YOSHIHIRO MIZUTA

On the boundary limits of harmonic functions with gradient in $L^p$


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ON THE BOUNDARY LIMITS
OF HARMONIC FUNCTIONS
WITH GRADIENT IN $L^p$

by Yoshihiro MIZUTA

1. Introduction.

Let $u$ be a function harmonic in the half space
$$\mathbb{R}^n_+ = \{ x = (x_1, \ldots, x_n) ; x_n > 0 \}$$
and satisfying the condition :
$$\int_G |\text{grad } u(x)|^p x_n^\alpha \, dx < \infty \quad (1)$$
for any bounded open set $G \subset \mathbb{R}^n_+, \text{ where } p > 1 \text{ and } \alpha < p - 1$. For $\xi \in \partial \mathbb{R}^n_+, \gamma \gg 1 \text{ and } a > 1$, set
$$T_\gamma(\xi, a) = \{ x \in \mathbb{R}^n_+ ; |x - \xi| < ax^{1/\gamma} \}.$$ The existence of nontangential limits of $u$, that is, the limit of $u(x)$ as $x \rightarrow \xi, \ x \in T_\gamma(\xi, a)$, was studied by Carleson [1] ($n = p = 2$ and $0 < \alpha < 1$), Wallin [10] ($p = 2$ and $0 < \alpha < 1$) and Mizuta [6] in the present situation.

Recently Cruzeiro [2] proved the existence of lim $u(x)$ as $x \rightarrow \xi, \ x \in T_\gamma(\xi, a)$, for a harmonic function $u$ satisfying (1) with $p = n$ and $\alpha = 0$. The existence of such limits for Green potentials in $\mathbb{R}^n_+$ was obtained by Wu [11]. Taking these results into account, we give the following theorem :

**Theorem.** — Let $u$ be a function harmonic in $\mathbb{R}^n_+$ and satisfying (1) with $p > 1$ and $\alpha < p - 1$.

(i) If $n - p + \alpha > 0$, then for each $\gamma > 1$ there exists a set $E_\gamma \subset \partial \mathbb{R}^n_+$ such that $H_{\gamma(n - p + \alpha)}(E_\gamma) = 0$ and
exists and is finite for any $\xi \in \partial \mathbb{R}^n - E_\gamma$ and any $a > 1$.

(ii) If $n - p + \alpha = 0$, then there exists a set $E \subset \partial \mathbb{R}^n$ such that $B_{n/p,p}(E) = 0$ and (2) exists and is finite for any $\xi \in \partial \mathbb{R}^n - E$, any $\gamma \geq 1$ and any $a > 1$.

(iii) If $n - p + \alpha < 0$, then $\lim_{x \to \xi, x \in \partial \mathbb{R}^n} u(x)$ exists and is finite for any $\xi \in \partial \mathbb{R}^n$.

Here $H_d$ denotes the $d$-dimensional Hausdorff measure, and $B_{\beta,p}$ the Bessel capacity of index $(\beta,p)$ (cf. [3]). In view of [3; Theorems 21 and 22], one notes the following results:

a) If $q > 1$ and $n - d \geq \beta q$, then $H_d(E) < \infty$ implies $B_{\beta,q}(E) = 0$;

b) If $q > 1$ and $n - d < \beta q$, then $B_{\beta,q}(E) = 0$ implies $H_d(E) = 0$.

Recently Nagel, Rudin and Shapiro [7] studied tangential behaviors of Poisson integrals of potential type functions (see Sec. 4, Remark 2). Their results are not applicable to our case unless $p = 2$.

Remark. — The same result as in the theorem is also valid for a domain $\Omega$ for which any function $v$ satisfying

$$\int_{\Omega} |\text{grad} v(x)|^p \delta(x)^\alpha \, dx < \infty, \quad p > 1, \alpha < p - 1, \quad (3)$$

can be extended to a function satisfying (3) with $\Omega$ replaced by $\mathbb{R}^n$, where $\delta(x)$ denotes the distance from $x$ to $\partial \Omega$. The special Lipschitz domains in [9; Chap. VI] are typical examples of $\Omega$.

2. Lemmas.

First we note the following result, which follows readily from the fact in [4; p. 165].

**Lemma 1.** — Let $f$ be a locally integrable function on $\mathbb{R}^n$. For $\beta > 0$, we set
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\[ A_\beta = \{ x \in \partial \mathbb{R}^n : \limsup_{r \to 0} r^{-\beta} \int_{B(x, r)} |f(y)| \, dy > 0 \} , \]

where \( B(x, r) \) denotes the open ball with center at \( x \) and radius \( r \). Then \( H_\beta (A_\beta) = 0 \).

By [3; Theorem 21] and the result of [4; p. 165], we have

**Lemma 2.** Let \( f \) be a locally integrable function on \( \mathbb{R}^n \).

For \( p > 1 \), we set

\[ B_p = \{ x : \limsup_{r \to 0} (\log r)^{-1/p} \int_{B(x, r)} |f(y)| \, dy > 0 \} . \]

Then \( B_{n/p, p}(B_p) = 0 \).

Next we prove the following technical result.

**Lemma 3.** Let \( c_1 > 0 \), \( c_2 > 0 \), \( \gamma > 1 \), \( p > 1 \) and \( p - n \leq \alpha < p - 1 \). Then

\[
\left\{ \int_{\{ y : c_1 x_n < |x - y| < c_2 x_n^{1/\gamma} \} } |x - y|^{p'(1-n)} |y_n|^{-\alpha p'/p} \, dy \right\}^{1/p'} \leq C \begin{cases} \frac{x_n^{(p-n-\alpha)/p}}{(\log x_n^{-1})^{1/p'}} & \text{if } n - p + \alpha > 0 , \\ \frac{x_n^{(p-n-\alpha)/p}}{(\log x_n^{-1})^{1/p'}} & \text{if } n - p + \alpha = 0 , \end{cases}
\]

where \( 1/p + 1/p' = 1 \) and \( C \) is a positive constant independent of \( x = (x_1, \ldots, x_n) \) with \( 0 < x_n < 1/2 \).

**Proof.** Let \( x^* = (0, \ldots, 0, 1) \). By change of variables we see that the left hand side is equal to

\[
x_n^{1-n-\alpha/p+n/p'} \left\{ \int_{E_1} |x^* - z|^{p'(1-n)} |z_n|^{-\alpha p'/p} \, dz \right\}^{1/p'} \leq C x_n^{(p-n-\alpha)/p} \left\{ \int_{E_2} (1 + |z|)^{p'(1-n)} |z_n|^{-\alpha p'/p} \, dz \right\}^{1/p'} ,
\]

where \( E_1 = \{ z : c_1 < |x^* - z| < c_2 x_n^{1/\gamma-1} \} \), \( E_2 = B(0, (c_2 + 1) x_n^{1/\gamma-1}) \),

\[
\begin{align*}
\int_{E_2} (1 + |z|)^{p'(1-n)} |z_n|^{-\alpha p'/p} \, dz &= \int_{E_2} (1 + |z|)^{p'(1-n)} \left( \frac{c_2 x_n^{1/\gamma-1}}{c_2 x_n^{1/\gamma-1}} \right)^{p'/p} \, dz \\
&= C x_n^{(p-n-\alpha)/p} \int_{E_2} (1 + |z|)^{p'(1-n)} |z_n|^{-\alpha p'/p} \, dz .
\end{align*}
\]
C is a positive constant independent of $x$ with $0 < x < 1$. The required inequalities are established by estimating the last integral.

In the same manner we can prove

**Lemma 4.** Let $p > 1$ and $\alpha < p - n$. Then

$$
\left\{ \int_{\{y : x_n/2 < |x - y| < |x|/2\}} |x - y|^{p'(1-n)} |y_n|^{-\alpha p'/p} \, dy \right\}^{1/p'} \leq C |x|^{(p-n-\alpha)/p}
$$

for any $x \in \mathbb{R}_n^+$, where $C$ is a positive constant independent of $x$.

Finally we borrow a result from [6; Lemma 4].

**Lemma 5.** Let $p > 1$, $\alpha < p - 1$ and $f$ be a measurable function on $\mathbb{R}^n$ such that $\int_G |f(y)|^p |y_n|^\alpha \, dy < \infty$ for any bounded open set $G \subset \mathbb{R}^n$. If we set

$$E' = \left\{ \xi \in \partial \mathbb{R}_n^+ ; \int_{B(\xi,1)} |\xi - y|^{1-n} |f(y)| \, dy = \infty \right\},$$

then $B_{1-\alpha/p,p}(E') = 0$.

**Remark.** If $p - \alpha > n$, then one sees that $E'$ is empty.

### 3. Proof of the theorem.

Take $q$ such that $q = p$ if $\alpha \leq 0$ and $1 < q < p/(\alpha + 1)$ if $\alpha > 0$. Then Hölder's inequality implies that

$$\int_G |\text{grad } u(x)|^q \, dx < \infty$$

for any bounded open set $G \subset \mathbb{R}_n^+$. Hence the function

$$v(x) = \begin{cases} u(x_1, \ldots, x_{n-1}, x_n) & \text{for } x_n > 0, \\ u(x_1, \ldots, x_{n-1}, -x_n) & \text{for } x_n < 0, \end{cases}$$

can be extended to a locally $q$-precise function $w$ on $\mathbb{R}^n$ in view of [8; Theorem 5.6] (for $q$-precise functions, see Ohtsuka [8; Chap. IV] and Ziemer [12]). Define
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E' = \left\{ \xi \in \partial \mathbb{R}_+^n ; \int_{B(\xi,1)} |\xi - y|^{1-n} |\nabla w(y)| \, dy = \infty \right\}.

Then we have $B_{1-\alpha/p,p}(E') = 0$ on account of Lemma 5. We also define $A_\beta$ and $B_p$ with $f(y) = |\nabla w(y)|^p |y_n|^{\alpha}$. By Lemmas 1 and 2, we see that $H_\beta(A_\beta) = 0$ and $B_{n/p,p}(B_p) = 0$.

First suppose $n - p + \alpha > 0$. Let $\gamma > 1$ be given. We shall show that (2) exists and is finite for any

$$\xi \in \partial \mathbb{R}_+^n - (E' \cup A_\gamma(n - p + \alpha))$$

and any $\alpha > 1$. Let $\xi \in \partial \mathbb{R}_+^n - (E' \cup A_\gamma(n - p + \alpha))$. Take $N > 1$ such that $\xi \in B(0, N)$, and let $\phi_N$ be a function in $C_0^\infty(\mathbb{R}^n)$ such that $\phi_N = 1$ on $B(0, 2N)$ and $\phi_N = 0$ on $\mathbb{R}^n - B(0, 3N)$. Set $w_N = \phi_N w$. Then $w_N$ is $q$-precise on $\mathbb{R}^n$ and satisfies

$$w_N(x) = c \sum_{i=1}^n \int_{B(x, x_n/2)} \left( \frac{\partial}{\partial x_i} R_2 \right)(x - y) \frac{\partial w_N}{\partial y_i} \, dy$$

for $x \in \mathbb{R}_+^n$,

where $c$ is a constant, $R_2(x) = \log(1/|x|)$ if $n = 2$ and $R_2(x) = |x|^{2-n}$ if $n \geq 3$. In fact, since $w_N$ is continuously differentiable on $\mathbb{R}_+^n$, the right hand side is continuous on $\mathbb{R}_+^n$, and the required equality holds for any $x \in \mathbb{R}_+^n$ on account of Ohtsuka [8; Theorem 9.11].

For $x \in \mathbb{R}_+^n$, we write

$$w_N(x) = w_{N,1}(x) + w_{N,2}(x) + w_{N,3}(x),$$

where

$$w_{N,1}(x) = c \sum_{i=1}^n \int_{B(x,x_n/2)} \left( \frac{\partial}{\partial x_i} R_2 \right)(x - y) \frac{\partial w_N}{\partial y_i} \, dy,$$

$$w_{N,2}(x) = c \sum_{i=1}^n \int_{B(x,|\xi - x|/2) - B(x,x_n/2)} \left( \frac{\partial}{\partial x_i} R_2 \right)(x - y) \frac{\partial w_N}{\partial y_i} \, dy,$$

$$w_{N,3}(x) = c \sum_{i=1}^n \int_{\mathbb{R}_+^n - B(x,|\xi - x|/2)} \left( \frac{\partial}{\partial x_i} R_2 \right)(x - y) \frac{\partial w_N}{\partial y_i} \, dy.$$
Since \( w_N \) is harmonic on \( B(0, 2N) \cap \mathbb{R}^n \), \( w_N, (x) = 0 \) for \( x \in B(0, N) \cap \mathbb{R}^n \). It follows from our assumption \( \xi \notin E' \) that \( \int |\xi - y|^{1-n} |\text{grad} \ w_N(y)| \, dy < \infty \). Hence Lebesgue's dominated convergence theorem gives

\[
\lim_{x \to \xi, x \in \mathbb{R}^n} w_{N, 3}(x) = c \sum_{i=1}^n \int \left( \frac{\partial}{\partial x_i} \text{R}_2 \right)(\xi - y) \frac{\partial w_N}{\partial y_i} \, dy.
\]

For \( w_{N, 2} \) we apply Hölder's inequality to obtain by Lemma 3,

\[
| w_{N, 2}(x) | \leq \text{const.} \left\{ \int_{\{ y : x_n/2 < |x - y| < |\xi - x|/2 \}} |x - y|^{p'(1-n)} |y_n|^{-\alpha p'/p} \, dy \right\}^{1/p'} \times \left\{ \int_{B(x, |\xi - x|/2)} |\text{grad} \ w_N(y)|^p |y_n|^\alpha \, dy \right\}^{1/p} \leq \text{const.} \left\{ |x_n|^{-\alpha - n} \int_{B(\xi, 2|\xi - x|/n)} |\text{grad} \ w(y)|^p |y_n|^\alpha \, dy \right\}^{1/p}
\]

for \( x \in B(0, N) \cap T_\gamma(\xi, a) \). Since \( \xi \notin A_\gamma(\gamma - p + \alpha) \),

\[
\lim_{x \to \xi, x \in T_\gamma(\xi, a)} w_{N, 2}(x) = 0.
\]

Thus

\[
\lim_{x \to \xi, x \in T_\gamma(\xi, a)} u(x) = \lim_{x \to \xi, x \in T_\gamma(\xi, a)} w_N(x) = \lim_{x \to \xi, x \in T_\gamma(\xi, a)} w_{N, 3}(x),
\]

which is finite.

If \( n - p + \alpha = 0 \), then we can prove that (2) exists and is finite for any \( \xi \in \partial \mathbb{R}^n - (E' \cup B_p) \), any \( \gamma \geq 1 \) and any \( a > 1 \).

If \( n - p + \alpha < 0 \), then \( E' \) is empty, so that similar arguments yield the required assertion with the aid of Lemma 4.

4. Remarks.

Remark 1. – If \( p > n, \ p - \alpha - n > 0 \) and \( u \) is a locally \( p \)-precise function on \( \mathbb{R}^n_+ \) satisfying (1), then \( u(x) \) has a finite limit as \( x \to \xi, x \in \mathbb{R}^n_+ \) for any \( \xi \in \partial \mathbb{R}^n_+ \).
Remark 2. — Nagel, Rudin and Shapiro [7] studied tangential behaviors of functions of the form

\[(P_n \ast (K \ast g))(x') = (x', x_n) \in \mathbb{R}^n_+ \]

where \(P\) is the Poisson kernel in \(\mathbb{R}^n_+\), \(K\) is a nonnegative kernel, which is radial and decreasing, and \(g\) is a function in \(L^p(\mathbb{R}^{n-1})\).

The function in our theorem has a boundary value in the Lipschitz space \(\Lambda^{p,p}_{\beta}(\mathbb{R}^{n-1})\) with \(\beta = 1 - (\alpha + 1)/p\) locally, provided \(-1 < \alpha < p + 1\) (cf. [9; Chap. VI, §4.3, 4.5]). We do not know whether functions \(f\) in \(\Lambda^{p,p}_{\beta}(\mathbb{R}^{n-1})\) can be written as \(f = K \ast g\), where \(K\) is an appropriate kernel function which is determined independently of \(f\) and \(g \in L^p(\mathbb{R}^{n-1})\).

If \(g \in L^p(\mathbb{R}^n), \beta = 1 - (\alpha + 1)/p\) and \(-1 < \alpha < p - 1\), then

\[F(x') = \int g_{1-\alpha/p} ((x', 0) - y) g(y) dy\]

belongs to \(\Lambda^{p,p}_{\beta}(\mathbb{R}^{n-1})\), where \(g_\ell\) denotes the Bessel kernel of order \(\ell\) (cf. [9; Chap. VI, §4.3]). Hence \(u(x) = P_n \ast F(x')\) satisfies

\[\int_0^\infty \left[ x_n^{k-\beta} \left\{ \int_{\mathbb{R}^{n-1}} \left| \frac{\partial}{\partial x_n} \right|^k u(x', x_n) \left| dx' \right|^{1/p} \right\}^p x_n^{-1} dx_n < \infty,\]

where \(k\) is an integer greater than \(\beta\), in view of [9; p. 152]. This implies, by the observation given after Lemma 4' in [9; Chap. V], that \(u\) satisfies

\[\int_{\mathbb{R}^n_+} |\text{grad } u(x)|^p x_n^\alpha dx < \infty. \tag{1}'\]

Thus our theorem is applicable to this function \(u\).

Remark 3. — In case \(p - \alpha - n = 0\), our theorem gives the best possible result as to the size of the exceptional set as the next proposition shows.

**Proposition.** — Let \(E\) be a compact set in \(\partial \mathbb{R}^n_+\) with \(B_{1-\alpha/p,p}(E) = 0\), where \(p > 1\) and \(-1 < \alpha < p - 1\). Then there exists a function \(u\) which is harmonic in \(\mathbb{R}^n_+\) and satisfies (1)' such that \(\lim_{x_n \to 0} u(x', x_n)\) does not exist for any \((x', 0) \in E\).
Proof. — Since $B_{1-a/p,p}(E) = 0$, we can find a nonnegative function $f \in L^p$ such that

$$F(x') = \int g_{1-a/p}(x', 0) f(y) dy = \infty$$

for any $(x', 0) \in E$. As seen above, $P_{x_n} * F(x')$ satisfies (1)'.

Take $a_1, b_1$, and $c_1$ such that

$$0 < a_1 < 1, \quad 0 < b_1 < c_1 < 1$$

and $P_{a_1} * F_1(x') > 1$ for $(x', 0) \in E$, where

$$F_1(x') = \int_{\{b_1 < |y_n| < c_1\}} g_{1-a/p}(x', 0) f(y) dy.$$

We proceed inductively and obtain $\{a_j\}, \{b_j\}$ and $\{c_j\}$ such that

$$0 < a_{j+1} < a_j, \quad 0 < c_{j+1} < b_j < c_j, \quad P_{a_k} * F_j(x') < 2^{-j}$$

if $k < j$ and $(x', 0) \in E$ and $P_{a_j} * F_j(x') > j + \sum_{k=1}^{j-1} M_k$ if $(x', 0) \in E$, where

$$F_j(x') = \int_{\{b_j < |y_n| < c_j\}} g_{1-a/p}(x', 0) f(y) dy$$

and

$$M_k = \max \{F_k(x'); (x', 0) \in \partial R^+_n\}.$$

Define

$$u(x', x_n) = \sum_{j=1}^{\infty} (-1)^j P_{x_n} * F_j(x').$$

Then one sees easily that $\{u(x', a_j)\}$ does not converge as $j \to \infty$ for any $(x', 0) \in E$. Since $u$ satisfies (1)' and is harmonic in $R^+_n$, $u$ is the required function.

Remark 4. — Let $p > 1$, $p - n < \alpha < p - 1$ and $\alpha > -1$. Then we can find a function $u$ harmonic in $R^+_n$ and satisfying the following conditions:

(i) $u$ has a finite nontangential limit at 0;

(ii) $\limsup_{x \to 0, x \in T_{\gamma}(0, a)} u(x) = \infty$ for any $\gamma > 1$ and any $a > 1$;

(iii) $\int_{R^+_n} |\text{grad } u(x)|^p x_n^\alpha dx < \infty$.

To see this, let $x^{(j)} = (2^{-j}, 0, \ldots, 0$ and define
\( f_j(y) = \begin{cases} a_j |x^{(j)} - y|^{-1} & \text{if } y \in B(x^{(j)}, 2^{-j-2}) - R^n, \\ 0 & \text{otherwise}, \end{cases} \)

where \( a_j \) is a positive number determined later. Setting

\[ u(x) = \int (x_n - y_n) |x - y|^{-n} f(y) \, dy, \]

where \( f = \sum_{j=1}^{\infty} f_j \), we note the following facts:

a) \( u(0) \leq \int |y|^{1-n} f(y) \, dy \leq C_1 \sum_{j=1}^{\infty} a_j; \)

b) If \( x = (2^{-j}, 0, \ldots, 0, x_n) \) and \( 0 < x_n < 2^{-j-1} \), then \( u(x) \geq C_2 a_j \log(2^{-j}/x_n); \)

c) \( \int f(y)^p |y_n|^{\alpha} \, dy \leq C_3 \sum_{j=1}^{\infty} a_j^p 2^{-j(n-p+\alpha)}, \)

where \( C_1, C_2 \) and \( C_3 \) are positive constants independent of \( x \) and \( j \).

Now we choose \( \{a_j\} \) so that (a) and (c) are finite but \( \limsup_j ja_j = \infty \). Then (a) implies that \( u \) has a nontangential limit at \( 0 \), and (iii) follows from (c) and [5; Lemma 6](*). By (b) and the construction of \( \{a_j\} \), (ii) is fulfilled. Thus \( u \) satisfies (i), (ii) and (iii).

**Remark 5.** — If \( p > 1 \) and \( \alpha = p - n > -1 \), then for each \( \gamma > 1 \) there exists a function \( u \) harmonic in \( R^n \) such that:

(i) \( u \) has a finite nontangential limit at \( 0 \);

(ii) \( \limsup_{x \to 0, x \in T_{\gamma'}(t, a)} u(x) = \infty \) for any \( \gamma' \geq \gamma \) and any \( a > 1 \);

(iii) \( \int_{R^n} |\text{grad } u(x)|^p x_n^\alpha \, dx < \infty \).

In fact we modify above \( f_j \) by setting

\( f_j(y) = \begin{cases} a_j |x^{(j)} - y|^{-1} & \text{if } y \in B(x^{(j)}, 2^{-j-2}) - B(x^{(j)}, 2^{-\gamma j}) - R^n, \\ 0 & \text{otherwise}, \end{cases} \)

(* One notes that the conclusion of [5; Lemma 6] is true in case \( p > 1 \) and \( -1 < \alpha < p - 1 \) if \( g \) in the lemma has compact support.)
and consider \( u(x) = \sum_{j=1}^{\infty} \int (x_n - y_n) |x - y|^{-n} f_j(y) dy \). Then as in Remark 3 we can choose \( \{a_j\} \) such that (i), (ii) and (iii) hold.

Remark 6. — Let \( p > 1 \) and \( h \) be a positive function on the interval \((0, \infty)\) such that \( \lim_{r \downarrow 0} h(r) (\log r^{-1})^{p-1} = 0 \). Define for \( \xi \in \partial \mathbb{R}^n_+ \),

\[
T_h(\xi) = \left\{ x \in \mathbb{R}^n_+ ; \log \frac{|x - \xi|}{x_n} \leq h(|x - \xi|)^{-1/(p-1)} \right\}.
\]

If \( u \) is a function harmonic in \( \mathbb{R}^n_+ \) and satisfying (1) with \( \alpha = p - n > -1 \), then \( \lim_{x \to \xi, x \in T_h(\xi)} u(x) \) exists and is finite for any \( \xi \in \partial \mathbb{R}^n_+ - (E' \cup B_h) \), where

\[
B_h = \left\{ \xi \in \partial \mathbb{R}^n_+ ; \limsup_{r \downarrow 0} h(r)^{-1} \int_{B(\xi, r) \cap \mathbb{R}^n_+} |\text{grad} \ u(x)|^p x_n^\alpha \ dx > 0 \right\}.
\]

BIBLIOGRAPHY


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Yoshihiro Mizuta,  
Department of Mathematics  
Faculty of Integrated Arts and Sciences  
Hiroshima University  
Hiroshima 730 (Japan).