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Estimates of one-dimensional oscillatory integrals


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ESTIMATES OF ONE-DIMENSIONAL OSCILLATORY INTEGRALS

by Detlef MÜLLER

1. Introduction.

If $U$ is an open domain in $\mathbb{R}^k$ and if $f$ is a smooth, real valued function on $U$, one may define the associated oscillatory integral as

$$E_f(\theta) = \int_U \theta(x)e^{2\pi i f(x)} dx,$$

where $\theta$ belongs to $\mathcal{D}(U)$, the space of test functions on $U$.

When $f$ has the form $f = \sum_{j=1}^n \eta_j \psi_j$, where the $\psi_j \in C^\infty(U)$ are real-valued functions and $\eta_j$ are real parameters, one is interested in the asymptotic behaviour of $E_{\Sigma \eta_j \psi_j}(\theta)$ as $(\eta_1, \ldots, \eta_n)$ tends to infinity, for several reasons.

For example, if $\mu$ is a smooth measure on a smooth submanifold of $\mathbb{R}^m$, and if the support of $\mu$ is sufficiently small, then the Fourier-Stieltjes transform $\hat{\mu}(\eta_1, \ldots, \eta_n)$ may always be written as $E_{\Sigma \eta_j \psi_j}(\theta)$ for certain functions $\psi_j$ and $\theta$.

Good information about the asymptotic behaviour of such Fourier-Stieltjes transforms is needed to solve the synthesis problem for smooth submanifolds of $\mathbb{R}^m$ (see e.g. [7]). And, as Professor Y. Domar has pointed out to me, such knowledge would also yield information about the decay at infinity of solutions of partial differential equations (see e.g. [5]).

As far as I know, satisfactory answers to the above problem have only been given for oscillatory integrals $E_{\Sigma \eta_j \psi_j}(\theta)$ with

$$\Sigma \eta_j \psi_j(x_1, \ldots, x_k) = \sum_{j=1}^k \eta_j x_j + \eta_{k+1} \psi_{k+1}(x_1, \ldots, x_k),$$
which correspond to surface carried measures (see [2], [4], [6]). In some sense, the other extreme is the case where $\sum \eta_j \psi_j$ is a function of only one real variable, which corresponds to measures on curves. For this case, we will prove some quite general results.

2.

Let $\psi \in C^\infty(1, \mathbb{R}^n)$, $\psi = (\psi_1, \ldots, \psi_n)$, where $I \neq \emptyset$ is some bounded open interval in $\mathbb{R}$. For $\xi$, $\eta \in \mathbb{R}^n$ let $\xi \cdot \eta$ denote the Euclidean inner product on $\mathbb{R}^n$, and correspondingly let

$$\eta \cdot \psi(x) = \sum_{j=1}^{n} \eta_j \psi_j(x).$$

Further let

$$|\eta| := \max_j |\eta_j| \quad \text{for} \quad \eta \in \mathbb{R}^n.$$

Define the torsion $\tau$ of $\psi$ by

$$\tau(x) = \det (\psi^{(i+1)}(x))_{i,j=1,\ldots,n} = \det (\psi''(x)\psi'''(x)\ldots\psi^{(n+1)}(x)),$$

where $\psi$ is regarded as a column vector and $\psi^{(k)}$ denotes the $k$th derivative of $\psi$. At least for $n = 2$ we have $\tau(x) = k(x)\psi''(x)^2$, where $k$ is the torsion of the curve $\gamma = \{(x,\psi(x)) : x \in I\}$ in $\mathbb{R}^{n+1}$. Let

$$e(t) = e^{2\pi it} \quad \text{for} \quad t \in \mathbb{R}, \quad \text{and} \quad e(g) = e \circ g$$

for $g \in C^\infty(I, \mathbb{R})$. If $\psi_0(x) = x$ for $x \in \mathbb{R}$, then for $\theta \in \mathcal{D}(I)$, $\eta_0 \in \mathbb{R}$ and $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n$, we have

$$E_{n} \sum_{\theta} \eta_j \psi_j^{(i)}(0) = (\theta e(\eta \cdot \psi))(\theta \eta_0).$$

So it will be slightly more general to study the behaviour of $|\theta e(\eta \cdot \psi)|_{PM}$ as $|\eta| \to \infty$, where

$$|\varphi|_{PM} = \sup_{t \in \mathbb{R}} |\varphi(t)|$$

for every $\varphi \in \mathcal{D}(\mathbb{R})$. 
For certain reasons (see [3]; [7], Th. 4.1), we will also study $|\Theta(e(\eta \cdot \psi))|_A$, where

$$|\varphi|_A = \int |\hat{\varphi}(t)| \, dt$$

for every $\varphi \in \mathcal{D}(\mathbb{R})$.

We will first state our main results and prove some corollaries:

**Theorem 1.** — Let $\Theta \in \mathcal{E}(I)$. Then

(i) $|\Theta(e(\eta \cdot \psi))|_A = 0(|\eta|^{1/2})$, as $|\eta| \to \infty$.

(ii) If for some subinterval $J$ of $I$ and some $\sigma > 0$

$$|\Theta(x)| \geq \sigma \text{ and } |\Theta(x) - \Theta(y)| < \sigma/2 \text{ for all } x, y \in J,$$

and if $\psi_1, \ldots, \psi_n$ are linearly independent modulo affine linear functions, then there is a constant $C > 0$, such that

$$|\Theta(e(\eta \cdot \psi))|_A \geq C(1 + |\eta|)^{1/2}$$

for all $\eta \in \mathbb{R}^n$.

**Corollary 1.** — The following two conditions are equivalent:

(i) For each $\Theta \in \mathcal{D}(\mathbb{R})$, $\Theta \neq 0$, there are constants $c > 0$, $C > 0$, such that for all $\eta \in \mathbb{R}^n$

$$c(1 + |\eta|)^{1/2} \leq |\Theta(e(\eta \cdot \psi))|_A \leq C(1 + |\eta|)^{1/2}.$$

(ii) $\psi_1, \ldots, \psi_n$ are linearly independent modulo affine linear functions on every non empty open subinterval of $I$.

**Proof of Corollary 1.** — (i) follows directly from (ii) by Theorem 1. Now suppose that there exists a vector $v \in \mathbb{R}^n$, $v \neq 0$, such that $v \cdot \psi$ is affine linear on some open subinterval $J \neq \emptyset$ of $I$. Then we have for any non-trivial $\Theta \in \mathcal{D}(\mathcal{I})$

$$|\Theta(e(sv \cdot \psi))|_A = |\Theta|_A \neq 0 \text{ for all } s \in \mathbb{R},$$

since $e(sv \cdot \psi)$ is the product of a unimodular complex number and a unitary character of $\mathbb{R}$.

Thus (i) is not fulfilled, q.e.d.
Remark. — Condition (ii) of Corollary 1 is clearly satisfied if $\tau^{-1}(\{0\})$ has empty interior. As will be shown later (Lemma 3), this is always the case if $\psi_1, \ldots, \psi_n$ are real analytic and linearly independent modulo affine mappings. However one should notice that global linear independence does not in general imply local linear independence.

**Theorem 2.** — (i) If $\tau^{-1}(\{0\}) = \emptyset$, then for $\theta \in \mathcal{D}(I)$

$$|\theta e(\eta \cdot \psi)|_{PM} = 0(|\eta|^{-1/(n+1)}) \quad \text{as} \quad |\eta| \to \infty.$$  

(ii) If $\theta \in \mathcal{D}(I)$, and if there exists an $x_0 \in I$ with $\theta(x_0) \neq 0$ and $\tau(x_0) \neq 0$, then there exists an $\varepsilon > 0$ and a function $\xi \in C^\infty((\varepsilon, \varepsilon), \mathbb{R}^n)$ with

$$\det(\xi(y)\xi'(y) \ldots \xi^{(n-1)}(y)) \neq 0 \quad \text{for all} \quad y \in (-\varepsilon, \varepsilon),$$

such that, for some $C > 0$,

$$|\theta e(s\xi(y) \cdot \psi)|_{PM} \geq C(1 + |s|)^{-1/(n+1)}$$

for all $s \in \mathbb{R}$ and $y \in (-\varepsilon, \varepsilon)$.

Assume that $\tau^{-1}(\{0\})$ has empty interior. Then we have

**Corollary 2.** — There exists a $\theta \in \mathcal{D}(I)$, $\theta \neq 0$, such that for all positive $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ with $\sum_1^n \alpha_j = (n+1)^{-1}$, there exists a constant $C = C(\alpha_1, \ldots, \alpha_n) > 0$ such that

$$|\theta e(\eta \cdot \psi)|_{PM} \leq C \prod_{j=1}^n |\eta_j|^{-\alpha_j}. \quad (2.1)$$

Conversely, if $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ are positive, and if there exists a $\theta \in \mathcal{D}(I)$, $\theta \neq 0$, and a $C > 0$ such that (2.1) holds, then

$$\sum_1^n \alpha_j = (n+1)^{-1}.$$

**Proof of Corollary 2.** — If $\tau^{-1}(\{0\})$ has empty interior, then there is of course an $x_0 \in I$ with $\tau(x_0) \neq 0$, and so, for $\theta \in \mathcal{D}(I)$ with sufficiently small support near $x_0$,

$$|\theta e(\eta \cdot \psi)|_{PM} \leq C(1 + |\eta|)^{-1/(n+1)}$$

by Theorem 2, (i).
If \( \alpha_1, \ldots, \alpha_n \) are positive and \( \Sigma \alpha_j \leq (n+1)^{-1} \), then
\[
\prod_j |\eta_j|^2 \leq |\eta|^{1/(n+1)} \quad \text{for} \quad |\eta| \geq 1,
\]
hence
\[
|\Theta_\epsilon(\eta \cdot \psi)|_{PM} \leq C \prod_j |\eta_j|^{-\eta_j} \quad \text{for} \quad |\eta| \geq 1,
\]
and the same estimate holds for all \( \eta \) if one replaces \( C \) by \( C + |\Theta_\epsilon|_{L^1} \).

Conversely, let now \( \Theta \in \mathcal{D}(I) \), \( \Theta \neq 0 \), such that (2.1) holds for some \( \alpha_j \geq 0 \), and assume
\[
\Sigma \alpha_j = (n+1)^{-1} + \delta, \quad \delta > 0.
\]
Since \( \tau^{-1}({\{0\}}) \) has empty interior, there is an \( x_0 \in I \) with \( \Theta(x_0) \neq 0 \) and \( \tau(x_0) \neq 0 \). Choose \( \epsilon > 0 \) and \( \xi \in \mathcal{C}^\infty((-\epsilon, \epsilon), \mathbb{R}^n) \) as in Theorem 2 (ii). Since \( \det (\xi(y)\xi'(y) \ldots \xi^{(n-1)}(y)) \neq 0 \) for all \( y \in (-\epsilon, \epsilon) \), there exists a \( y_0 \in (-\epsilon, \epsilon) \) with
\[
\xi_j(y_0) \neq 0 \quad \text{for} \quad j = 1, \ldots, n.
\]
It follows
\[
|\Theta_\epsilon(s\xi_0(y_0) \cdot \psi)|_{PM} \geq C'(1 + |s|)^{-1/(n+1)}.
\]
On the other hand, (2.1) yields
\[
|\Theta_\epsilon(s\xi_0(y_0) \cdot \psi)|_{PM} \leq C \prod_j |s\xi_j(y_0)|^{-\eta_j} = \left( C \prod_j |\xi_j(y_0)|^{-\eta_j} \right)|s|^{-1/(n+1)}|s|^{-\delta}.
\]
For \( |s| \) sufficiently large this leads to a contradiction to (2.2), q.e.d.

Corollary 2 demonstrates that the result in Theorem 2 is in some sense best possible.

3.

Before we start to prove the theorems above we will state some lemmas. The first one is due to J.-E. Björk and is cited in [3], Lemma 1.6:

**Lemma 1.** — Let \( I \neq \emptyset \) be a bounded, open interval in \( \mathbb{R} \), and let \( \varphi \in \mathcal{D}(I) \), \( g \in \mathcal{C}^p(I) \) with
\[
0 < C_1 \leq |g'(x)| + |g''(x)| + \cdots + |g^{(p)}(x)| \leq C_2
\]
if \( x \in \bar{I} \), where \( C_1 \) and \( C_2 \) are constants and \( p \) is a positive integer. Then there exists a constant \( C \) not depending on \( g \), such that

\[
\left| \int \varphi(x)e^{2\pi i g(x)} \, dx \right| \leq C(1 + |t|)^{-1/p}
\]

for every \( t \in \mathbb{R} \).

The second lemma will be used to prove the remark following Corollary 1. I would like to thank Professor H. Leptin for pointing out to me a shorter proof than my original one. By \( \langle \wedge \rangle \) we denote the exterior product in the Grassmann algebra \( \Lambda(\mathbb{R}^n) \).

**Lemma 2.** — Let \( \psi \in C^\infty(\bar{I},\mathbb{R}^n) \). Then

\[
\psi(x) \wedge \psi'(x) \ldots \wedge \psi^{(n-1)}(x) = 0
\]

for all \( x \in \bar{I} \) implies

\[
\psi^{(k_1)}(x) \wedge \psi^{(k_2)}(x) \wedge \ldots \wedge \psi^{(k_n)}(x) = 0
\]

for all \( x \in \bar{I} \) and \( k_1, \ldots, k_n \in \mathbb{N}_0 \).

**Proof.** — Fix \( x_0 \in \bar{I} \), and assume first \( \psi(x_0) \neq 0 \). If \( u \in C^\infty(\bar{I},\mathbb{R}) \), then

\[
(u\psi)^{(k)} = \sum_{j=0}^{k} \binom{k}{j} u^{(k-j)} \psi^{(j)},
\]

so \( \psi \wedge \psi' \wedge \ldots \wedge \psi^{(n-1)} \equiv 0 \) implies

\[
(u\psi) \wedge (u\psi)' \wedge \ldots \wedge (u\psi)^{(n-1)} \equiv 0.
\]

So, it is no loss of generality to assume

\[
\psi_n(x) = 1 \quad \text{for} \quad x \in \bar{I}.
\]

If \( \{e_j\}_j \) denotes the canonical basis of \( \mathbb{R}^n \), we may thus write

\[
\psi(x) = \sum_{j=1}^{n-1} \psi_j(x)e_j + e_n = \rho(x) + e_n, \quad \text{where} \quad \rho(x) \in \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n.
\]

This yields

\[
0 = \psi(x) \wedge \psi'(x) \wedge \ldots \wedge \psi^{n-1}(x) = \rho(x) \wedge \rho'(x) \wedge \ldots \wedge \rho^{(n-1)}(x) + e_n \wedge \rho'(x) \wedge \ldots \wedge \rho^{(n-1)}(x),
\]

\[
\rightarrow e_n \wedge \rho'(x) \wedge \ldots \wedge \rho^{(n-1)}(x) = 0.
\]
and since $p(x), p'(x), \ldots, p^{(n-1)}(x)$ are clearly linearly dependent, we get

$$0 = p'(x) \wedge p''(x) \wedge \ldots \wedge p^{(n-1)}(x).$$

By induction over $n$, we now may assume

$$0 = p^{(k_2)}(x) \wedge p^{(k_3)}(x) \wedge \ldots \wedge p^{(k_n)}(x)$$

for $x \in I$ and $k_j \geq 1$.

This implies

$$\psi^{(k_1)}(x) \wedge \ldots \wedge \psi^{(k_n)}(x) = e^{(k_1)}_n(x) \wedge p^{(k_2)}(x) \wedge \ldots \wedge p^{(k_n)}(x) = 0$$

for $0 \leq k_1 < k_2 < \ldots < k_n$, where we considered $e_n$ as the function $e_n(x) = e_n$.

Thus we have proved

$$\psi^{(k_1)}(x_0) \wedge \psi^{(k_2)}(x_0) \wedge \ldots \wedge \psi^{(k_n)}(x_0) = 0$$

for all $x_0 \in I_0 = \{x \in I : \psi(x) \neq 0\}$ and $k_j \geq 0$. By continuity, the same holds true for $x_0 \in I_0 \cap I$, hence for all $x_0 \in I$, since for $y \in I \setminus I_0$ clearly $\psi^{(k)}(y) = 0$ for every $k \in \mathbb{N}_0$.

**Lemma 3.** - If $\psi = (\psi_1, \ldots, \psi_n) \in C^\infty(I, \mathbb{R}^n)$ is real analytic, and if $\psi_1, \ldots, \psi_n$ are linearly independent modulo affine mappings, then $\tau^{-1}(\{0\})$ has empty interior, where $\tau$ denotes the torsion of $\psi$.

**Proof.** - Assume $\tau(x) = 0$ for every $x$ in some nonempty open interval $J \subset I$. Fix $x_0 \in J$. Then, passing to a possibly smaller interval, we may assume that $\psi_j$ has an absolute convergent series expansion

$$\psi_j(x) = \sum_{k=0}^{\infty} a_j^k (x-x_0)^k, \quad j = 1, \ldots, n, \quad x \in J.$$

Define vectors

$$a_k = (a_k^j)_{j=1,\ldots,n} \in \mathbb{R}^n$$

and

$$a^j = (a_k^j)_{k=2,\ldots,\infty} \in \mathbb{R}^{n_1}, \quad N_1 = N \setminus \{0,1\}.$$

By Lemma 2, $\psi^{(k_1)}(x_0), \ldots, \psi^{(k_n)}(x_0)$ are linearly dependent for any $k_j \in \mathbb{N}$ with $2 \leq k_1 < \ldots < k_n$, i.e. $a_1, \ldots, a_n$ are linearly dependent for $2 \leq k_1 < \ldots < k_n$. But this implies that $a^1, \ldots, a^n$ are linearly
dependent, i.e. there exist \( v_1, \ldots, v_n \in \mathbb{R} \), not all zero, with
\[
0 = \sum_j v_j a^j, \quad \text{i.e.}
\]
\[
\sum_j v_j \psi_j(x) = \sum_j v_j a^j_0 + v_j a^j_1 (x - x_0) \quad \text{for} \quad x \in J.
\]
But, since \( \psi \) is real analytic, this equation holds for all \( x \in I \), i.e. \( \sum_j v_j \psi_j \)
is affine linear.

4.

Proof of Theorem 1. — It is well-known (see e.g. [1], [7]) that for \( \varphi \in \mathcal{D}(\mathbb{R}) \) one has the estimate
\[
|\varphi|_\Lambda \leq \left\{ 2 |\text{supp } \varphi| |\varphi|_\infty |\varphi'|_\infty \right\}^{1/2},
\]
where \( |\text{supp } \varphi| \) denotes the Lebesgue measure of the support of \( \varphi \). From (4.1) one immediately gets (i) of Theorem 1.

Now, suppose there exists a subinterval \( J \) in \( I \) and a \( \sigma > 0 \) such that
\[ |\vartheta(x)| \geq \sigma \quad \text{and} \quad |\vartheta(x) - \vartheta(y)| < \sigma/2 \quad \text{for } x, y \in J, \]
and such that \( \psi_1, \ldots, \psi_n \) are linearly independent modulo affine mappings on \( J \). Then a simple compactness argument yields:

There are constants \( \varepsilon > 0, \delta > 0 \), such that for every \( \eta \in \mathbb{R}^n \) with \( |\eta| = 1 \) there is an interval \( J_\eta \) of length \( 2\varepsilon \) in \( J \) with
\[
|\eta \cdot \psi(x)| \geq \delta \quad \text{for all} \quad x \in J_\eta.
\]
Now choose \( \varphi \in \mathcal{D}(-\varepsilon, \varepsilon), \varphi \geq 0, \) with \( \int \varphi(x) \, dx = 1 \). For fixed \( \eta \in \mathbb{R}^n, \eta \neq 0 \), set \( \eta' = |\eta|^{-1} \eta \), and choose \( J_{\eta'} \) as in (4.2). Let \( \tilde{\varphi} \) be a suitable translate of \( \varphi \) such that \( \text{supp } \tilde{\varphi} \subset J_{\eta'} \). Then we get
\[
0 < \sigma/2 \leq \left| \int \vartheta(x) \tilde{\varphi}(x) \, dx \right|
\]
\[
= \left| \int \vartheta(x) e(\eta \cdot \psi)(x) \tilde{\varphi}(x) e(-\eta \cdot \psi)(x) \, dx \right|
\]
\[
\leq |\vartheta e(\eta \cdot \psi)|_\Lambda |\tilde{\varphi} e(-\eta \cdot \psi)|_{PM},
\]
since \( J_{\eta'} \subset J \).
For \( \xi \in \mathbb{R} \) one has

\[
(\phi e(\eta \cdot \psi))^{\sim}(-\xi) = \int \tilde{\phi}(x)e(-\xi x - \eta \cdot \psi(x)) \, dx = \int \varphi(x)e(-|\eta|g(x)) \, dx,
\]
where \( g \) is a function on \([-\varepsilon, \varepsilon]\) which is a certain translate of the function

\[
x \mapsto \xi'x + \eta' \cdot \psi(x) \quad \text{on} \quad J''_{\eta},
\]
where \( \xi' = |\eta|^{-1} \xi \).

But (4.2) implies

\[
\delta \leq |g''(x)| \quad \text{for every} \quad x \in [-\varepsilon, \varepsilon].
\]

Moreover, if we set \( A = 2 \sup_{x \in J} |\psi'(x)|, \quad B = \sup_{x \in J} |\psi''(x)| \), then for \( |\xi| \leq A|\eta| \):

\[
|g'(x)| + |g''(x)| \leq |\xi'| + |\eta'| (A + B) \leq 2A + B
\]
for every \( x \in [-\varepsilon, \varepsilon] \).

Thus, by Lemma 1, there exists a \( C > 0 \), such that for \( |\xi| \leq A|\eta| \)

\[
(4.4) \quad \left| \int \tilde{\phi}(x)e(-\xi x - \eta \cdot \psi(x)) \, dx \right| \leq C(1 + |\eta|)^{-1/2}.
\]

And, if \( |\xi| > A|\eta| \), then integration by parts yields

\[
(4.5) \quad \left| \int \tilde{\phi}(x)e(-\xi x - \eta \cdot \psi(x)) \, dx \right| = \left| \int e(-|\eta|g(x)) \left( \frac{\varphi}{2\pi i |\eta|g'}(x) \right) \, dx \right| \leq (2\pi |\eta|)^{-1} \left\{ \frac{|\varphi'(x)|}{|g'(x)|} + \frac{|\varphi(x)||g''(x)|}{|g'(x)|^2} \right\} \, dx \leq C'|\eta|^{-1},
\]
where \( C' \) is some constant depending on \( \varphi, \psi \) and \( A \) only, since for \( x \in [-\varepsilon, \varepsilon] \) we have \( |g''(x)| \leq B \) and \( |g'(x)| = |\xi' + \eta' \psi(y)| \geq A - A/2 \) for some \( y \in J \).
Now, by (4.4), (4.5),

$$|\Phi e(-\eta \cdot \psi)_{PM} \leq (C + C')|\eta|^{-1/2}$$ if $|\eta| \geq 1,$

which together with (4.3) proves Theorem 1 (ii).

Proof of Theorem 2. — Assume $\tau(x) \neq 0$ for every $x \in I,$ and let $\vartheta \in \mathcal{D}(I),$ $\vartheta \neq 0.$ Passing to a smaller interval, we may even assume that $I$ is closed.

Set $A = 2 \sup_{x \in I} |\psi'(x)|,$ and for $\xi' \in \mathbb{R},$ $|\xi'| \leq A,$ $\eta' \in \mathbb{R}^n,$ $|\eta'| = 1,$ $x \in I$ let

$$Q_{\xi', \eta'}(x) = \sum_{j=1}^{n+1} |(\xi' x + \eta' \cdot \psi(x))^{(j)}(x)|.$$

Since $\tau^{-1}(\{0\}) = \emptyset,$ we have $Q_{\xi', \eta'}(x) \neq 0$ for every $x \in I,$ and since $Q_{\xi', \eta'}(x)$ is continuous in $\xi',$ $\eta'$ and $x$ on the compact space $[-A, A] \times \{ \eta' \in \mathbb{R}^n: |\eta'| = 1 \} \times I,$ there exist constants $C_1 > 0,$ $C_2 > 0,$ such that

$$Q_{\xi', \eta'}(x) \leq C_1 \leq C_2$$

for all $x \in I,$ $\xi', \eta'$ with $|\xi'| \leq A,$ $|\eta'| = 1.$

So, using quite the same arguments as in the proof of Theorem 1 (ii), we can deduce from (4.6) by Lemma 1:

$$|\delta e(\eta \cdot \psi)|_{PM} \leq C(1 + |\eta|)^{-1/(\alpha + 1)}$$

for some constant $C > 0,$ which proves (i).

To prove (ii), we will assume, for convenience, $x_0 = 0,$ i.e. $0 \in I,$ and $\vartheta(0) \neq 0,$ $\tau(0) \neq 0.$

Let $\varepsilon > 0$ such that $\tau(x) \neq 0$ for $x \in [-\varepsilon, \varepsilon].$

Since $\psi''(x),$ $\psi'''(x),$ ..., $\psi^{(\alpha + 1)}(x)$ are linearly independent for $x \in [-\varepsilon, \varepsilon],$ there exists a function $\xi \in C^\infty([-\varepsilon, \varepsilon], \mathbb{R}^n),$ such that for every $x \in [-\varepsilon, \varepsilon]

$$\xi(x) \cdot \psi^{(j)}(x) = 0, \quad j = 2, \ldots, n,$$

and

$$\xi(x) \cdot \tilde{\psi}^{(\alpha + 1)}(x) = 1.$$
Differentiating (4.7) and inserting (4.8), we get
\[ \xi'(x) \cdot \psi^{(j)}(x) = 0 \quad \text{for} \quad j = 2, \ldots, n - 1, \]
and
\[ \xi'(x) \psi^{(n)}(x) = -1. \]

Repeating this process, one inductively obtains for \( k = 0, \ldots, n - 1 \)
\[
\begin{cases} 
\xi^{(k)}(x) \cdot \psi^{(j)}(x) = 0 & \text{for} \quad j = 2, \ldots, n - k, \\
\xi^{(k)}(x) \cdot \psi^{(n+1-k)}(x) = (-1)^k.
\end{cases}
\]

So, if we define matrices
\[
S(x) = (\xi^{(n-l)}(x))_{i,j=1,\ldots,n}, \quad T(x) = (\psi^{(l+1)}(x))_{i,j=1,\ldots,n},
\]
then (4.9) means that \( S(x)T(x) \) is an upper triangular matrix with diagonal elements 1 or \(-1\), which yields
\[
|\det (\xi(x) \cdot \xi'(x) \cdot \ldots \cdot \xi^{(n-1)}(x))| = |\det S(x)| = |\tau(x)|^{-1} \neq 0
\]
for all \( x \in [-\varepsilon, \varepsilon] \).

We now claim:

There is a constant \( C > 0 \), such that for all \( y \in (-\varepsilon, \varepsilon) \) and \( s \in \mathbb{R} \)
\[
|\mathfrak{g}(s\xi'(y) \cdot \psi)|_{PM} \geq C(1 + |s|)^{-1/(n+1)}.
\]
Choose \( y \in (-\varepsilon, \varepsilon) \). Then by (4.7), \( (\xi(y) \cdot \psi)^{(j)}(y) = \delta_{j,n+1} \) for \( j = 2, \ldots, n + 1 \), and so a Taylor expansion of \( \xi(y) \cdot \psi \) yields (for \( \varepsilon \) small enough)
\[
(\xi(y) \cdot \psi)(x) = \alpha + \beta x + (x-y)^{n+1}g(x) \quad \text{for} \quad x \in (-2\varepsilon, 2\varepsilon),
\]
where \( g \) is some smooth function on \( (-2\varepsilon, 2\varepsilon) \) which depends on \( y \), and where \( \alpha \) and \( \beta \) are some real numbers.

Let us remark here that although \( g = g_y \) depends on \( y \), \( \sup_{|x|<2\varepsilon} |g'_y(x)| \) is uniformly bounded for \( y \in (-\varepsilon, \varepsilon) \).

Now take \( \rho \in \mathcal{D}(\mathbb{R}) \) with \( \text{supp} \rho \subset (-\varepsilon, \varepsilon) \), \( \rho \geq 0 \) and
\[
\int \rho(x) \, dx = 1, \quad \text{and set} \quad \tilde{\rho}(x) = \rho(|s|^{1/(n+1)}(x-y)).
\]
If we choose $\varepsilon$ small enough such that

$$|\mathcal{G}(0) - \mathcal{G}(x)| < \frac{1}{2} |\mathcal{G}(0)|$$

for $x \in (-2\varepsilon, 2\varepsilon)$, then we get

$$\left| \int \mathcal{G}(x) \tilde{p}(x) \, dx \right| = \left| \int \mathcal{G}(s^{-1/(n+1)}x + y) \rho(x) \, dx \right| |s|^{-1/(n+1)}$$

$$\geq \frac{1}{2} |\mathcal{G}(0)| |s|^{-1/(n+1)}, \quad \text{if } |s| \geq 1;$$

and since

$$\left| \int \mathcal{G}(x) \tilde{p}(x) \, dx \right| = \left| \int \mathcal{G}(x) e(s\xi(y) \cdot \Psi) \tilde{p}(x) e(-s\xi(y) \cdot \Psi) \, dx \right|$$

$$\leq |\mathcal{G}e(s\xi(y) \cdot \Psi)|_{PM} |\tilde{p}e(-s\xi(y) \cdot \Psi)|_{A},$$

(4.11) will follow if we can show that $|\tilde{p}e(-s\xi(y) \cdot \Psi)|_A$ is uniformly bounded for $y \in (-\varepsilon, \varepsilon)$ and $|s| \geq 1$.

Now, regular affine mappings of $\mathbb{R}$ induce isometries of the Fourier algebra $A = A(\mathbb{R})$, thus

$$|\tilde{p}e(-s\xi(y) \cdot \Psi)|_A = |\rho e(-s\xi(y) \cdot \Psi)|_A,$$

where $\Psi(x) = \psi(|s|^{-1/(n+1)}x + y)$.

Since for $x \in \text{supp} \rho$ and $|s| \geq 1$,

$$|s|^{-1/(n+1)}x + y \in (-2\varepsilon, 2\varepsilon),$$

(4.12) yields

$$\xi(y) \cdot \Psi(x) = \alpha + \beta y + \beta |s|^{-1/(n+1)}x + |s|^{-1}x^{n+1} g(|s|^{-1/(n+1)}x + y).$$

Thus

$$|\tilde{p}e(-s\xi(y) \cdot \Psi)|_A = |\rho e(h)|_A,$$

where $h(x) = -s |s|^{-1}x^{n+1} g(|s|^{-1/(n+1)}x + y)$. If we again apply estimate (4.1), we easily see that $|\rho e(h)|_A$ is uniformly bounded for $y \in (-\varepsilon, \varepsilon)$ and $|s| \geq 1$, q.e.d.
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