functions of u_0. If $f_0, f_1 \ldots f_n$ denote the non-tangential boundary functions of $u_0, u_1 \ldots u_n$ and $g_0, g_1 \ldots g_n$ is another such system of boundary functions such that $g_k \in L^2 \cap L^\infty(\mathbf{R}^n)$, $k = 0, 1 \ldots n$, then*

$$(A)\int (f_k g_0 + f_0 g_k)\, dx = 0, \quad k = 1, 2 \ldots n.$$

In section 3, we shall prove a similar result for singular integral transforms, using real variable methods, and the fact that a certain set of transforms forms a conjugate system does not play any essential role. Our result then contains the above result of Alexandrov.

2.

The $H^{1,\infty}_{(0,(0))}$ spaces have been defined above by means of a non-tangential maximal function with respect to a cone of aperture 1. But this is in fact not a restriction, and we have

PROPOSITION 1. — *Let $u(x, y)$ be any continuous function on $\mathbf{R}^{n+1}_+$. Then the following are equivalent :*

1) $u^\star(x) = \sup\limits_{\Gamma(x)} |u(x',y)| \in L^{1,\infty}_{(0,(0))}(\mathbf{R}^n)$

2) $u_N^\star(x) = \sup_{\Gamma_N(x)} |u(x',y)| \in L^{1,\infty}_{(0,(0))}(\mathbf{R}^n)$

3) $u^{\star\star}(x) = \sup_{(x',y)\in\mathbf{R}^{n+1}_+} |u(x',y)|\left(\frac{y}{|x-x'|+y}\right)^M \in L^{1,\infty}_{(0,(0))}(\mathbf{R}^n)$

where $M > n$.

The proof of this proposition is only a slight modification of the proof of lemma 1 of [3], where the equivalence of $L^p(\mathbf{R}^n)$ $(0<p<\infty)$ norms of these functions has been proved.

Further, these spaces can also be characterized using the area function,

$$S_a(u)(x) = \left(\int_{\Gamma_a(x)}\int |\nabla(x',y)|^2 y^{1-n}\, dx\, dy\right)^{1/2}$$

as a consequence of the following inequality [3]

$$\lambda_{S(u)}(\alpha) < C\left\{\lambda_{u^\star}(\alpha) + \frac{1}{\alpha^2}\int_0^\alpha \beta\lambda_{u^\star}(\beta)\, d\beta\right\}$$

and a corresponding inequality with the roles of $S(u)$ and $u^\star$ interchanged. These inequalities have been proved in [3] for harmonic functions $u(x,y)$ which are Poisson Integrals of L^2-functions, a restriction which can easily be removed. Also the restriction on the cones can be removed using Proposition 1. A similar characterization also holds for the radial maximal function $u^+(x) = \sup_{y>0} |u(x,y)|$ and for the g-function

$$g(u)(x) = \left(\int_0^\infty |\nabla u(x,y)|^2 y\, dy\right)^{1/2}$$

(see [5] for details). We summarize these results in

PROPOSITION 2. — *Let* $u(x,y)$ *be a harmonic function on* $\mathbf{R}^{n+1}_+$. *Then the following are equivalent :*

1) $u^\star \in L^{1,\infty}_{(0,(0))}(\mathbf{R}^n)$

2) $u^+ \in L^{1,\infty}_{(0,(0))}(\mathbf{R}^n)$

3) $S(u) \in L^{1,\infty}_{(0,(0))}(\mathbf{R}^n)$

4) $g(u) \in L^{1,\infty}_{(0,(0))}(\mathbf{R}^n)$.

It is well-known that if $u(x,y)$ is the Poisson integral of a bounded measure $\left(\text{i.e. } u(x,y) = P_y \star \mu(x) = C_n \int_{\mathbf{R}^n} \frac{y}{(|x-t|^2+y^2)^{\frac{n+1}{2}}} d\mu(t)\right)$ then $u \in H^{1,\infty}$ [6] and μ is absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}^n$ if and only if $u \in H_0^{1,\infty}$ [4]. It is not difficult to see that not every function of $H^{1,\infty}$(resp. $H_0^{1,\infty}$) can be obtained in this way. In the following proposition we characterize those bounded measures on $\mathbf{R}^n$ whose Poisson integrals are in $H_{00}^{1,\infty}$.

PROPOSITION 3. — *Let* μ *be a bounded measure on* $\mathbf{R}^n$ *and let* $u(x,y) = P_y \star \mu(x)$ *be its harmonic extension to* $\mathbf{R}_+^{n+1}$.

Then $\lim_{\delta \to 0_+} \delta m\{u^\star > \delta\} = 0$ *if and only if* $\int_{\mathbf{R}^n} d\mu(x) = 0$.

Proof. — It is well-known that

$$\int_{\mathbf{R}^n} d\mu(x) = \lim_{y \to \infty} C_n y^n u(0,y).$$

From this it follows immediately that for δ small enough

$$\delta m\{u^\star > \delta\} \geqslant C \left| \int_{\mathbf{R}^n} d\mu(x) \right|.$$

Conversely, let $\int d\mu(x) = 0$. By an easy reduction we may assume that μ has compact support and that μ is supported on the unit cube Q_0 in $\mathbf{R}^n$.

$$\begin{aligned} u(x,y) &= C_n \int_{\mathbf{R}^n} \frac{y}{(|x-t|^2+y^2)^{\frac{n+1}{2}}} d\mu(t) \\ &= \int_{\mathbf{R}^n} [P_y(x-t) - P_y(x)]\, d\mu(t). \end{aligned}$$

Hence $|u(x,y)| < C_n \|\mu\| \sup_{t \in Q_0} |P_y(x-t) - P_y(x)|$.

If $|x|$ is large, then the supremum on the right hand side of the above inequality $\sim \frac{y|x|^n}{|x|^{2(n+1)}}$. Also since $u^\star \in H^{1,\infty}$, for $(x,y) \in \mathbf{R}_+^{n+1}$ fixed, the ball in $\mathbf{R}^n$ with center x and radius y is contained in the set $\{u^\star > |u(x,y)|\}$. Therefore

$$K \geqslant |u(x,y)| m\{u^\star > |u(x,y)|\} \geqslant C|u(x,y)|y^n$$
$$\text{i.e. } |u(x,y)| \leqslant C/y^n .$$

Consequently,

$$\{(x,y) \in \mathbf{R}^{n+1}_+ . |u(x,y)| > \delta\} \subseteq \{(x,y) : |x| \leqslant 1/\delta^{1/n(n+2)},\ y \leqslant C/\delta^{1/n}\} .$$

Hence

$$\delta m\{u^+(x) > \delta\} \leqslant C\|\mu\|\, \delta^{\frac{n+1}{n+2}} = o(1) \quad \text{as} \quad \delta \to 0 .$$

This with Proposition 2 completes the proof.

COROLLARY. — $H^{1,\infty}_{00} \cap \{P_y \star \mu(x);\ \mu \text{ a bounded measure}\}$

$$= \{P_y \star f(x) : f \in L^1(\mathbf{R}^n), \int f(x)\, dx = 0\} .$$

In the next proposition, we prove that if $u \in H^{1,\infty}$ then $u(\cdot,y)$ converges in the sense of tempered distributions as $y \to 0$. The proof of the corresponding result for the H^p spaces [3] does not directly apply since in this case the fact that $u^\star \in L^{1,\infty}(\mathbf{R}^n)$ does not necessarily imply that for $y > 0$, $u(\cdot,y) \in L^1(\mathbf{R}^n)$.

PROPOSITION. — *Let* $u \in H^{1,\infty}$. *Then* $\lim_{y \to 0} u(\cdot,y) = f$ *exists in the sense of tempered distribution.*

Proof. — We have seen above that $u^\star \in L^{1,\infty}$ implies that $|u(x,y)| \leqslant C/y^n$. Hence for every $y > 0$, the function $u_y(x) = u(x,y) \in L^2(\mathbf{R}^n)$ and

$$\|u_y\|_2^2 = \int_{\mathbf{R}^n} |u(x,y)|^2\, dx$$
$$= \int_{\{|u_y| \leqslant Cy^{-n}\}} |u(x,y)|^2\, dx \leqslant \int_0^{Cy^{-n}} \beta \lambda_{u_y}(\beta)\, d\beta = C/y^n .$$

Now for $\delta > 0$ fixed we define a function almost everywhere by

$$\hat{u}_0(\xi) = \hat{u}(\xi,\delta) e^{2\Pi|\xi|\delta},$$

$\xi \in \mathbf{R}^n$ where $\hat{u}(\xi,\delta)$ is the Plancherel transform of $u_\delta(x)$. Since $u(x,y)$ is a harmonic function, we have $\hat{u}(\cdot,\delta') = \hat{u}(\cdot,\delta)e^{2\bar{\lambda}|\cdot|(\delta'-\delta)}$, $\delta, \delta' > 0$; hence the definition of $\hat{u}_0$ does not depend on the choice of δ. It is clear that $\hat{u}_0$ defines a distribution, denoted by $T_{\hat{u}_0}$. To show that this distribution is in fact tempered, it is enough to prove that for every rapidly decreasing C^∞ function $\psi(h)$ on $\mathbf{R}^n$, the distributions $\psi(h)\tau_h T_{\hat{u}_0}$ are bounded in the space of distributions (here τ_h is the translation by h). Let φ be a C^∞ function with compact support (say Q), then

$$|\langle\psi(h)\tau_h T_{\hat{u}_0},\varphi\rangle| \leqslant |\psi(h)| \int_Q |\hat{u}(\xi,\delta)| e^{2\bar{\lambda}|\xi|\delta} |\varphi(\xi+h)|\, d\xi .$$

Choose $\delta = 1/K(1+|h|)$ where K is a suitable constant depending on the support of φ then

$$\begin{aligned}|\langle\psi(h)\tau_h T_{\hat{u}_0},\varphi\rangle| &\leqslant C'|\psi(h)|\ \|\hat{u}_\delta\|_2\ \|\varphi\|_2 \\ &\leqslant C|\psi(h)|(1+|h|)^{n/2}\|\varphi\|_2 \leqslant C\|\varphi\|_2 .\end{aligned}$$

This proves that $T_{\hat{u}_0}$ is a tempered distribution. Let $f = \mathscr{F}^{-1}(\hat{u}_0)$ (the inverse Fourier transform of $T_{\hat{u}_0}$). Then, if φ is in the Schwarz class $\mathscr{S}$,

$$\begin{aligned}\int u(x,y)\overline{\varphi(x)}\, dx &= \int \hat{u}(\xi,y)\hat{\varphi}(\xi)\, d\xi \\ &= \int \hat{u}_0(\xi)e^{-2\bar{\lambda}|\xi|y}\hat{\varphi}(\xi)\, d\xi \\ &\xrightarrow[y\to 0]{\mathscr{S}'} \langle T_{\hat{u}_0},\hat{\varphi}\rangle = \langle f,\varphi\rangle\end{aligned}$$

so that $u(\cdot,y) \to f$ as $y \to 0$ in the sense of tempered distributions.

We shall not go into the details, but with the estimates proved in [3] for H^p spaces $(0<p<\infty)$ it can be shown that the $H^{1,\infty}_{(0,(0))}$ spaces can be realized as certain spaces of tempered distributions:

Let: $\varphi \in \mathscr{S}$, $\int_{\mathbf{R}^n} \varphi(x)\, dx = 1$ and $\varphi_t(x) = t^{-n}\varphi(x/t)$. Then if $H^{1,\infty}_{(0,(0))}$ is identified with the space of boundary distributions (Proposition 3), we have

$$H^{1,\infty}_{(0,(0))} = \{f \in \mathscr{S}' : \sup_{\Gamma(x)} |\varphi_t \star f(x')| \in L^{1,\infty}_{(0,(0))}(\mathbf{R}^n)\}$$

(for details, see theorem 11 in [3]).

3. The A-integral.

Let K be a tempered distribution on $\mathbf{R}^n$, which is C^1 away from the origin and

(i) $|\hat{K}(\xi)| \leqslant B < \infty$

(ii) $|\nabla K(x)| \leqslant C|x|^{-n-1}$.

For $f \in L^1(\mathbf{R}^n)$, $Tf = K \star f$ (which exists as a limit) is a tempered distribution and belongs to $H_0^{1,\infty}$ i.e. it arises as the boundary distribution of a harmonic function $v(x,y)$ such that $v^\star \in L_0^1(\mathbf{R}^n)$. We let Tf also denote the non-tangential boundary function of $v(x,y)$. Further, if $\int_{\mathbf{R}^n} f(x)\,dx = 0$ (i.e. the associated harmonic function is in $H_{00}^{1,\infty}$) then $Tf \in L_{00}^{1,\infty}(\mathbf{R}^n)$.

THEOREM. – *Let* $f \in L^1(\mathbf{R}^n)$, $\int f(x)\,dx = 0$, *and let* Tf *be as defined above. If* $\psi \in L^2 \cap L^\infty(\mathbf{R}^n)$ *is such that* $T\psi \in L^2 \cap L^\infty(\mathbf{R}^n)$, *then*

$$\text{(A)} \quad \int_{\mathbf{R}^n} Tf(x)\psi(x) = -\int_{\mathbf{R}^n} f(x)T\psi(x)\,dx.$$

Proof. – Let $M = \max(\|\psi\|_2, \|\psi\|_\infty, \|T\psi\|_2, \|T\psi\|_\infty)$ and suppose $\varepsilon > 0$ is small and $\alpha > 0$ is large

$$\text{(1)} \quad \int_{\mathbf{R}^n} [Tf]_{\varepsilon,\alpha}(x)\,dx = \int_{\{\varepsilon<u^\star\leqslant\alpha\}} [Tf\psi]_{\varepsilon,\alpha}\,dx$$
$$+ \int_{\{u^\star\leqslant\varepsilon\}} [Tf\psi]_{\varepsilon,\alpha}\,dx + \int_{\{u^\star>\alpha\}} [Tf\psi]_{\varepsilon,\alpha}\,dx$$
$$= I_1 + I_2 + I_3.$$

Clearly

$$\text{(2)} \quad |I_3| \leqslant \alpha m\{u^\star>\alpha\} = o(1) \text{ as } \alpha\to\infty, \text{ uniformely in } \varepsilon.$$

To estimate I_1 and I_2 we do a Calderon Zygmund decomposition at the level α. Then f can be written as $f(x) = g(x) + b(x)$, where

$|g(x)| \leqslant C\alpha$ and $\|g\|_1 \leqslant \|f\|_1$ (hence $\|g\|_2^2 \leqslant C\alpha\|f\|_1$), and the function b satisfies

$$\int b(x)\,dx = 0$$

$$\|b\|_1 \leqslant \int_{\{u^\star > \alpha\}} |f(x)|\,dx + C\alpha m\{u^\star > \alpha\}$$

$$\int_{\{u^\star \leqslant \alpha\}} |Tb(x)| \leqslant C\alpha m\{u^\star > \alpha\}. \tag{3}$$

Consider the integral

$$\begin{aligned} I_1 &= \int_{F_{\varepsilon,\alpha}} [Tf\psi]_{\varepsilon,\alpha}\,dx, \quad \text{where} \quad F_{\varepsilon,\alpha} = \{x : \varepsilon < u^\star(x) \leqslant \alpha\} \\ &= \int_{F_{\varepsilon,\alpha}} Tf\psi\,dx - \int_{F_{\varepsilon,\alpha}\cap\{|Tf\psi|\leqslant\varepsilon\}} Tf\psi\,dx - \int_{F_{\varepsilon,\alpha}\cap\{|Tf\psi|>\alpha\}} Tf\psi\,dx \\ &= \int_{F_{\varepsilon,\alpha}} Tf\psi\,dx - J_1 - J_2. \end{aligned}$$

We have $|J_1| < \varepsilon m\{u^\star > \varepsilon\} = o(1)$ as $\varepsilon \to 0$, uniformely in α and

$$\begin{aligned} |J_2| &\leqslant \int_{F_{\varepsilon,\alpha}\cap\{|Tf\psi|>\alpha\}} |Tg\,\psi|\,dx + \int_{\{u^\star\leqslant\alpha\}} |Tb\,\psi|\,dx \\ &\leqslant C\|Tg\|_2\,\|\psi\chi_{\{|Tf\psi|>\alpha\}}\|_2 + C\alpha m\{u^\star > \alpha\} \end{aligned}$$

using Holder's inequality and (3). But since g is in L^2 and T is a bounded operator on L^2,

$$\begin{aligned} |J_2| &\leqslant C\|g\|_2 M(m\{|Tf\psi| > \alpha^2\})^{1/2} + C\alpha m\{u^\star > \alpha\} \\ &\leqslant CM\|f\|_1(\alpha m\{|Tf\psi| > \alpha\})^{1/2} + C\alpha m\{u^\star > \alpha\} \\ &= o(1) \quad \text{as} \quad \alpha \to \infty \qquad \text{uniformely in } \varepsilon. \end{aligned}$$

Hence we get

$$\begin{aligned} I_1 &= \int_{F_{\varepsilon,\alpha}} Tf(x)\psi(x)\,dx + o(1), \quad \alpha \to \infty, \quad \varepsilon \to 0_+ \\ &= \int_{F_{\varepsilon,\alpha}} Tg(x)\psi(x)\,dx + o(1), \quad \alpha \to \infty, \quad \varepsilon \to 0_+. \end{aligned} \tag{4}$$

It remains to evaluate I_2. Let $F_\varepsilon = \{u^\star \leqslant \varepsilon\}$

$$\begin{aligned} I_2 &= \int_{F_\varepsilon} [Tf\psi]_{\varepsilon,\alpha}(x)\,dx \\ &= \int_{F_\varepsilon} Tf\psi\,dx - \int_{F_\varepsilon \cap \{|Tf\psi| \leqslant \varepsilon\}} Tf\psi\,dx - \int_{F_\varepsilon \cap \{|Tf\psi| > \alpha\}} Tf\psi\,dx \\ &= \int_{F_\varepsilon} Tf\psi\,dx - K_1 - K_2 . \end{aligned}$$

K_2 can be estimated in the same way as J_2 and we get $|K_2| = o(1)$ as $\alpha \to \infty$ uniformely in ε.

Note that K_1 is independent of α; to estimate we do a Calderon-Zymund decomposition of f at a level α_0 chosen large enough depending on ε. Write $f = g_0 + b_0$ with g_0 and b_0 as above with respect to α_0. Then

$$\begin{aligned} |K_1| &\leqslant \int_{\{|Tf\psi| \leqslant \varepsilon,\, |Tg_0\psi| > \varepsilon\}} |Tf\psi|\,dx + \int_{\{|Tf\psi| < \varepsilon,\, |Tg_0\psi| \leqslant \varepsilon\} \cap F_\varepsilon} |Tf\psi|\,dx \\ &\leqslant \varepsilon m\{|Tg_0\psi| > \varepsilon\} + \int_{\{|Tg_0\psi| \leqslant \varepsilon\}} |Tg_0\psi|\,dx + \int_{\{u^\star \leqslant \varepsilon\}} |Tb_0\psi|\,dx \\ &= o(1) \quad \text{as} \quad \varepsilon \to 0 . \end{aligned}$$

Hence

$$|I_2| = \int_{\{u^\star \leqslant \varepsilon\}} Tg\psi\,dx + o(1), \quad \alpha \to \infty, \quad \varepsilon \to 0. \tag{5}$$

Combining (2), (4) and (5),

$$\begin{aligned} \int_{\mathbf{R}^n} [Tf\psi]_{\varepsilon,\alpha}\,dx &= \int_{\{u^\star \leqslant \alpha\}} Tg\psi\,dx + o(1) \\ &= \int_{\mathbf{R}^n} Tg(x)\psi(x)\,dx + o(1) \\ &= -\int g(x)T\psi(x)\,dx + o(1) \\ &= -\int f(x)T\psi(x)\,dx + o(1), \quad \alpha \to \infty,\ \varepsilon \to 0 . \end{aligned}$$

In the last step we have used the estimate

$$\|b\|_1 \leqslant \int_{\{u^\star>\alpha\}} |f(x)|\,dx + C\alpha m\{u^\star>\alpha\} = o(1) \quad \text{as} \quad \alpha \to \infty .$$

This completes the proof of the theorem.

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Shobha MADAN,

Department of Mathematics
Indian Statistical Institute
7 S.J.S. Sansanwal Marg
New Delhi 110016 (India).