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SUBGROUPS OF CONTINUOUS GROUPS ACTING DIFFERENTIABLY ON THE HALF-LINE

by J.F. PLANTE

1. Introduction.

We consider groups of diffeomorphisms of the closed half-line $[0,\infty)$. Denote the group of C^k diffeomorphisms $1 \leqslant k \leqslant \infty$ by $\mathrm{Diff}^k[0,\infty)$ and the group of homeomorphisms by $\mathrm{Homeo}[0,\infty)$. Clearly, $\mathrm{Homeo}[0,\infty)$ is isomorphic to the group of all orientation preserving homeomorphisms of the line since $(0,\infty)$ is homeomorphic to the full-line. In the case of diffeomorphisms, the situation is different in that the differentiability at 0 can be expected to play a role in determining the structure of the group. If Γ is a subgroup of $\mathrm{Diff}^k[0,\infty)$, we would like to determine necessary and/or sufficient conditions for Γ to be a subgroup of a Lie group which acts at least continuously on $[0,\infty)$. More precisely, is there a Lie group G, an embedding $\Gamma \longrightarrow G$, and an action $G \longrightarrow \mathrm{Homeo}[0,\infty)$ such that the following diagram commutes ?

$$\begin{array}{ccc} \Gamma & \longrightarrow & \mathrm{Diff}^k(0,\infty) \\ \big\downarrow & & \big\downarrow \text{inclusion} \\ G & \longrightarrow & \mathrm{Homeo}[0,\infty) \end{array}$$

Recall that the affine group of the line is the group of diffeomorphisms of the line of the form $x \longmapsto ax + b\,(a \neq 0)$. This group has two components. Denote the identity component $(a > 0)$ by $\mathfrak{A}^+$. Up to isomorphism, $\mathfrak{A}^+$ is the only non-abelian connected 2-dimensional Lie group.

THEOREM A. – *Let* G *be a simply connected Lie group which acts in a* C^∞ *effective manner on* $[0, \infty)$. *If* 0 *is the only fixed point of the action (thus,* G *is transitive on* $(0, \infty)$*), then* G *is isomorphic (as an abstract group) to a subgroup of* $\mathfrak{A}^+$.

The proof of Theorem A, which rests mainly on a classical result of Lie, is given in Section 2. At this stage, it is worth observing that the result is not vacuous by constructing a C^∞ action of $\mathfrak{A}^+$ on $[0, \infty)$ with only 2 orbits. The only problem is achieving differentiability at 0 since $\mathfrak{A}^+$ acts on the line which is diffeomorphic to $(0, \infty)$. Let $\phi : \mathbf{R} \longrightarrow \mathbf{R}^+$ be a C^∞ diffeomorphism such that

$$\phi(t) = \frac{1}{1-t} \qquad -\infty < t \leqslant 0$$

$$\phi(t) = t \qquad 2 \leqslant t < \infty .$$

Computation shows that for $t \leqslant 0$,

$$\phi_*\left[\frac{\partial}{\partial t}\right] = t^2 \frac{\partial}{\partial t} \quad \text{and} \quad \phi_*\left[t \frac{\partial}{\partial t}\right] = (t^2 - t)\frac{\partial}{\partial t} \cdot$$

Thus, the standard action of $\mathfrak{A}^+$ on $\mathbf{R}$ induces an action of $\mathfrak{A}^+$ on $[0, \infty)$ which is C^∞ even at 0.

Theorem A suggests what to expect as a sufficient condition that a subgroup of $\mathrm{Diff}[0, \infty)$ embed in a continuous group of homeomorphisms. Since $\mathfrak{A}^+$ is solvable, it seems reasonable to consider groups which are solvable and occur as subgroups of Lie groups. When the group Γ is discrete, this can be done. It is known [11] that the groups which occur as discrete subgroups of solvable Lie groups are precisely the polycyclic groups.

THEOREM B. – *If* $\Gamma \subset \mathrm{Diff}^2[0, \infty)$ *is a polycyclic group which has* 0 *as its unique fixed point, then* Γ *embeds in a continuous action of* $\mathfrak{A}^+$ *on* $[0, \infty)$ *whose restriction to* $(0, \infty)$ *is topologically equivalent to the standard action of* $\mathfrak{A}^+$ *on* $\mathbf{R}$.

The proof of Theorem B is given in Section 3. In Section 4, we describe some applications of these results to codimension one foliations and Lie group actions.

The author is grateful to R. Gardner for helpful discussions about Lie algebras of vector fields on the line which provided the stimulus for writing this paper.

2. Proof of Theorem A.

It is a classical result of Lie that the only simply connected Lie groups which can act transitively on the line are $\mathbf{R}$, $\mathfrak{A}^+$, and $\widetilde{\mathrm{SL}}(2,\mathbf{R})$ (see for example [1] page 333). The proof given in [1] assumes that the (infinitesimal) action is real analytic. This smoothness assumption can easily be reduced to C^∞ by the following result which was communicated to the author by R. Bryant.

LEMMA. – *Let* $\mathfrak{g}$ *be a finite dimensional Lie algebra of* C^∞ *vector fields on the line such that for every point* $p \in \mathbf{R}$ *there is a vector field* $X \in \mathfrak{g}$ *such that* $X(p) \neq 0$. *If* $Z \in \mathfrak{g}$ *vanishes to infinite order at some point, then* $Z \equiv 0$.

Proof. – For $p \in \mathbf{R}$, let $\mathfrak{g}_p \subset \mathfrak{g}$ be the subalgebra of vector fields which vanish to infinite order at p. Let $S_p \subset \mathbf{R}$ be the set of points where *every* element of $\mathfrak{g}_p$ vanishes. S_p is a non-empty closed set since $\mathfrak{g}_p$ is finite dimensional. The proof of the lemma will be completed by showing that S_p is also open. Let $q \in S_p$ and $X \in \mathfrak{g}$ such that $X(q) \neq 0$. Choose coordinates for an open interval containing q so that $X = \dfrac{\partial}{\partial t}$. Let $Z_1, \dots, Z_k$ be a basis for $\mathfrak{g}_p$. There are functions $f_1, \dots, f_k$ which vanish to infinite order at q such that $Z_i = f_i \dfrac{\partial}{\partial t}$; $i = 1, \dots, k$. $\mathfrak{g}_p$ contains the vector field

$$[X, Z_i] = \left[\frac{\partial}{\partial t}, f_i \frac{\partial}{\partial t}\right] = f_i' \frac{\partial}{\partial t}.$$

Therefore, there exist constants c_{ij} such that

$$[X, Z_i] = \sum_{j=1}^{k} c_{ij} Z_j .$$

In other words, on an interval containing q the functions $f_1, \ldots, f_k$ are the (unique) solutions to the linear initial value problem

$$\frac{dx_i}{dt} = \sum_{j=1}^{k} c_{ij} x_j \qquad i, j = 1, \ldots, k$$

$$x_i(q) = 0 .$$

Therefore, $f_i \equiv 0$ in a neighborhood of q for all i which means that S_p is open.

We return now to the proof of Theorem A. In this case, we have a simply connected Lie group G acting on $[0, \infty)$ which acts transitively when restricted to $(0, \infty)$. Since $(0, \infty)$ is diffeomorphic to the line, G is either $\widetilde{SL}(2, \mathbf{R})$ or a subgroup of $\mathfrak{A}^+$. However, an action of $\widetilde{SL}(2, \mathbf{R})$ on $\mathbf{R}^+$ cannot be extended to $[0, \infty)$ in even a C^1 fashion by Thurston's generalized Reeb Stability Theorem, since $\widetilde{SL}(2, \mathbf{R})$ has a uniform discrete subgroup Γ such that $\mathrm{Hom}(\Gamma; \mathbf{R}) = 0$, and the action by Γ doesn't extend [10]. Therefore, G must be isomorphic to a subgroup of $\mathfrak{A}^+$.

3. Proof of Theorem B.

We begin with a result which suggests how the proof is organized.

LEMMA. – *If* $\Gamma \subset \mathrm{Diff}^2[0, \infty)$ *is a polycyclic group, then precisely one of the following possibilities occurs*

(i) Γ *is a finitely generated free abelian group.*

(ii) Γ *has exponential growth and a subgroup of finite index* Γ_0, *which in turn has a normal subgroup* N *such that* N *is a free abelian group of* rank $\geqslant 2$ *and* Γ_0/N *is a free abelian group of* rank $\geqslant 1$.

Proof. – If Γ has polynomial growth, then it is finitely generated and free abelian by (4.6) of [9]. The only other alternative according to [11] is that Γ has exponential growth. Since Γ is polycyclic, there is a subgroup Γ_0 of finite index and a finitely generated nilpotent subgroup N which is normal in Γ_0 such that Γ_0/N is finitely generated free abelian [11]. Since $N \subset \mathrm{Diff}^2[0, \infty)$, (4.5) of [9] says that N is actually free abelian. We assume that Γ, and hence Γ_0, have exponential growth. N and Γ_0/N must be non-trivial since nilpotent groups have polynomial growth [11]. It remains to be shown that $\operatorname{rank} N > 1$. If this were not the case, N would be generated by a single element h and for $g \in \Gamma_0$, we would have either $ghg^{-1} = h$ or $ghg^{-1} = h^{-1}$. Let Γ_0' be the subgroup of index at most 2 in Γ_0 consisting of those g such that $ghg^{-1} = h$. Then N is central in Γ_0' and Γ_0'/N is abelian which implies that Γ_0' is nilpotent and that Γ_0 and Γ have polynomial growth. Since this is not the case, $\operatorname{rank} N \geqslant 2$ and the lemma is proved.

In order to prove Theorem B, it suffices to show that the restriction of Γ to $(0, \infty)$ is conjugate to an action by a subgroup of $\mathcal{A}^+$. According to (4.6) of [8], any polycyclic group of homeomorphisms of the line which acts minimally (every orbit dense) is conjugate to a sub-group of the affine group of the line. Furthermore, if the group preserves orientation, it is conjugate to a subgroup of $\mathcal{A}^+$. Because of the endpoint 0, Γ preserves the orientation of $(0, \infty)$. Since $(0, \infty)$ is homeomorphic to the line, we see that Γ embeds in a continuous action of $\mathcal{A}^+$ on $[0, \infty)$ whenever Γ is polycyclic and the restriction to $(0, \infty)$ is minimal. The proof of Theorem B will be completed by showing that, in most cases, the action on $(0, \infty)$ is minimal. This requires the differentiability hypothesis and the hypothesis that 0 is the only fixed point.

Suppose $\Gamma \subset \operatorname{Diff}^2[0, \infty)$ is a free abelian group of rank > 1 which has no fixed points other than 0. We claim that *each element* of Γ fixes only 0. Suppose otherwise. Since $\operatorname{rank} \Gamma > 1$, let $s > 0$ and g, $h \in \Gamma$ be elements such that $g(s) = s$, $h(s) < s$. Let $s_0 = \lim_{n \to \infty} h^n(s)$. Clearly, s_0 is fixed by both g and h (since g and h commute). The points $h^n(s)$ are fixed by g and converge to s_0, and h has no fixed points in some interval of the form $(s_0, s_0 + \epsilon)$. Since g and h are C^2, this contradicts a result of Kopell [5] so the claim is proved.

We now claim that $\operatorname{rank} \Gamma > 1$ implies that it acts minimally on $(0, \infty)$. Select a fixed $g \in \Gamma$ and let $\langle g \rangle$ be the infinite cyclic group generated by g. The quotient group $\Gamma/\langle g \rangle$ is non-trivial, finitely generated, free abelian, and acts in the obvious way on the quotient space $(0, \infty)/t \sim g(t)$. Since no element of Γ has fixed points in $(0, \infty)$ this quotient space is C^2 diffeomorphic to the circle, and $\Gamma/\langle g \rangle$ acts freely and of class C^2. By a result of

Denjoy [3], this action is minimal and so, therefore, is the original action of Γ on $(0,\infty)$.

Theorem B is now proved in case (i) of the Lemma when $\operatorname{rank}\Gamma > 1$. In case (ii), we consider instead the free abelian group N. As before, we find that N acts minimally on $(0,\infty)$. Since N is a subgroup of Γ, Γ itself acts minimally on $(0,\infty)$ so Theorem B is proved in case (ii).

The only remaining case is case (i) with $\operatorname{rank}\Gamma = 1$. In this case, Γ is infinite cyclic generated a diffeomorphism g which clearly fixes only 0. Suppose, without loss of generality, that $g(t) > t$ for all $t > 0$. Fix $s > 0$ and select a homeomorphism $h : [s, g(s)) \longrightarrow [0,1)$. h extends in an obvious way to a homeomorphism $(0,\infty) \longrightarrow \mathbf{R}$ such that hgh^{-1} is the translation $t \longrightarrow t+1$. Thus, Γ embeds in a continuous action of $\mathfrak{A}^+$ on $[0,\infty)$ and the proof of Theorem B is complete.

Remark. – Theorem B can be extended in a manner which will be useful in the next section. Call a group Γ *virtually polycyclic* if it has a polycyclic subgroup of finite index. Theorem B is valid for virtually polycyclic subgroups of $\operatorname{Diff}^2[0,\infty)$. To see this, let Γ_0 be a polycyclic subgroup of finite index in Γ and let $\Gamma_1 \subset \Gamma$ be the intersection of all conjugates of Γ_0 in Γ. Γ_1 is a polycyclic *normal* subgroup of finite index in Γ. This means that virtually polycyclic groups are contained in the class $\mathfrak{S}$ of groups allowed in (4.6) of [8] and so the proof given above also works for virtually polycyclic groups.

4. Applications to codimension one foliations and Lie group actions.

Let $\mathfrak{F}$ be a transversely oriented codimension one foliation and L a compact leaf of $\mathfrak{F}$ which is isolated from other compact leaves on a side. We say that L has *affine holonomy* (on the side in question) if there is an arc τ transverse to $\mathfrak{F}$ which intersects L in a single endpoint x_0 such that the holonomy maps of τ determined by some generating set of $\pi_1(\mathrm{L}, x_0)$ are affine in suitable coordinates. By this, we mean that there is an embedding $f : \operatorname{int}\tau \longrightarrow \mathbf{R}$ such that $\lim_{t\to x_0} f(t) = -\infty$ and if h is a holonomy map (a homeomorphism between neighborhoods of x_0 in τ), then fhf^{-1} (which is defined on an interval of the form $(-\infty, b)$) is affine. The arguments used to prove Theorem B can be easily adapted to prove the following.

THEOREM. – *If* $\mathfrak{F}$ *is a transversely oriented codimension one foliation of class* C^2, L *is a compact leaf of* $\mathfrak{F}$ *which is isolated from other compact leaves on a side and* $\pi_1(L)$ *is virtually polycyclic then* L *has affine holonomy on that side.*

COROLLARY. – *Let* G *be a connected amenable Lie group which acts locally freely of class* C^2 *on an orientable manifold so that the orbit foliation has codimension one. If* L *is a compact orbit which is isolated on a side then* L *has affine holonomy on that side.*

The corollary follows immediately from the theorem and the following result from [6] which also helps explain the hypothesis.

LEMMA. – *If* G *is a connected Lie group, then the following statements are equivalent:*

a) G *is amenable.*

b) *Every discrete subgroup of* G *is virtually polycyclic.*

c) G *contains a solvable normal subgroup with compact quotient.*

When G is nilpotent, the holonomy in the conclusion of the corollary is actually equivalent to translations [2, 4, 9]. Using the construction from Section 1, we now show that codimension one solvable group actions do not always have such simple holonomy.

Let $\psi : \mathbf{R} \longrightarrow (0,1)$ be a C^∞ diffeomorphism such that

$$\psi(t) = \frac{1}{1-t} \qquad -\infty < t \leqslant -2$$

$$\psi(t) = \frac{t}{1+t} \qquad 2 \leqslant t < \infty .$$

The standard action of $\mathfrak{A}^+$ on $\mathbf{R}$ induces, via ψ, a C^∞ action of $\mathfrak{A}^+$ on $(0,1)$ which extends to be C^∞ on $[0,1]$. In this case, 0 and 1 are fixed and the action is transitive on $(0,1)$. We will construct a C^∞ codimension one foliation of $M \times [0,1]$ where M is a compact 3-manifold such that the only compact leaves are $M \times \{0\}$ and $M \times \{1\}$ and all other leaves are dense. The leaves of the codimension one foliation will be everywhere transverse to the arcs $\{x\} \times [0,1]$. Countably many of the leaves will have infinite cyclic holonomy groups and the foliation can be parametrized by the action of a 3-dimensional solvable Lie group.

Let M be the quotient of $\mathbf{R}^3$ by the fixed point free group Γ of transformations generated by A, B, C, where

$$A(x,y,z) = (2x+y, x+y, z-1)$$
$$B(x,y,z) = (x+1, y, z)$$
$$C(x,y,z) = (x, y+1, z).$$

Let X, Y, Z be vector fields on M defined by

$$X(x,y,z) = (2\lambda^{-z}, (\sqrt{5}-1)\lambda^{-z}, 0)$$
$$Y(x,y,z) = (-2\lambda^{z}, (\sqrt{5}+1)\ \lambda^{z}, 0)$$
$$Z(x,y,z) = (0,0,1)$$

where $\lambda = \frac{1}{2}(3+\sqrt{5})$. These vector fields satisfy the relations

$$[X,Y] = 0$$
$$[X,Z] = (\log\lambda)\, X$$
$$[Y,Z] = (-\log\lambda)\, Y$$

which implies that M is a solvmanifold.

Γ is the group of covering transformations of $\mathbf{R}^3$ over M. We let Γ act affinely on $\mathbf{R}$ as follows:

$$A : t \longmapsto \lambda t$$
$$B : t \longmapsto t + b$$
$$C : t \longmapsto t + c$$

where b, c are positive numbers such that $b^2 + c^2 = 1$, $(\sqrt{5}-1)\, b = 2c$. We leave it to the reader to check that the appropriate relations are satisfied

$$(ABA^{-1}B = BC = CB,\ ACA^{-1}C^{-1} = B).$$

Now ψ induces an action of Γ on $[0,1]$. The combined action of Γ on $\mathbf{R}^3 \times [0,1]$ preserves the codimension one foliation having leaves of the form $\mathbf{R}^3 \times \{t\}$ and, hence, induces the desired foliation $\mathcal{F}$ of $M \times [0,1] = \mathbf{R}^3 \times [0,1]/\Gamma$. The vectors fields X, Y, Z lift to vector fields tangent to the leaves of $\mathcal{F}$ which generate a solvable Lie group action having $\mathcal{F}$ as orbit foliation.

Remarks. – (i) This example shows that the orbit foliations of codimension one locally free actions by unimodular solvable groups need not be « almost without holonomy » which is in sharp contrast with the nilpotent case [2, 4]. The theorem of this section shows that the phenomenon of this example occurs whenever the action is C^2 and has an isolated compact leaf whose holonomy group has exponential growth.

(ii) It is probably the case that C^2 codimension one actions by amenable Lie groups can't have exceptional orbits. When there are no compact orbits and the group is unimodular, if follows from [7] that every orbit is dense. In the non-unimodular case, the same conclusion may follow from unpublished results of Duminy. When there are compact orbits (in which case the group must be unimodular), the results of the present paper should be useful.

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