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SUNAO OUCHI

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CHARACTERISTIC CAUCHY PROBLEMS AND SOLUTIONS OF FORMAL POWER SERIES

by Sunao ŌUCHI

1. Introduction and preliminaries.

Let C^{n+1} be the $(n+1)$-dimensional complex space. For the point in C^{n+1} we make use of the notation

$$z = (z_0, z_1, \ldots, z_n) = (z_0, z').$$

We employ the notation $\partial_{z_i} = \dfrac{\partial}{\partial z_i}$, $\partial_z = (\partial_{z_0}, \partial_{z_1}, \ldots, \partial_{z_n}) = (\partial_{z_0}, \partial_{z'})$ and $(\partial_z)^\alpha = (\partial_{z_0})^{\alpha_0}(\partial_{z'})^{\alpha'} = (\partial_{z_0})^{\alpha_0}(\partial_{z_1})^{\alpha_1}, \ldots, (\partial_{z_n})^{\alpha_n}$, where multi-index $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n) = (\alpha_0, \alpha')$ is an $(n+1)$-tuple of non-negative integers. For multi-index α, $|\alpha| = \alpha_0 + \alpha_1 + \ldots + \alpha_n$. We denote the dual variable by $\xi = (\xi_0, \xi_1, \ldots, \xi_n) = (\xi_0, \xi')$. For a linear partial differential operator $a(z, \partial_z)$ we denote by $a(z, \xi)$ its total symbol and by P.S. (a) (z, ξ) its principal symbol. We denote by $\mathcal{O}(\Omega)$ the totality of holomorphic functions in a domain Ω. For a real number a, $[a]$ means the integral part of a. For two natural numbers a, b, (a, b) means the greatest common diviser.

Now let us consider Cauchy problem

$$\left.\begin{aligned} &L(z, \partial_z)\, u(z) = \{(\partial_{z_0})^k - A(z, \partial_z)\}\, u(z) = f(z), \\ &(\partial_{z_0})^i\, u(0, z') = \hat{u}_i(z'), \quad 0 \leqslant i \leqslant k-1, \end{aligned}\right\} \qquad (1.1)$$

where

$$A(z, \partial_z) = \sum_{i=0}^{k-1} A_i(z, \partial_{z'})\, (\partial_{z_0})^i \qquad (1.2)$$

and its order is m and its coefficients and $f(z)$ belong to $\mathcal{O}(\Omega)$ for a neighbourhood Ω of $z = 0$ and $\hat{u}_i(z')$ $(0 \leqslant i \leqslant k-1)$ are holomorphic in $\Omega' = \Omega \cap \{z_0 = 0\}$. We can easily find out a formal solution of the form

$$\hat{u}(z) = \sum_{n=0}^{\infty} \hat{u}_n(z')\,(z_0)^n/n!\ , \tag{1.3}$$

where $\hat{u}_n(z')$ $(n \geqslant k)$ are successively and uniquely determined from (1.1). It follows from well-known Cauchy-Kovalevskaja theorem that whenever $m \leqslant k$, $\hat{u}(z)$ converges and is a unique holomorphic solution of (1.1). When $m > k$, $\hat{u}(z)$ does not always converge, that is, generally $\hat{u}(z)$ is a divergent series (see Mizohata [5]).

The purpose of this paper is to give an analytical interpretation of $u(z)$, when $m > k$. One of the results in this paper is the following:

Under some condition on $L(z, \partial_z)$, there is a function $u_S(z)$ holomorphic in a neighbourhood U of $z = 0$ except on $\{z_0 = 0\}$ such that

$$L(z, \partial_z)\, u_S(z) = f(z) \quad \text{in} \quad U - \{z_0 = 0\} \tag{1.4}$$

and it has the asymptotic expansion

$$u_S(z) \sim \hat{u}(z) = \sum_{n=0}^{\infty} \hat{u}_n(z')\,(z_0)^n/n!\ , \tag{1.5}$$

as $z_0 \longrightarrow 0$ in the sector $S = \{z_0\,;\ a < \arg z_0 < b\}$, where $(b-a)$ is less than a constant determined by $L(z, \partial_z)$.

Some of results in this paper are announced in Ōuchi [8].

Now let us give some definitions and lemmas to state the results in detail. The proofs of these lemmas will be given in § 2. We write $A(z, \partial_z)$ in the form different from (1.2):

$$A(z, \partial_z) = \sum_{i=0}^{m} \sum_{\ell=s_i}^{i} a_{i,\ell}(z, \partial_{z'})\,(\partial_{z_0})^{i-\ell}, \tag{1.6}$$

where $a_{i,\ell}(z, \xi')$ is homogeneous in ξ' with degree ℓ, $a_{i,s_i}(z, \xi') \not\equiv 0$ and if $a_{i,\ell}(z, \xi') \equiv 0$ for all ℓ, we put $s_i = +\infty$. We have

$$A_i(z, \partial_{z'}) = \sum_{(h,\ell),\,h-\ell=i} a_{h,\ell}(z, \partial_{z'}). \tag{1.7}$$

Let us define some quantities associated with $L(z, \partial_z)$. To do so let us expand $A_i(z, \xi')$ and $a_{i,\ell}(z, \xi')$ with respect to z_0,

$$A_i(z, \xi') = \sum_{j=0}^{\infty} A_{i,j}(z', \xi')\,(z_0)^j, \tag{1.8}$$

$$a_{i,\ell}(z, \xi') = \sum_{j=0}^{\infty} a_{i,\ell,j}(z', \xi')\,(z_0)^j. \tag{1.9}$$

From (1.7) we have

$$A_{i,j}(z', \xi') = \sum_{(h,\ell),\, h-\ell=i} a_{h,\ell,j}(z', \xi'). \tag{1.10}$$

Set $M_{i,j} = \operatorname{ord} A_{i,j}(z', \partial_{z'})$ and

$$\left.\begin{aligned} d_i &= \min\{(\ell + j)\,;\ a_{i,\ell,j}(z', \xi') \not\equiv 0\} \quad (i > k), \\ d_k &= 0. \end{aligned}\right\} \tag{1.11}$$

If $s_i = +\infty$, we put $d_i = +\infty$. Let us define

$$\beta = \max\{1, (M_{i,j} + j)/(k - i + j);\ 0 \leqslant i \leqslant k-1,\ j \geqslant 0\}. \tag{1.12}$$

We have

LEMMA 1.1. –

(i) *If* $a_{i,s_i}(0, z', \xi') \not\equiv 0$, *then* $d_i = s_i$.

(ii) $L(z, \partial_z)$ *is non-characteristic with respect to the surface* $\{z_0 = 0\}$ *if and only if* $\beta = 1$.

In the following of this paper we assume that the surface $\{z_0 = 0\}$ is characteristic, that is, $m > k$. Let us define other quantities σ_i $(0 \leqslant i \leqslant \ell + 1)$, which we call characteristic indices. Consider the set of points $P = \{P_j = (j, d_j)\,;\ k \leqslant j \leqslant m\}$. Let $\hat{P}$ be the convex envelope of the set P. The lower convex part of the boundary of $\hat{P}$ consists of segments Σ_i $(1 \leqslant i \leqslant \ell)$. We denote by Δ the set of extremal points (vertices) of Σ_i $(1 \leqslant i \leqslant \ell)$. Put

$$\Delta = \{(j_i, d_{j_i}) : i = 0, 1, \ldots, \ell\},$$

where $m = j_0 > j_1 > \ldots > j_\ell = k$ (see fig. 1.1).

DEFINITION 1.2. – *The i-th characteristic index* σ_i *of* $L(z, \partial_z)$ *is defined by*

$$\left.\begin{aligned} \sigma_0 &= +\infty \\ \sigma_i &= (d_{j_{i-1}} - d_{j_i})/(j_{i-1} - j_i), \quad 1 \leqslant i \leqslant \ell, \\ \sigma_{\ell+1} &= 1. \end{aligned}\right\} \tag{1.13}$$

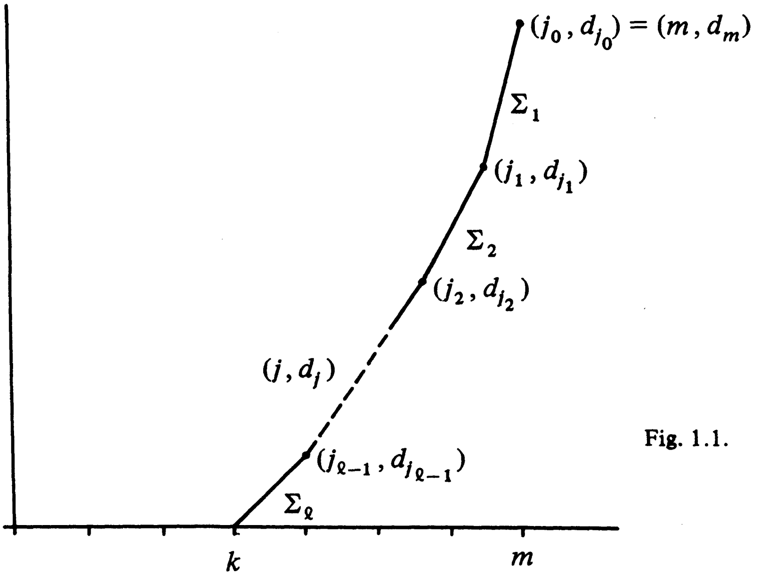

Fig. 1.1.

$\sigma_i (1 \leqslant i \leqslant \ell)$ *is the slope of the segment* Σ_i. *We put*

$$\left.\begin{aligned} &\gamma_0 = +\infty, \\ &\gamma_i = \sigma_{\ell+1-i}/(\sigma_{\ell+1-i} - 1), \quad 1 \leqslant i \leqslant \ell, \\ &\gamma_{\ell+1} = 1 . \end{aligned}\right\} \tag{1.14}$$

Remark 1.3. – σ_1 is a generalization of the irregularity of characteristic elements in Komatsu [4]. Characteristic indices can be more generally defined. They will be investigated elsewhere.

We have

LEMMA 1.4. –

(i) $+\infty = \sigma_0 > \sigma_1 > \ldots > \sigma_{\ell+1} = 1$,

(ii) $+\infty = \gamma_0 > \gamma_1 > \ldots > \gamma_{\ell+1} = 1$,

(iii) $\beta = \gamma_1$.

Later we shall deal with functions of several complex variables which have an asymptotic expansion with respect to one of them. Let $S = S(a, b) = \{z_0 \in C^1 ;\ a < \arg z_0 < b\}$ be a sectorial domain in C^1 and $U = \{z \in C^{n+1} ;\ |z_0| < r_0 ,\ |z_i| < r,\ 1 \leqslant i \leqslant n\}$ be a domain in C^{n+1}. Put $U' = \{z' \in C^n ;\ |z_i| < r\}$ and

$$U_S = \{S \cap (|z_0| < r_0)\} \times U'.$$

DEFINITION 1.5. – *Let* $f(z)$ *be holomorphic in* U_S. *A formal series* $\sum_{n=0}^{\infty} a_n(z)\,(z_0)^n/n!$, *where* $a_n(z)$ $(n = 0, 1, \ldots)$ *are holomorphic in* U, *is said to represent* $f(z)$ *asymptotically in* U_S, *if for any* N

$$|z_0|^{-N} \left| f(z) - \sum_{n=0}^{N} a_n(z)\,(z_0)^n/n! \right| \tag{1.15}$$

tends to zero uniformly on any compact set in U', *as* z_0 *tends to zero in* S.

The asymptotic relationship of the definition is usually written in the form

$$f(z) \sim \sum_{n=0}^{\infty} a_n(z)\,(z_0)^n/n!\,, \text{ as } z_0 \longrightarrow 0 \text{ in } U_S. \tag{1.16}$$

By expanding $a_n(z)$ with respect to z_0, we have

$$f(z) \sim \sum_{n=0}^{\infty} b_n(z')\,(z_0)^n/n! \text{ as } z_0 \longrightarrow 0 \text{ in } U_S. \tag{1.17}$$

In the following of this paper we often use expansions such as (1.16). For asymptotic series of functions we refer to Wasow [11]. We only give a proposition concerning differentiation of asymptotic series.

PROPOSITION 1.6. – *Suppose that* $f(z)$ *is holomorphic in* U_S *and possesses an asymptotic expansion of the form (1.17). Then we have*

$$(\partial_{z'})^{\alpha'} f(z) \sim \sum_{n=0}^{\infty} (\partial_{z'})^{\alpha'} b_n(z')\,(z_0)^n/n!\,, \text{ as } z_0 \longrightarrow 0 \text{ in } U_S \tag{1.18}$$

and for any proper subsector S_0 of S

$$(\partial_z)^{\alpha} f(z) \sim \sum_{n=0}^{\infty} (\partial_{z'})^{\alpha'} b_{n+\alpha_0}(z')\,(z_0)^n/n!, \text{ as } z_0 \longrightarrow 0 \text{ in } U_{S_0}. \tag{1.19}$$

By $\widetilde{\mathcal{O}}(\Omega - \{z_0 = 0\})$ we denote the set of holomorphic functions on the universal covering space of $\Omega - \{z_0 = 0\}$. Later we shall use functions of $(n+2)$-variables (z, λ). By $\widetilde{\mathcal{O}}(\Omega \times (|\lambda| > \Lambda))$ we denote the set of holomorphic functions of (z, λ) on the universal covering space of $\Omega \times (|\lambda| > \Lambda)$. By $C(d, \theta)$ or simply $C(\theta)$ we denote a path in λ-space defined as follows: Set

$$\begin{cases} C^-(d,\theta) = \{\lambda = r\exp(i(-\pi+\theta))\,;\ d \leqslant r < \infty\} \\ C^0(d,\theta) = \{\lambda = d\exp(i\rho);\ -\pi+\theta \leqslant \rho \leqslant \pi+\theta\} \\ C^+(d,\theta) = \{\lambda = r\exp(i(\pi+\theta))\,;\ d \leqslant r < \infty\}. \end{cases} \tag{1.20}$$

$C(\theta) = C^-(d, \theta) \cup C^0(d, \theta) \cup C^+(d, \theta)$ is a path which starts at $\infty \exp(i(-\pi + \theta))$, goes to $d \exp(i(-\pi + \theta))$ on $C^-(d, \theta)$, goes around the origin once on $C^0(d, \theta)$ and ends to $\infty \exp(i(\pi + \theta))$ on $C^+(d, \theta)$.

Now let us state some of results.

THEOREM 1.7. – *Let* $S = S(a, b)$ *be a sector with*

$$(b - a) < \pi/(\sigma_\ell - 1) = \pi(\gamma_1 - 1)$$

and θ_1 *be a number with* $(\pi + b - a)/2 < \theta_1 < (\pi\gamma_1)/2$, $\gamma_1 = \sigma_\ell/(\sigma_\ell - 1)$. *Then there are a neighbourhood* U *of* $z = 0$ *and functions*

$$u_{0,S}(z),\ g_{1,S}(z) \in \widetilde{\mathfrak{O}}(U - \{z_0 = 0\})$$

such that

$$\left.\begin{aligned} &L(z, \partial_z)\, u_{0,S}(z) = f(z) + g_{1,S}(z), \\ &u_{0,S}(z) \sim \hat{u}(z), \quad \textit{as } z_0 \longrightarrow 0 \textit{ in } U_S, \\ &g_{1,S}(z) \sim 0, \quad \textit{as } z_0 \longrightarrow 0 \textit{ in } U_S. \end{aligned}\right\} \tag{1.21}$$

Here $g_{1,S}(z)$ *is represented in the form, if* $|\arg z_0 + \theta| < \pi/2$,

$$g_{1,S}(z) = \int_{C(\theta)} \exp(\lambda z_0)\, G_{1,S}(z, \lambda)\, d\lambda, \tag{1.22}$$

where $G_{1,S}(z, \lambda) \in \widetilde{\mathfrak{O}}(U \times (|\lambda| > 1))$ *and satisfies*

$$\sup_{z \in U} |G_{1,S}(z, \lambda)| \leqslant A \exp(c' |\lambda|^{1/\gamma_1}) \tag{1.23}$$

and if $|\arg \lambda + (a + b)/2| \leqslant \theta_1$,

$$\sup_{z \in U} |G_{1,S}(z, \lambda)| \leqslant A \exp(-c\, |\lambda|^{1/\gamma_1}). \tag{1.24}$$

A, c' *and* c *are positive constants.*

Remark 1.8. – It follows from well-known Borel-Ritt theorem for asymptotic series that there are $u_{0,S}(z)$ and $g_{1,S}(z)$ satisfying (1.21) for arbitrary S, but it is important in Theorem 1.7 that $g_{1,S}(z)$ is represented in the form (1.22) with the estimates (1.23) and (1.24)

Now let us give an exact solution. To cancel $g_{1,S}(z)$, we put a sufficient condition on $L(z, \partial_z)$:

Condition I. $(i, s_i) \in \hat{P}$ $(i > k)$, and for $(i, d_i) \in \Delta$ (necessarily $d_i = s_i$)

$$\prod_{\substack{(i, s_i) \in \Delta \\ i > k}} a_{i, s_i}(0, \xi') \not\equiv 0. \tag{1.25}$$

THEOREM 1.9. – *Suppose that* $L(z, \partial_z)$ *satisfies condition I. Let* $S = S(a, b)$ *be a sector with* $(b - a) < \pi/(\sigma_1 - 1)$. *Then there is a function* $u_S(z) \in \widetilde{\mathcal{O}}(U - \{z_0 = 0\})$ *in a neighbourhood* U *of* $z = 0$ *such that*

$$\left.\begin{aligned} &L(z, \partial_z)\, u_S(z) = f(z), \\ &u_S(z) \sim \hat{u}(z), \ \text{as} \ z_0 \longrightarrow 0 \ \text{in} \ U_S . \end{aligned}\right\} \qquad (1.26)$$

We give an application of Theorem 1.9. Let us regard the operator $L(z, \partial_z)$ as an operator $L(x, \partial_x)$ with analytic coefficients on a domain $\Omega_R = \Omega \cap \{\operatorname{Im} z = 0\}$ in R^{n+1} by the restriction. We denote by x the point in R^{n+1}. We consider a characteristic Cauchy problem in Ω_R,

$$\left.\begin{aligned} &L(x, \partial_x)\, u(x) = f(x), \\ &(\partial_{x_0})^i\, u(0, x') = u_i(x'), \ \ 0 \leqslant i \leqslant k-1 . \end{aligned}\right\} \qquad (1.27)$$

In general (1.27) is not solvable. But we have

THEOREM 1.10. – *Suppose that* $L(x, \partial_x)$ *satisfies Condition I and* $f(x)$ *and* $u_i(x')$ $(0 \leqslant i \leqslant k-1)$ *are analytic in* x *and* x' *respectively in a neighbourhood of the origin. Then Cauchy problem (1.27) has a solution* $u(x)$ *in a neighbourhood* V *of* $x = 0$, *which is* C^∞ *in* V *and analytic in* $V - \{x_0 = 0\}$. *Moreover* $u(x)$ *has estimates*

$$|(\partial_{x_0})^{\alpha_0} (\partial_{x'})^{\alpha'} u(x)| \leqslant A\, C^{|\alpha|} (\alpha_0 !)^{\gamma_1} (\alpha' !) \quad \text{for } x \in V, \quad (1.28)$$

where A *and* C *are independent of* α.

In the following sections we shall use operators with a parameter λ in order to prove Theorem 1.7 and 1.9. Let us summerize what we shall need. Let $M(\lambda ; z, \partial_z)$ be an operator of the form

$$M(\lambda ; z, \partial_z) = \sum_{r=0}^{m} \lambda^r M_r(z, \partial_z), \ M_m(z, \partial_z) \not\equiv 0, \qquad (1.29)$$

where $M_r(z, \partial_z)$ is a linear partial differential operator of order t_r defined in Ω. Let us define quantities ν_i $(0 \leqslant i \leqslant m_0 + 1)$ associated with $M(\lambda ; z, \partial_z)$. Consider the set of points

$$P(\lambda) = \{(r, t_r) ;\ 0 \leqslant r \leqslant m\} .$$

Let $\hat{P}(\lambda)$ be the convex envelope of the set $P(\lambda)$. We assume that the upper convex part of the boundary of $\hat{P}(\lambda)$ consists of segments $\Sigma_i(\lambda)$ $(1 \leqslant i \leqslant m')$ (see fig. 1.2).

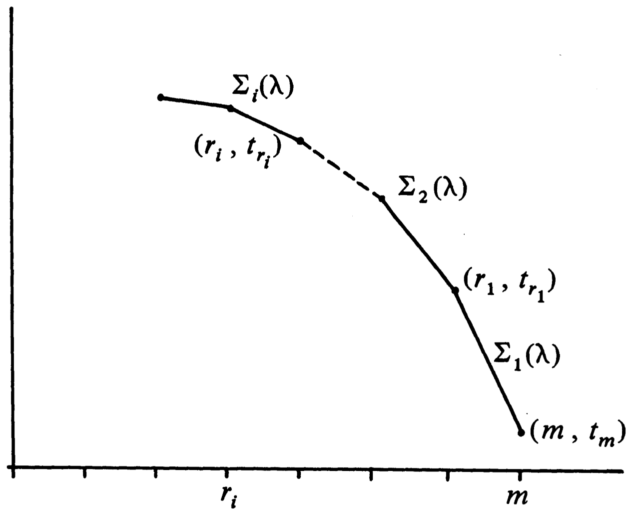

Fig. 1.2

We denote by $\Delta'(\lambda)$ the set of vertices of segments $\Sigma_i(\lambda)$ $(1 \leqslant i \leqslant m')$. Set $\Delta'(\lambda) = \{(r_i, t_{r_i})\,;\ i = 0, 1, \ldots, m'\}$, where $m = r_0 > r_1 > r_{m'}$. Put $\nu_0 = +\infty$ and

$$\nu_i = \max\{-(t_{r_{i-1}} - t_{r_i})/(r_{i-1} - r_i),\ 1\}. \tag{1.30}$$

We may assume that

$$\nu_1 > \nu_2 > \ldots > \nu_{m_0} > 1 = \nu_{m_0+1}, \qquad 0 \leqslant m_0 \leqslant m'.$$

We set $\Delta(\lambda) = \{(r_i, t_{r_i})\,;\ 0 \leqslant i \leqslant m_0\}$.

DEFINITION 1.11. – ν_i $(0 \leqslant i \leqslant m_0 + 1)$ *defined by (1.30) is said to be the i-th* λ*-characteristic index of* $M(\lambda\,; z, \partial_z)$.

LEMMA 1.12. – *Let* $(r, t_r) = (r_i, t_{r_i}) \in \Delta(\lambda)$. *Then if* $j > r$,

$$(t_j - t_r)/(j - r) \leqslant -\nu_i \tag{1.31}$$

and if $j < r$

$$(t_j - t_r)/(j - r) \geqslant -\nu_{i+1}. \tag{1.32}$$

We shall use Lemma 1.12 to show next theorem. Now let us consider an equation for $M(\lambda\,; z, \partial_z)$,

$$M(\lambda\,; z, \partial_z)\ V(z, \lambda) = G(z, \lambda), \tag{1.33}$$

where $G(z, \lambda) \in \widetilde{\mathcal{O}}(\Omega \times (|\lambda| > \Lambda))$. Set $M(\lambda) = \sup_{z \in \Omega} |G(z, \lambda)|$.

We shall not construct an exact solution, but construct a function $V(z, \lambda)$ which satisfies (1.33) asymptotically as $\lambda \longrightarrow \infty$ in some sector.

Let $\nu_i = p_i/q_i$, $p_i, q_i \in \mathrm{N}$, $(p_i, q_i) = 1$.

THEOREM 1.13 – *Suppose that* $\mathrm{P.S.}(\mathrm{M}_{r_i})\ (0, \xi) \not\equiv 0$ *and* $0 < \hat{\theta} < \pi\nu_{i+1}/2$. *Then there are a function*

$$V(\tau; z, \lambda) \in \widetilde{\mathcal{O}}(\Omega_0 \times (|\lambda| > \Lambda)),\ \Omega \supset \Omega_0,$$

with a parameter $\tau > 0$ *and a constant* $\hat{\tau} > 0$ *dependent on* $\hat{\theta}$ *such that for* $0 < \tau \leqslant \hat{\tau}$

$$M(\lambda; z, \partial_z)\, V(\tau; z, \lambda) = G(z, \lambda) + \exp(-\tau\lambda^{1/\nu_{i+1}})\, H(\tau; z, \lambda), \quad (1.34)$$

$$\sup_{z \in \Omega_0} |V(\tau; z, \lambda)| \leqslant AM(\lambda)\, \exp(c' |\lambda|^{1/\nu_{i+1}}), \quad (1.35)$$

and if $|\arg \lambda| \leqslant \hat{\theta}$,

$$\sup_{z \in \Omega_0} |V(\tau; z, \lambda)| \leqslant AM(\lambda)\, \exp(c\,|z|\,|\lambda|^{1/\nu_i}). \quad (1.36)$$

Here $H(\tau; z, \lambda) \in \widetilde{\mathcal{O}}(\Omega_0 \times (|\lambda| > \Lambda))$ *and satisfies*

$$|H(\tau; z, \lambda)| \leqslant \begin{cases} AM(\lambda)\, \exp(\tau^{q_{i+1}/(q_{i+1}-1)}\, b\, |\lambda|^{1/\nu_{i+1}}), & q_{i+1} > 1, \\ AM(\lambda)\, (1 + |\lambda|)^{N}, & q_{i+1} = 1. \end{cases} \quad (1.37)$$

Constants b, c', c, A *and* N *are independent of* τ.

In this paper we only consider an operator $L(\lambda; z, \partial_z)$ induced from $L(z, \partial_z)$,

$$L(z, \partial_z)\, \exp(\lambda z_0)\, V(z, \lambda) = \exp(\lambda z_0) \sum_{r=0}^{k} \lambda^r L_r(z, \partial_z)\, V(z, \lambda)$$
$$= \exp(\lambda z_0)\, L(\lambda; z, \partial_z)\, V(z, \lambda). \quad (1.38)$$

In view of (1.6), we obtain

$$L_r(z, \partial_z) = \sum_{\substack{(i,p) \\ i-p \geqslant r}} \frac{(i-p)!}{(i-p-r)!\, r!}\, a_{i,p}(z, \partial_{z'})\, (\partial_{z_0})^{i-p-r}. \quad (1.39)$$

We can also define $\Delta(\lambda) = \Delta(\lambda, L)$ and $\nu_i = \nu_i(L)$ for $L(\lambda; z, \partial_z)$ as above. Put $\Delta(\lambda, L) = \{(r_i, t_{r_i});\ 0 \leqslant i \leqslant k_0\}$. Now we have

LEMMA 1.14. – *Suppose that* $L(z, \partial_z)$ *satisfies Condition I. Then* $k_0 = \ell$, *where* ℓ *is that in Definition 1.2,*

$\Delta(\lambda, L) = \{(i - s_i, s_i);\ (i, s_i) \in \Delta\}$, P.S. $(L_{i-s_i})(z, \xi) = a_{i,s_i}(z, \xi')$ *and* $\nu_{\ell+1-i}(L) = \gamma_{\ell+1-i}\ (= \sigma_i/(\sigma_i - 1))\ (1 \leqslant i \leqslant \ell)$.

Let us state the contents of the following sections. In § 2 we shall give proofs of lemmas in § 1. In § 3 we shall show how to construct the function $u_{0,S}(z)$ in Theorem 1.7. In § 4 we shall investigate equations with a parameter λ and construct solutions with singularities on $\{z_0 = 0\}$ for $L(z, \partial_z)$. In § 5 by making use of results obtained, we shall show how to construct $u_S(z)$ in Theorem 1.9. In § 6 we shall give estimates of functions constructed in § 3 and § 4. In § 7 we shall study functions defined by integrals. Asymptotic expansions of functions will be investigated. By applying them, we shall complete the proofs of theorems.

2. Proofs of lemmas in § 1.

In § 2 we shall prove Lemma 1.1, 1.4, 1.12 and 1.14.

Proof of Lemma 1.1. –

(i) From the condition $a_{i,s_i}(0, z', \xi') = a_{i,s_i,0}(z', \xi') \not\equiv 0$ we have $d_i = s_i$.

(ii) $\beta = 1$ holds if and only if $M_{i,j} + j \leqslant k - i + j$. Hence $M_{i,j} \leqslant k - i$. This implies $\{z_0 = 0\}$ is non-characteristic.

Proof of Lemma 1.4. –

(i) follows from lower convexity of $\hat{P}$. (ii) is obvious. Let us show (iii). First we show $\beta \leqslant \gamma_1$. Suppose that $M_{i_0,j_0} = \ell_0$. By putting $h_0 = i_0 + \ell_0$, we have

$$(M_{i_0,j_0} + j_0)/(k - i_0 + j_0) = (\ell_0 + j_0)/(\ell_0 + j_0 - h_0 + k). \quad (2.1)$$

If $h_0 \leqslant k$, we have $(\ell_0 + j_0)/(\ell_0 + j_0 - h_0 + k) \leqslant 1 \leqslant \gamma_1$. If $h_0 > k$, then from (1.10) we have $d_{h_0} \leqslant \ell_0 + j_0$. Hence we obtain $(\ell_0 + j_0)/(h_0 - k) \geqslant d_{h_0}/(h_0 - k) \geqslant \sigma_\ell = \gamma_1/(\gamma_1 - 1)$. So in view of (2.1), we get $(M_{i_0,j_0} + j_0)/(k - i_0 + j_0) \leqslant \gamma_1$. This implies $\beta \leqslant \gamma_1$.

Next we show $\beta \geqslant \gamma_1$. Let ℓ_1, j_1 and i_1 be integers such that $a_{i_1,\ell_1,j_1}(z', \xi') \not\equiv 0$, $a_{i_1,\ell,j}(z, \xi') \equiv 0$ for $\ell + j < \ell_1 + j_1$ and $d_{i_1}/(i_1 - k) = (\ell_1 + j_1)/(i_1 - k) = \sigma_\ell$. Since

$$M_{i_1-\ell_1,j_1} = \max\{\ell\,;\, a_{h,\ell,j_1}(z',\xi') \not\equiv 0\,,\; h = \ell + i_1 - \ell_1\} \geqslant \ell_1\,,$$

we have

$$\beta \geqslant (M_{i_1-\ell_1,j_1} + j_1)/(k - i_1 + \ell_1 + j_1)$$
$$\geqslant (\ell_1 + j_1)/(k - i_1 + \ell_1 + j_1) = \gamma_1\,.$$

This completes the proof.

Proof of Lemma 1.12. – (1.31) and (1.32) follow from upper convexity of $\hat{P}(\lambda)$.

Proof of Lemma 1.14. – Put

$$A = \{(i\,, s_i)\,;\; k \leqslant i \leqslant m\,,\; s_i \neq +\infty\} \cup (0\,,0) \cup (m\,,m)\,.$$

Let $\hat{A}$ be the convex envelop of A. In view of Lemma 1.1 the set of extremal points of $\hat{A}$ consists of Δ defined for $L(z\,,\partial_z)$, $(0\,,0)$ and $(m\,,m)$. So from geometrical consideration of $\hat{A}$ and (1.39) we have $\Delta(\lambda\,,L) = \{(i - s_i\,, s_i)\,;\; (i\,, s_i) \in \Delta\}$ and

$$\text{P.S.}\,(L_{i-s_i})\;(z\,,\xi) = a_{i,s_i}(z\,,\xi')$$

(see fig. 2.1.). Let $\Delta = \{(j_i\,, s_{j_i})\,;\; 0 \leqslant i \leqslant \ell\}$ and

$$\Delta(\lambda\,,L) = \{(r_i\,, t_{r_i})\,;\; 0 \leqslant i \leqslant \ell\}\,.$$

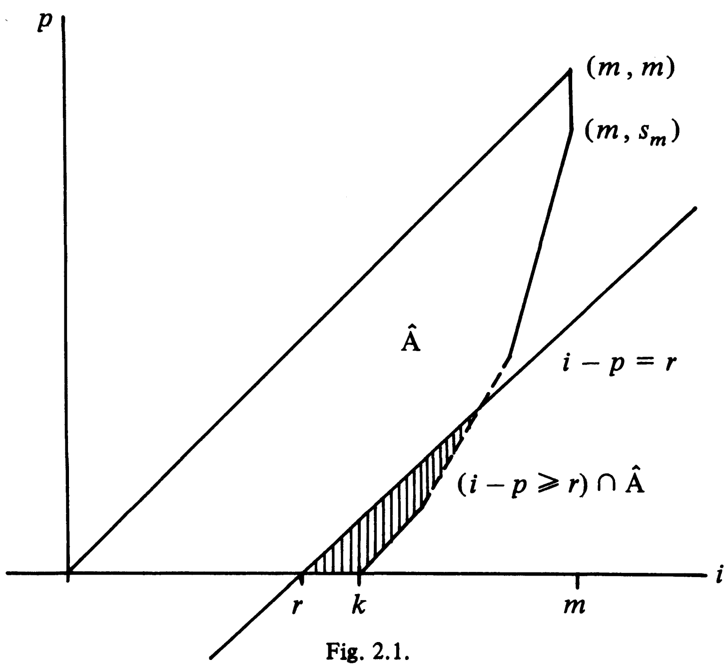

Fig. 2.1.

We have $r_i = j_{\ell-i} - s_{j_{\ell-i}}$ and $t_{r_i} = s_{j_{\ell-i}}$. Thus we get

$$\nu_i = (s_{j_{\ell+1-i}} - s_{j_{\ell-i}})/(j_{\ell-i} - s_{j_{\ell-i}} - j_{\ell+1-i} + s_{j_{\ell+1-i}}) = \sigma_{\ell+1-i}/(\sigma_{\ell+1-i} - 1) = \gamma_i .$$

3. Construction of solutions I.

In § 3 we construct the function $u_{0,S}(z)$ in Theorem 1.7. We only show construction of $u_{0,S}(z)$. Estimates of functions appearing in construction and asymptotic behaviour of them will be investigated in the later sections. In the sequel we denote $u_{0,S}(z)$ by $u_0(z)$ and assume $\hat{u}_i(z') \equiv 0$ $(0 \leqslant i \leqslant k-1)$ and

$$S = \{z_0 \in C^1 ; \ |\arg z_0| < \omega\}, \ \omega < \pi(\gamma_1 - 1)/2 .$$

We seek for $u_0(z)$ in the form

$$u_0(z) = \frac{1}{2\pi i} \int_{C(\theta)} \exp(\lambda z_0)\, W(z, \lambda)\, d\lambda , \tag{3.1}$$

where

$$W(z, \lambda) = \lambda^{(1-p)/p} \int_0^{\tau\lambda^\delta} \exp(-\lambda^{1/p} \zeta)\, w(z, \zeta)\, d\zeta . \tag{3.2}$$

Here we recall that $\gamma_1 = \sigma_\ell/(\sigma_\ell - 1) = p/q$, $\delta = (q-1)/p$, p and q are natural numbers with $(p, q) = 1$ and τ is a positive constant, which will be determined later. The path $C(\theta)$ is defined in § 1.

Now let us give an equation which $w(z, \zeta)$ satisfies. Our calculations are formal, but by obtaining estimates we shall be able to justify them. First we introduce some notions. Let $v(\zeta)$ be holomorphic in $\{\zeta \in C^1 ; |\zeta| < R\}$. Define

$$\left.\begin{aligned} (\partial_\zeta)^s v(\zeta) &= \left(\frac{d}{d\zeta}\right)^s v(\zeta), \quad s \geqslant 0, \\ (\partial_\zeta)^s v(\zeta) &= \int_0^\zeta (\partial_\zeta)^{s+1} v(\zeta)\, d\zeta, \quad s < 0 . \end{aligned}\right\} \tag{3.3}$$

DEFINITION 3.1. – *A linear operator* $H(z, \zeta, \partial_z, \partial_\zeta)$ *is said to be an integro-differential operator on* $\Omega \times \{|\zeta| < R\}$; $(0 < R \leqslant \infty)$, *if it has the form*

$$H(z, \zeta, \partial_z, \partial_\zeta) = \sum_{|j| \leqslant J} H_j(z, \zeta, \partial_z)\, (\partial_\zeta)^j , \tag{3.4}$$

where $H_j(z, \zeta, \partial_z)$ $(|j| \leqslant J)$ *are linear partial differential operators* *in* Ω *with coefficients holomorphic in* $\Omega \times \{|\zeta| < R\}$. *If coefficients of* $H_j(z, \zeta, \partial_z)$ $(|j| \leqslant J)$ *are polynomials of* ζ, $H(z, \zeta, \partial_z, \partial_\zeta)$ *is said to be an integro-differential operator of polynomial type.*

DEFINITION 3.2. – *Let* $v(z, \lambda, \zeta)$ *be holomorphic in*

$$\Omega \times \{|\lambda| > \Lambda\} \times \{|\zeta| < R\}.$$

(i) *A function* $h(z, \lambda, \zeta)$ *is said to belong to* Err. (v), *if* $h(z, \lambda, \zeta)$ *has an expression*

$$h(z, \lambda, \zeta) = \sum_{n=1}^{N} \lambda^{h_n} H^n(z, \zeta, \partial_z, \partial_\zeta)\, v(z, \lambda, \zeta), \tag{3.5}$$

where $H^n(z, \zeta, \partial_z, \partial_\zeta)$ $(n = 1, 2, \ldots, N)$ *are integro-differential operators of polynomial type and* h_n $(n = 1, 2, \ldots, N)$ *are constants.*

(ii) *A function* $f(z, \lambda)$ *is said to belong to* Err $(v, c\lambda^a)$ $(a \geqslant 0)$, *if there is a function* $h(z, \lambda, \zeta) \in$ Err (v) *such that*

$$f(z, \lambda) = h(z, \lambda, \zeta)\big|_{\zeta = c\lambda^a},$$

where if $a = 0$, $|c| < R$ *and if* $a > 0$, $R = +\infty$.

We need some properties of functions defined by integrals. Put

$$V(z, \lambda) = \lambda^{(1/p-1)} \int_0^{\tau\lambda^\delta} \exp(-\lambda^{1/p}\zeta)\, v(z, \zeta)\, d\zeta, \tag{3.6}$$

where $v(z, \zeta)$ is holomorphic in $\Omega \times \{|\zeta| < R\}$ and if $\delta > 0$, $R = +\infty$, if $\delta = 0$, $0 < \tau < R$. Let us recall $\delta = (q-1)/p$. We have

LEMMA 3.3. –

$$\{-(1/p - 1)(\partial_\zeta)^{-p} + (1/p)(\partial_\zeta)^{-p+1}\zeta\}\, v(z, \zeta) = (\zeta/p)(\partial_\zeta)^{-p+1} v(z, \zeta). \tag{3.7}$$

Proof. – By expanding $v(z, \zeta)$ with respect to ζ, we have only to show this lemma for functions ζ^m $(m = 0, 1, \ldots)$. We have

$$\begin{aligned} &-(1/p - 1)(\partial_\zeta)^{-p}\zeta^m + (1/p)(\partial_\zeta)^{-p+1}\zeta^{m+1} \\ &\qquad = -(1/p - 1)\,\zeta^{m+p}/(m+p)\ldots(m+1) \\ &\qquad + \zeta^{m+p}/p(m+p)(m+p-1)\ldots(m+2) \\ &\qquad = \zeta^{m+p}/p(m+p-1)\ldots(m+1) = (\zeta/p)(\partial_\zeta)^{-p+1}\zeta^m. \end{aligned}$$

Hence we get

PROPOSITION 3.4. –

(i) *Let* $(\partial_\zeta)^h v(z,0) = 0$ *for* $0 \leqslant h \leqslant p-1$. *Then*

$$\lambda V(z,\lambda) = \lambda^{(1/p-1)} \int_0^{\tau\lambda^\delta} \exp(-\lambda^{1/p}\zeta)(\partial_\zeta)^p v(z,\zeta)\,d\zeta + \exp(-\tau\lambda^{q/p}) V_1(z,\lambda). \tag{3.8}$$

(ii)

$$-\frac{\partial V}{\partial \lambda} = \lambda^{(1/p-1)} \int_0^{\tau\lambda^\delta} \exp(-\lambda^{1/p}\zeta)(\zeta/p)(\partial_\zeta)^{-p+1} v(z,\zeta)\,d\zeta + \exp(-\tau\lambda^{q/p}) V_2(z,\lambda). \tag{3.9}$$

Here $V_1(z,\lambda)$, $V_2(z,\lambda) \in \mathrm{Err}\,(v, \tau\lambda^\delta)$.

Proof. – (i) follows from integration by parts (ii) follows from Lemma 3.3.

Now let us give an equation for $w(z,\zeta)$. Let J be an integer such that $\beta > \max\{(M_{i,j}+j)/(k-i+j)\,;\ 0 \leqslant i \leqslant k-1,\ j \geqslant J\}$. We fix J. We have

$$A_i(z,\xi') = \sum_{j=0}^{J-1} A_{i,j}(z',\xi')(z_0)^j + A_{i,J}(z,\xi')(z_0)^J. \tag{3.10}$$

In the following we use (3.10) instead of (1.8). So we denote

$$A_{i,j}(z',\xi')\ (0 \leqslant i \leqslant J-1) \text{ by } A_{i,j}(z,\xi')$$

and put $M_{i,J} = \operatorname{ord} A_{i,J}(z,\partial_{z'})$. Recall that the initial values $\hat{u}_i(z')$ $(0 \leqslant i \leqslant k-1)$ are assumed to be zero.

By Leibniz formula we get

$$(\partial_{z_0})^h \exp(\lambda z_0) W(z,\lambda) = \exp(\lambda z_0) \sum_{s=0}^{h} \frac{h!}{(h-s)!\,s!} \lambda^s (\partial_{z_0})^{h-s} W(z,\lambda). \tag{3.11}$$

Hence we obtain

$$L(z,\partial_z)(\exp(\lambda z_0) W(z,\lambda)) = \exp(\lambda z_0) \sum_{s=0}^{k} \frac{k!}{(k-s)!\,s!} \lambda^s (\partial_{z_0})^{k-s} - \sum_{i=0}^{k-1} \sum_{j=0}^{J} A_{i,j}(z,\partial_{z'})(z_0)^j \left\{ \sum_{s=0}^{i} \frac{i!}{(i-s)!\,s!} \lambda^s (\partial_{z_0})^{i-s} \right\} W(z,\lambda) = \exp(\lambda z_0) L(\lambda\,; z, \partial_z) W(z,\lambda). \tag{3.12}$$

Thus, from (3.1) and (3.12), we have

$$L(z,\partial_z) u_0(z) = \frac{1}{2\pi i} \int_{C(\theta)} \exp(\lambda z_0) L(\lambda\,; z,\partial_z) W(z,\lambda)\,d\lambda. \tag{3.13}$$

By integration by parts with respect to λ, we have

$$L(z, \partial_z)\, u_0(z) = \frac{1}{2\pi i}\int_{C(\theta)} \exp(\lambda z_0)\, L(\lambda, \partial_\lambda, z, \partial_z)\, W(z, \lambda)\, d\lambda, \tag{3.14}$$

where

$$L(\lambda, \partial_\lambda, z, \partial_z) = \sum_{s=0}^{k} \frac{k!}{(k-s)!\,s!}\, \lambda^s (\partial_{z_0})^{k-s} - \sum_{i=0}^{k-1} \left\{ \sum_{j=0}^{J} A_{i,j}(z, \partial_{z'})\, (-\partial_\lambda)^j \right\} \left\{ \sum_{s=0}^{i} \frac{i!}{(i-s)!\,s!}\, \lambda^s (\partial_{z_0})^{i-s} \right\}. \tag{3.15}$$

Since we assume that $W(z, \lambda)$ has the form (3.2), we can apply Proposition 3.4. Thus we have

$$L(\lambda, \partial_\lambda, z, \partial_z)\, W(z, \lambda) = \lambda^{(1/p-1)} \int_0^{\tau\lambda^\delta} \exp(-\lambda^{1/p}\zeta)\, \mathscr{L}(z, \partial_z, \partial_\zeta)\, w(z, \zeta)\, d\zeta + \exp(-\tau\lambda^{q/p})\, V_1(z, \lambda), \tag{3.16}$$

where

$$\mathscr{L}(z, \partial_z, \partial_\zeta) = \sum_{s=0}^{k} \frac{k!}{(k-s)!\,s!}\, (\partial_{z_0})^{k-s} (\partial_\zeta)^{ps} - \sum_{i=0}^{k-1} \left\{ \sum_{j=0}^{J} A_{i,j}(z, \partial_{z'})\, (\zeta(\partial_\zeta)^{1-p}/p)^j \right\} \left\{ \sum_{s=0}^{i} \frac{i!}{(i-s)!\,s!}\, (\partial_\zeta)^{ps} (\partial_{z_0})^{i-s} \right\} \tag{3.17}$$

and $V_1(z, \lambda) \in \mathrm{Err}\,(w, \tau\lambda^\delta)$.

On the other hand

$$f(z) = \frac{1}{2\pi i}\int_{C(\theta)} \exp(\lambda z_0)\, \lambda^{1/p-1}\, d\lambda \int_0^{\tau\lambda^\delta} \exp(-\lambda^{1/p}\zeta)\, f(z)\, d\zeta + \frac{1}{2\pi i}\int_{C(\theta)} \exp(\lambda z_0 - \tau\lambda^{q/p})\, f(z)/\lambda\, d\lambda. \tag{3.18}$$

Consequently we obtain an equation for $w(z, \zeta)$,

$$\mathscr{L}(z, \partial_z, \partial_\zeta)\, w(z, \zeta) = f(z). \tag{3.19}$$

Hence, if $w(z, \zeta)$ satisfies (3.19), we shall have

$$L(z, \partial_z)\, u_0(z) = f(z) + g_1(z), \tag{3.20}$$

where

$$g_1(z) = \int_{C(\theta)} \exp(\lambda z_0)\, G_1(z, \lambda)\, d\lambda \tag{3.21}$$

and $G_1(z, \lambda)\exp(\tau\lambda^{q/p}) \in \mathrm{Err}\,(w, \tau\lambda^\delta)$.

Let us construct a solution $w(z, \zeta)$ of (3.19) in the form

$$w(z, \zeta) = \sum_{n=k}^{\infty} w_n(z)\, \zeta^{np}/\Gamma(np+1). \qquad (3.22)$$

Substituting (3.22) into (3.19), we have

$$\mathscr{L}(z, \partial_z, \partial_\zeta)\, w(z, \zeta) = \sum_{n=0}^{\infty} \Big(\sum_{s=0}^{k} \frac{k!}{(k-s)!\,s!} (\partial_{z_0})^{k-s} w_{n+s}(z)$$

$$- \sum_{i=0}^{k-1} \sum_{\substack{0 \leq j \leq J \\ 0 \leq s \leq i}} \frac{i!}{(i-s)!\,s!} \frac{n!}{(n-j)!} A_{i,j}(z, \partial_{z'}) (\partial_{z_0})^{i-s} w_{n+s-j}(z) \Big)$$

$$\zeta^{np}/\Gamma(np+1) = \delta_{n,0} f(z), \qquad (3.23)$$

where $\delta_{i,j}$ is Kronecker's delta. Since $w_n(z) = 0$ for $n < k$, we have

$$w_{n+k}(z) = - \sum_{s=0}^{k-1} \frac{k!}{(k-s)!\,s!} (\partial_{z_0})^{k-s} w_{n+s}(z)$$

$$+ \sum_{i=0}^{k-1} \sum_{\substack{0 \leq s \leq i \\ 0 \leq j \leq J}} \frac{n!}{(n-j)!} \frac{i!}{(i-s)!\,s!} A_{i,j}(z, \partial_{z'}) (\partial_{z_0})^{i-s} w_{n+s-j}(z)$$

$$+ \delta_{n,0} f(z). \qquad (3.24)$$

Thus we can determine $w_n(z)$ successively from (3.24).

PROPOSITION 3.5. – *There are constants* A *and* C *and a neighbourhood* Ω_0 *of* $z = 0$ *such that for* $z \in \Omega_0$

$$|w_n(z)| \leq AC^n\, \Gamma(n\beta + 1), \quad \beta = \gamma_1 = p/q = \sigma_{\mathfrak{L}}/(\sigma_{\mathfrak{L}} - 1). \qquad (3.25)$$

This proposition will be proved in § 6 with other estimates. In view of Proposition 3.5, we can show convergence of $w(z, \zeta)$.

PROPOSITION 3.6. –

(i) *If* $q > 1$, *then* $w(z, \zeta)$ *is an entire function of* ζ *and there are constants* A *and* b *such that*

$$|w(z, \zeta)| \leq A \exp(b|\zeta|^{q/(q-1)}) \quad \textit{for} \quad z \in \Omega_0. \qquad (3.26)$$

(ii) *If* $q = 1$, *then* $w(z, \zeta)$ *is a holomorphic function of* ζ *in* $\{\zeta\,;\, |\zeta| \leq R_0\}$ *for some* R_0.

Proof. – Recall that $\gamma_1 = \beta = p/q > 1$. It follows form Proposition 3.5 that for $z \in \Omega_0$ there is a constant B such that

$$|w_n(z)\, \zeta^{np}/\Gamma(np+1)| \leq AC^n \Gamma((np/q) + 1)\, |\zeta|^{np}/\Gamma(np+1)$$

$$\leq A(B|\zeta|)^{np}/\Gamma(np(1 - q^{-1}) + 1).$$

Hence if $q > 1$, we have for a constant b

$$|w(z,\zeta)| \leqslant A \sum_{n=0}^{\infty} (B|\zeta|)^{np}/\Gamma(np(1-q^{-1})+1)$$
$$\leqslant A \exp(b|\zeta|^{q/(q-1)}). \tag{3.27}$$

If $q = 1$, by putting $R_0 = (2B)^{-1}$, we can show that $w(z,\zeta)$ converges on $\{\zeta ; |\zeta| \leqslant R_0\}$ and is holomorphic and bounded.

Concerning $V_1(z,\lambda) = \exp(\tau|\lambda|^{1/\beta})\, G_1(z,\lambda)$ (see (3.21)) we have

PROPOSITION 3.7. – *There are constants* A, C *and* h *such that for* $z \in \Omega_0$ *and* $|\lambda| \geqslant 1$, *if* $q > 1$,

$$|V_1(z,\lambda)| \leqslant A(1+|\lambda|)^h \exp(b\tau^{q/(q-1)}|\lambda|^{1/\beta}) \tag{3.28}$$

and if $q = 1$

$$|V_1(z,\lambda)| \leqslant A(1+|\lambda|)^h, \tag{3.29}$$

where b *is the same constant in Proposition 3.6.*

Proof. – It follows from Proposition 3.6 that there are constants $N(s)$ and $C_{\alpha,s}$ such that

$$|(\partial_z)^\alpha(\partial_\zeta)^s w(z,\zeta)| \leqslant \begin{cases} C_{\alpha,s}(1+|\zeta|)^{N(s)} \exp(b|\zeta|^{q/(q-1)}) & (q>1), \\ C_{\alpha,s} & (q=1, |\zeta| \leqslant R_0), \end{cases} \tag{3.30}$$

in a neighbourhood Ω_0 of $z = 0$. Noting that

$$V_1(z,\lambda) \in \mathrm{Err}\,(w, \tau\lambda^\delta)$$

and if $q > 1$, $\exp(b|\zeta|^{q/(q-1)})|_{\zeta=\tau\lambda^\delta} = \exp(b\tau^{q/(q-1)}|\lambda|^{1/\beta})$, we have (3.28) and (3.29).

From these propositions, $u_0(z)$ is well-defined. By varying θ in the path $C(\theta)$, we can show that $u_0(z) \in \widetilde{\mathcal{O}}(\Omega_0 - \{z_0 = 0\})$. Thus we have

PROPOSITION 3.8. – $u_0(z) \in \widetilde{\mathcal{O}}(\Omega_0 - \{z_0 = 0\})$ *in a neighbourhood* Ω_0 *of* $z = 0$ *and satisfies*

$$L(z,\partial_z)\,u_0(z) = f(z) + g_1(z), \tag{3.31}$$

where

$$g_1(z) = \int_{C(\theta)} \exp(\lambda z_0 - \tau\lambda^{1/\beta})\, V_1(z,\lambda)\, d\lambda, \tag{3.32}$$

and $V_1(z,\lambda)$ *satisfies (3.28) or (3.29).*

In § 7 we shall show that $u_0(z) \sim \hat{u}(z)$ and $g_1(z) \sim 0$, as $z_0 \longrightarrow 0$ in $S = S(a, b)$ after determination of τ.

4. Equations with a parameter λ.

In order to get Theorem 1.9 we have to cancel $g_1(z)$ in Theorem 1.7. In other words we have to find out a function $\overline{u}(z)$ so as to satisfy $L(z, \partial_z)\,\overline{u}(z) = g_1(z)$ and $\overline{u}(z) \sim 0$ as $z_0 \longrightarrow 0$ in some sector. As mentioned in § 1, to do so we investigate equations with a parameter λ. In § 4 we construct $V(z, \lambda)$ in Theorem 1.13.

Now let $M(\lambda; z, \partial_z) = \sum_{r=0}^{m} \lambda^r M_r(z, \partial_z)$ be an operator with a parameter λ. Let us recall $\Delta(\lambda)$, ν_i $(0 \leq i \leq m_0)$ and $t_r = \operatorname{ord} M_r(z, \partial_z)$ (see § 1). Assume that

$$\text{P.S.}(M_r)\ (0, \xi) \not\equiv 0 \quad \text{for some} \quad (r, t_r) \in \Delta(\lambda). \tag{4.1}$$

So there exists a segment $\Sigma_{l+1}(\lambda)$ with $(r, t_r) = (r_l, t_{r_l})$. Now consider an equation

$$M(\lambda; z, \partial_z)\ \hat{V}(z, \lambda) = G(z, \lambda) \tag{4.2}$$

under the assumption (4.1). Let us construct $\hat{V}(z, \lambda)$ in the form $\hat{V}(z, \lambda) = \lambda^{-r} \sum_{n=0}^{\infty} v_n(z, \lambda)$ so as to formally satisfy (4.2). We may assume that $\text{P.S.}\, M_r(0, \hat{\xi}) \neq 0$, $\hat{\xi} = (0, 0, \ldots, 0, 1)$ and z'' denotes $(z_0, z_1, \ldots, z_{n-1})$. Let us define $v_n(z, \lambda)$ as follows:

$$\begin{cases} M_r(z, \partial_z)\, v_0 + \sum_{j>r} \lambda^{j-r} M_j(z, \partial_z)\, v_0 = G(z, \lambda), \\ (\partial_{z_n})^h\, v_0(z'', 0) = k_h(z''), \quad 0 \leq h \leq t_r - 1, \end{cases} \tag{4.3$_0$}$$

$$\begin{cases} M_r(z, \partial_z) v_n + \sum_{j>r} \lambda^{j-r} M_j(z, \partial_z) v_n + \sum_{j<r} \lambda^{j-r} M_j(z, \partial_z) v_{n-r+j} = 0 \\ (\partial_{z_n})^h\, v_n(z'', 0) = 0, \quad 0 \leq h \leq t_r - 1, \end{cases} \tag{4.3$_n$}$$

where $k_h(z'')$ $(0 \leq h \leq t_r - 1)$ are holomorphic in a neighbourhood of $z'' = 0$. We seek for $v_n(z, \lambda)$ of the form

$$v_n(z, \lambda) = \sum_{s=-n}^{\infty} v_{n,s}(z, \lambda)\, \lambda^s. \tag{4.4}$$

So substituting (4.4) into (4.3), we determine $v_{n,s}(z,\lambda)$ in the following way:

$$\begin{cases} M_r(z,\partial_z)v_{n,s} + \sum_{j>r} M_j(z,\partial_z)v_{n,s-j+r} + \sum_{j<r} M_j(z,\partial_z)v_{n-r+j,s-j+r} \\ \qquad\qquad = \delta_{n,0}\,\delta_{s,0}\,G(z,\lambda), \\ (\partial_{z_n})^h v_{n,s}(z'',0) = \delta_{n,0}\,\delta_{s,0}\,k_h(z''), \quad 0 \leqslant h \leqslant t_r - 1. \end{cases} \tag{4.5}$$

Equation (4.5) has a unique solution $v_{n,s}(z,\lambda)$ holomorphic in z in a neighbourhood Ω_1 of $z=0$, which is independent of n and s in view of Cauchy-Kovalevskaja theorem.

Let us estimate $v_{n,s}(z,\lambda)$. Put $M(\lambda) = \sup_{z\in\Omega} |G(z,\lambda)|$. We have, by using λ-characteristic indices ν_i and ν_{i+1},

Proposition 4.1. – *There are constants* A, B *and* C *and a neighbourhood* Ω_0 *of* $z=0$ *such that for* $z\in\Omega_0$, *if* $i\geqslant 1$,

$$|v_{n,s}(z,\lambda)| \leqslant AM(\lambda)\,B^{n+s}\,C^n\,|z|^{N(n,s)}\,\Gamma(n\nu_{i+1}+1)/\Gamma((n+s)\nu_i+1), \tag{4.6}$$

where $N(n,s) = \max\{[(\nu_i - \nu_{i+1})n + \nu_i s], 0\}$ *and if* $i = 0$,

$$\begin{cases} v_{n,s}(z,\lambda) = 0, \quad s \neq -n, \\ |v_{n,-n}(z,\lambda)| \leqslant AM(\lambda)\,C^n\,\Gamma(n\nu_1+1). \end{cases} \tag{4.7}$$

The proof of Proposition 4.1 will be given in § 6. It follows immediately from Proposition 4.1 that $v_n(z,\lambda) = \sum_{s=-n}^{\infty} v_{n,s}(z,\lambda)\,\lambda^s$ converges. Put

$$\hat{v}_s(z,\lambda) = \sum_{n=\max(0,-s)}^{\infty} v_{n,s}(z,\lambda). \tag{4.8}$$

We have

Proposition 4.2. – $\hat{v}_s(z,\lambda)$ *converges absolutely and uniformly in* $z\in\Omega_0$ *and for* $i\geqslant 1$ *estimates*

$$|\hat{v}_s(z,\lambda)| \leqslant A_1 M(\lambda)\,B_1^s\,|z|^{[s\nu_i]}/\Gamma(s\nu_i+1), \quad s\geqslant 0 \tag{4.9}$$

and

$$|\hat{v}_s(z,\lambda)| \leqslant A_1 M(\lambda)\,B_1^{|s|}\,\Gamma(|s|\,\nu_{i+1}+1), \quad s\leqslant 0 \tag{4.10}$$

hold for some constants A_1 *and* B_1.

Proof. – Let $s \geq 0$. We have for $z \in \Omega_0$,

$$\sum_{n=0}^{\infty} |v_{n,s}(z,\lambda)| \leq \mathrm{AM}(\lambda)\, \mathrm{B}^s |z|^{[s\nu_i]} \Big\{ \sum_{n=0}^{\infty} (\mathrm{BC})^n |z|^{[n(\nu_i - \nu_{i+1})]}$$

$$\times \Gamma(n\nu_{i+1} + 1)/\Gamma((n+s)\nu_i + 1) \Big\} \leq \mathrm{AM}(\lambda)\, \mathrm{B}_1^s |z|^{[s\nu_i]} \Gamma(s\nu_i + 1)^{-1}$$

$$\times \Big\{ \sum_{n=0}^{\infty} \mathrm{C}_1^n |z|^{[n(\nu_i - \nu_{i+1})]} \Gamma(n(\nu_i - \nu_{i+1}) + 1)^{-1} \Big\}$$

$$\leq \mathrm{A}_1 \mathrm{M}(\lambda)\, \mathrm{B}_1^s |z|^{[s\nu_i]} \Gamma(s\nu_i + 1)^{-1}.$$

Let $s = -h < 0$. We have

$$\sum_{n=-s}^{\infty} |v_{n,s}(z,\lambda)| \leq \mathrm{AM}(\lambda)\, \mathrm{B}_2^h \sum_{n=h}^{\infty} \mathrm{C}_2^n \Gamma(n\nu_{i+1} + 1)/\Gamma((n-h)\nu_i + 1)$$

$$\leq \mathrm{A}_1 \mathrm{M}(\lambda)\, \mathrm{B}_1^h \Gamma(|s|\nu_{i+1} + 1).$$

Thus a formal sum

$$\hat{\mathrm{V}}(z,\lambda) = \lambda^{-r}\Big(\sum_{s=-\infty}^{\infty} \hat{v}_s(z,\lambda)\lambda^s\Big) \tag{4.11}$$

formally satisfies equation (4.2). By making use of $\hat{\mathrm{V}}(z,\lambda)$, we can prove Theorem 1.13 in § 1. First let us introduce auxiliary functions $f_j(\zeta)$ $(-\infty < j < \infty)$ used in Hamada [1], Wagschal [10] and others:

$$\begin{cases} f_j(\zeta) = (-1)^{j+1}\Gamma(|j|)\,\zeta^j/2\pi i, & j \leq -1, \\ f_0(\zeta) = \log \zeta/2\pi i \\ f_j(\zeta) = \zeta^j(\log \zeta - (1 + 1/2 + 1/3 + \dots + 1/j))/(2\pi i\,\Gamma(j+1)), & j \geq 1. \end{cases} \tag{4.12}$$

Let us remark an important relation $df_{j+1}(\zeta)/d\zeta = f_j(\zeta)$. Let us put $\nu_{i+1} = p_{i+1}/q_{i+1}$, $p_{i+1}, q_{i+1} \in \mathrm{N}$, $(p_{i+1}, q_{i+1}) = 1$ and put $\delta_{i+1} = (q_{i+1} - 1)/p_{i+1}$. In the following of this section we denote p_{i+1}, q_{i+1} and δ_{i+1} simply by p, q and δ respectively.

Define

$$h(z,\lambda,\zeta) = \sum_{s=1}^{\infty} \hat{v}_{-s}(z,\lambda)\,\zeta^{(s-1)p}/((s-1)p)!\,, \tag{4.13}$$

$$v^-(z,\lambda,\zeta) = \sum_{s=1}^{\infty} \hat{v}_{-s}(z,\lambda)\, f_{(s-1)p}(\zeta) \tag{4.14}$$

and

$$\mathrm{V}^+(z,\lambda) = \sum_{s=0}^{\infty} \hat{v}_s(z,\lambda)\lambda^s. \tag{4.15}$$

We have

LEMMA 4.3. – *There exist constants* A, κ *and* $\hat{\zeta}$ *such that if* $q > 1$,

$$|h(z, \lambda, \zeta)| \leqslant \mathrm{AM}(\lambda) \exp(\kappa |\zeta|^{q/(q-1)}) \tag{4.16}$$

and

$$|v^-(z, \lambda, \zeta)| \leqslant \mathrm{AM}(\lambda)(1 + |\log \zeta|) \exp(\kappa |\zeta|^{q/(q-1)}), \tag{4.17}$$

if $q = 1$, *for* $\zeta \in \{\zeta \in \mathrm{C}^1 ; |\zeta| \leqslant \hat{\zeta}\}$

$$|h(z, \lambda, \zeta)| \leqslant \mathrm{AM}(\lambda) \tag{4.18}$$

and

$$|v^-(z, \lambda, \zeta)| \leqslant \mathrm{AM}(\lambda)(1 + |\log \zeta|). \tag{4.19}$$

Proof. – In view of (4.10), we have

$$\sum_{s=1}^{\infty} |\hat{v}_{-s}(z, \lambda)| \, |\zeta|^{(s-1)p}/\Gamma((s-1)p+1) \leqslant \mathrm{A}_1 \mathrm{M}(\lambda) \left\{ \sum_{s=1}^{\infty} \mathrm{B}_1^s |\zeta|^{(s-1)p} \times \Gamma(s(p/q)+1)/\Gamma((s-1)p+1) \right\}$$
$$\leqslant \mathrm{A}_2 \mathrm{M}(\lambda) \left\{ \sum_{s=1}^{\infty} \mathrm{B}_2^s |\zeta|^{(s-1)p} \Gamma((sp(q-1)/q+1))^{-1} \right\}.$$

Hence, if $q > 1$, we have (4.16) and if $q = 1$, by putting $\hat{\zeta} = (2\mathrm{B}_2)^{-1/p}$ we have (4.18). By the similar way we have (4.17) and (4.19).

LEMMA 4.4. – $\mathrm{V}^+(z, \lambda)$ *converges and there are constants* A *and* c *such that for* $z \in \Omega_0$

$$|\mathrm{V}^+(z, \lambda)| \leqslant \mathrm{AM}(\lambda) \exp(c|z| \, |\lambda|^{1/\nu_l}). \tag{4.20}$$

Proof. – In view of (4.9), we have

$$\sum_{s=0}^{\infty} |\hat{v}_s(z, \lambda)| \, |\lambda|^s \leqslant \mathrm{A}_1 \mathrm{M}(\lambda) \sum_{s=0}^{\infty} \mathrm{B}_1^s |z|^{[s\nu_l]} |\lambda|^s/\Gamma(s\nu_l+1)$$
$$\leqslant \mathrm{AM}(\lambda) \exp(c|z| \, |\lambda|^{1/\nu_l}).$$

Now let us define another path $\overline{\mathrm{C}}(\eta)$. $\overline{\mathrm{C}}(\eta)$ is a path which starts at $\zeta = \eta$ and goes around $\zeta = 0$ once on $|\zeta| = |\eta|$.

LEMMA 4.5. –

(i) *The following equality holds:*

$$\int_{\overline{\mathrm{C}}(\eta)} \exp(-\lambda^{1/p}\zeta) \, v^-(z, \lambda, \zeta) \, d\zeta = \int_0^{\eta} \exp(-\lambda^{1/p}\zeta) \, h(z, \lambda, \zeta) \, d\zeta. \tag{4.21}$$

(ii) *Let* $|\arg\lambda| \leqslant \hat{\theta}$, $\hat{\theta} < \pi\nu_{l+1}/2$. *Then there are positive constants* $\hat{\tau}$ *and* K *dependent on* $\hat{\theta}$ *such that if* $\hat{\tau} \geqslant \tau > 0$,

$$\sup_{z\in\Omega_0}\left|\int_0^{\tau\lambda^\delta}\exp(-\lambda^{1/p}\zeta)\,h(z,\lambda,\zeta)\,d\zeta\right| \leqslant \mathrm{K}M(\lambda)\,|\lambda|^{-1/p}. \quad (4.22)$$

Proof. –

(i) By the relation $\int_{\overline{C}(\eta)} f_j(\zeta)\,d\zeta = \int_0^\eta (\zeta^j/j!)\,d\zeta$ for $j \geqslant 0$, we have (4.21).

(ii) If $q = 1$, then $\delta = 0$ and (ii) is clear. Let $q > 1$ and put $\zeta = t\tau\lambda^\delta$, $0 \leqslant t \leqslant 1$. Since $|\arg\lambda| \leqslant \hat{\theta} < \pi\nu_{l+1}/2$, there is a $c > 0$ such that $\mathrm{Re}\,\lambda^{1/p}\zeta \leqslant t\tau c\,|\lambda|^{q/p}$. Therefore from lemma 4.3, we have

$$\left|\int_0^{\tau\lambda^\delta}\exp(-\lambda^{1/p}\zeta)\,h(z,\lambda,\zeta)\,d\zeta\right| \quad (4.23)$$
$$\leqslant AM(\lambda)\,\tau\,|\lambda|^\delta\int_0^1\exp((-t\tau c + (t\tau)^{q/(q-1)}\kappa)\,|\lambda|^{q/p})\,dt.$$

There is a $\hat{\tau} > 0$ such that $(t\tau c)/2 \geqslant (t\tau)^{q/(q-1)}\kappa$ for $0 < \tau \leqslant \hat{\tau}$. Thus we have, if $\hat{\tau} \geqslant \tau > 0$,

$$\left|\int_0^{\tau\lambda^\delta}\exp(-\lambda^{1/p}\zeta)\,h(z,\lambda,\zeta)\,d\zeta\right|$$
$$\leqslant AM(\lambda)\,\tau\,|\lambda|^\delta\int_0^1\exp(-t\tau c\,|\lambda|^{q/p}/2)\,dt \leqslant \mathrm{K}M(\lambda)\,|\lambda|^{-1/p}.$$

Now let us prove Theorem 1.13. Put

$$V(\tau;z,\lambda) = \lambda^{-r}\sum_{s=0}^{\infty}\hat{v}_s(z,\lambda)\,\lambda^s$$
$$+\,\lambda^{1/p-r-1}\int_{\overline{C}(\tau\lambda^\delta)}\exp(-\lambda^{1/p}\zeta)\,v^-(z,\lambda,\zeta)\,d\zeta, \quad (4.24)$$

where $\tau > 0$ and if $q = 1$, $\tau < \hat{\zeta}$. τ will be determined later. We have

$$V(\tau;z,\lambda) = \lambda^{1/p-r-1}\sum_{s=0}^{\infty}\int_{\overline{C}(\tau\lambda^\delta)}\exp(-\lambda^{1/p}\zeta)\,\hat{v}_s(z,\lambda)\,f_{-(s+1)p}(\zeta)\,d\zeta$$
$$+\,\lambda^{1/p-r-1}\sum_{s=-\infty}^{-1}\int_{\overline{C}(\tau\lambda^\delta)}\exp(-\lambda^{1/p}\zeta)\,\hat{v}_s(z,\lambda)\,f_{-(s+1)p}(\zeta)\,d\zeta. \quad (4.25)$$

By operating $M(\lambda;z,\partial_z)$ to $V(\tau;z,\lambda)$, we have, by integrations by parts and Lemma 4.5 (i),

$$M(\lambda;z,\partial_z)\,V(\tau;z,\lambda) = I_1 + I_2, \quad (4.26)$$

where

$$I_1 = \lambda^{1/p-r-1} \sum_{s=0}^{\infty} \int_{\overline{C}(\tau\lambda^\delta)} \exp(-\lambda^{1/p}\zeta) \left\{ \sum_{j=0}^{m} M_j(z, \partial_z) \hat{v}_s(z, \lambda) \times f_{-(s+j+1)p}(\zeta) \right\} d\zeta \quad (4.27)$$

and

$$I_2 = \lambda^{1/p-r-1} \sum_{s=-\infty}^{-1} \int_{\overline{C}(\tau\lambda^\delta)} \exp(-\lambda^{1/p}\zeta) \left\{ \sum_{j=0}^{m} M_j(z, \partial_z) \hat{v}_s(z, \lambda) \times f_{-(s+j+1)p}(\zeta) \right\} d\zeta + \exp(-\tau\lambda^{q/p}) H(\tau; z, \lambda). \quad (4.28)$$

From (4.13) and (4.14), $H(\tau; z, \lambda) \in \mathrm{Err}(h, \tau\lambda^\delta)$. Hence we obtain

$$I_1 + I_2 = \lambda^{1/p-r-1} \sum_{s=-\infty}^{\infty} \int_{\overline{C}(\tau\lambda^\delta)} \exp(-\lambda^{1/p}\zeta) \left\{ \sum_{j=0}^{m} M_j(z, \partial_z) \hat{v}_{s-r+j}(z, \lambda) \times f_{-(s+r+1)p}(\zeta) \right\} d\zeta + \exp(-\tau\lambda^{q/p}) H(\tau; z, \lambda). \quad (4.29)$$

It follows from (4.5) that

$$\sum_{j=0}^{m} M_j(z, \partial_z) \hat{v}_{s-j+r}(z, \lambda) = \delta_{s,0} G(z, \lambda). \quad (4.30)$$

So we have

$$\begin{aligned} M(\lambda; z, \partial_z) V(\tau; z, \lambda) & \quad (4.31) \\ &= \lambda^{1/p-r-1} \int_{\overline{C}(\tau\lambda^\delta)} \exp(-\lambda^{1/p}\zeta) G(z, \lambda) f_{-(r+1)p}(\zeta) d\zeta \\ &+ \exp(-\tau\lambda^{q/p}) H(\tau; z, \lambda) = G(z, \lambda) + \exp(-\tau\lambda^{q/p}) H(\tau; z, \lambda). \end{aligned}$$

This implies (1.34) in Theorem 1.13. It follows from Lemma 4.3 ~ 4.5 that there are positive constants a, b, c, A and $\hat{\tau}$ such that for τ with $0 < \tau \leqslant \hat{\tau}$ and $(z, \lambda) \in \Omega_0 \times \{|\lambda| > \Lambda\}$

$$|V(\tau; z, \lambda)| \leqslant AM(\lambda) \exp(a|\lambda|^{1/\nu_{i+1}}) \quad (4.32)$$

and if $|\arg\lambda| \leqslant \hat{\theta}$,

$$|V(\tau; z, \lambda)| \leqslant AM(\lambda) \exp(c|z|\,|\lambda|^{1/\nu_i}). \quad (4.33)$$

Since $H(\tau; z, \lambda) \in \mathrm{Err}(h, \tau\lambda^\delta)$, it follows from Lemma 4.3 that

$$|H(\tau; z, \lambda)| \leqslant \begin{cases} AM(\lambda) \exp(b\tau^{q/(q-1)}|\lambda|^{1/\nu_{i+1}}), & q = q_{i+1} > 1 \\ AM(\lambda)(1 + |\lambda|)^N, & q = q_{i+1} = 1. \end{cases} \quad (4.34)$$

Thus we have Theorem 1.13.

Now let us construct solutions with singularity on the characteristic surface $\{z_0 = 0\}$ for $L(z, \partial_z)$. Let us return to Proposition 4.2. Assume that $\mathrm{P.S.}(M_{m_0})(0, \hat{\xi}) \neq 0$. Put $i = m_0$ in (4.9) and (4.10). Since $\nu_{m_0+1} = 1$, we have

$$|\hat{v}_s(z, \lambda)| \leqslant A_1 B_1^s M(\lambda) |z|^{[s\nu_{m_0}]}/\Gamma(s\nu_{m_0} + 1), \quad s \geqslant 0, \tag{4.35}$$

and

$$|\hat{v}_s(z, \lambda)| \leqslant A_1 B_1^{|s|} M(\lambda) (|s|!), \quad s < 0. \tag{4.36}$$

Let us define, by using $\hat{v}_s(z, \lambda)$ for $i = m_0$,

$$v(z) = \int_{C(\theta)} \exp(\lambda z_0) V^+(z, \lambda) \lambda^{-r} \hat{\varphi}(\lambda) \, d\lambda + \sum_{s=1}^{\infty} \int_{C(\theta)} \exp(\lambda z_0) v_{-s}(z, \lambda) \lambda^{-r-s} \hat{\varphi}(\lambda) \, d\lambda. \tag{4.37}$$

We set conditions on $M(\lambda)$ and $\hat{\varphi}(\lambda)$ in order that $v(z)$ converges:

Condition II.

(i) $\hat{\varphi}(\lambda) \in \widetilde{\mathcal{O}}(C^1 - \{|\lambda| > \Lambda\})$.

(ii) For any a, b $(b > a)$ and any $\epsilon > 0$, there is a constant $C(a, b, \epsilon)$ such that for $\lambda \in \{\lambda ; a < \arg \lambda < b, |\lambda| > \Lambda\}$

$$\left\{ \begin{array}{l} |M(\lambda)| \\ |\hat{\varphi}(\lambda)| \end{array} \right. \leqslant C(a, b, \epsilon) \exp(\epsilon |\lambda|). \tag{4.38}$$

Then we have

THEOREM 4.6. – *Suppose that* $\mathrm{P.S.}(M_{m_0})(0, \hat{\xi}) \neq 0$ *and Condition II holds. Then* $v(z)$ *defined by (4.37) is a function in* $\widetilde{\mathcal{O}}(\Omega_1 - \{z_0 = 0\})$ *for a neighbourhood* Ω_1 *of* $z = 0$.

Proof. – From (4.35), (4.36) and (4.38) we have for

$$\lambda \in \{\lambda ; a < \arg \lambda < b, \ |\lambda| > \Lambda\}$$

$$|V^+(z, \lambda) \hat{\varphi}(\lambda)| \leqslant C(a, b, \epsilon) \exp(\epsilon |\lambda|) \tag{4.39}$$

and

$$|\hat{v}_{-s}(z, \lambda) \hat{\varphi}(\lambda)| \leqslant C(a, b, \epsilon) \exp(\epsilon |\lambda|) B_1^s s!, \quad s > 0. \tag{4.40}$$

It follows from (4.39) that the first term of the right hand side of (4.37) converges. By the method similar to that used in Ōuchi [6, 7] (see also § 7 in this paper) we can show from (4.40) that $v(z)$ converges in a small neighbourhood Ω_1 of $z = 0$ except on $\{z_0 = 0\}$ and belongs to $\widetilde{\mathcal{O}}(\Omega_1 - \{z_0 = 0\})$.

We apply Theorem 4.6 to the operator $L(\lambda ; z , \partial_z)$ induced from $L(z , \partial_z)$ (see (1.38)), that is, we put $M(\lambda ; z , \partial_z) = L(\lambda ; z , \partial_z)$. Hence $m_0 = \ell$, $\nu_{\ell+1} = 1$ and $\nu_\ell = \sigma_1/(\sigma_1 - 1)$ (see § 1). $\text{P.S.}(M_{m_0})\ (0 , \hat{\xi}) \neq 0$ implies that

$$\text{P.S.}\, L_{m-s_m}(0 , \hat{\xi}) = a_{m-s_m}(0 , \hat{\xi}') \neq 0 \,, \ \hat{\xi}' = (0 , 0 , \ldots , 1) . \quad (4.41)$$

THEOREM 4.7. – *Suppose that Condition II and (4.41) hold. Then $v(z)$ defined by (4.37) for $L(\lambda ; z , \partial_z)$ satisfies*

$$L(z , \partial_z)\, v(z) = g_\varphi(z) \quad (4.42)$$

and

$$(\partial_{z_n})^h\, v(z'', 0) = k_h(z'')\ \varphi(z_0) , \quad 0 \leqslant h \leqslant m - s_m - 1 , \quad (4.43)$$

where

$$g_\varphi(z) = \int_{C(\theta)} \exp(\lambda z_0)\ G(z , \lambda)\ \hat{\varphi}(\lambda)\, d\lambda \quad (4.44)$$

and

$$\varphi(z_0) = \int_{C(\theta)} \exp(\lambda z_0)\, \lambda^{-r}\, \hat{\varphi}(\lambda)\, d\lambda . \quad (4.45)$$

Proof. – We have

$$\begin{aligned} L(z , \partial_z)\, v(z) &= \int_{C(\theta)} \exp(\lambda z_\theta)\ L(\lambda ; z , \partial_z)\ V^+(z , \lambda)\, \lambda^{-r}\, \hat{\varphi}(\lambda)\, d\lambda \\ &\quad + \sum_{s=1}^{\infty} \int_{C(\theta)} \exp(\lambda z_0)\ L(\lambda ; z , \partial_z)\ \hat{v}_{-s}(z , \lambda)\, \lambda^{-r-s}\, \hat{\varphi}(\lambda)\, d\lambda \\ &= \int_{C(\theta)} \exp(\lambda z_0)\ G(z , \lambda)\ \hat{\varphi}(\lambda)\, d\lambda = g_\varphi(z) \end{aligned} \quad (4.46)$$

and

$$\begin{aligned} \left(\frac{\partial}{\partial z_n}\right)^h v(z'', 0) &= \int_{C(\theta)} \exp(\lambda z_0) \left(\frac{\partial}{\partial z_n}\right)^h V^+(z'', 0 , \lambda)\, \lambda^{-r}\, \hat{\varphi}(\lambda)\, d\lambda \\ &\quad + \sum_{s=1}^{\infty} \int_{C(\theta)} \exp(\lambda z_0) \left(\frac{\partial}{\partial z_n}\right)^h v_{-s}(z'', 0 , \lambda)\, \lambda^{-r-s}\, \hat{\varphi}(\lambda)\, d\lambda \\ &= k_h(z'') \int_{C(\theta)} \exp(\lambda z_0)\, \lambda^{-r}\, \hat{\varphi}(\lambda)\, d\lambda . \end{aligned} \quad (4.47)$$

Remark 4.8. –

(i) In the construction of $\hat{V}(z , \lambda)$, the initial values $k_h(z'')$ $(0 \leqslant h \leqslant t_r - 1)$ in $(4.3)_0$ may depend on λ. So, by assuming that $k_h(z'', \lambda)$ satisfies Condition II, we can generalize Theorem 4.6 and 4.7.

(ii) We have $g_\varphi(z) \in \widetilde{\mathcal{O}}(\Omega_1 - \{z_0 = 0\})$ and

$$\varphi(z_0) \in \widetilde{\mathcal{O}}(C^1 - \{z_0 = 0\}) .$$

So Theorem 4.7 is an existence theorem of solutions with singularity on characteristic surface $\{z_0 = 0\}$ of the equation $L(z, \partial_z)\, v(z) = g_\varphi(z)$. By choosing $\hat{\varphi}(\lambda)$ or $k_h(z'', \lambda)$ suitably (see § 7 in Ōuchi [7]), we have many solutions. This is a generalization of Hamada, Leray and Wagschal [3] and Persson [9].

5. Construction of solutions II.

In this section we shall construct $\overline{u}(z)$ so as to satisfy

$$L(z, \partial_z)\, u(z) = g_1(z) \tag{5.1}$$

and

$$\overline{u}(z) \sim 0 \quad \text{as} \quad z_0 \longrightarrow 0 \quad \text{in} \quad U_{S(\omega)}, \tag{5.2}$$

where $S(\omega) = \{z_0 \in C^1;\ |\arg z_0| < \omega\}$, $\omega < \pi/2(\sigma_1 - 1)$. If such $\overline{u}(z)$ exists, $u_S(z) = u_0(z) - \overline{u}(z)$ is a desired solution of Theorem 1.9. We shall apply the results in § 4 to the operator $L(\lambda; z, \partial_z)$ induced from $L(z, \partial_z)$. We find out $\overline{u}(z)$ under Condition I. Let θ_i $(1 \leqslant i \leqslant \ell)$ be positive numbers such that $\theta_1 > \theta_2 > \ldots > \theta_\ell > \pi/2$ and $\theta_i < \pi\gamma_i/2$. First let us recall what we shall need. $L(\lambda; z, \partial_z)$ is an operator with a parameter λ defined by

$$\begin{aligned} L(\lambda; z, \partial_z)\, v(z, \lambda) &= \exp(-\lambda z_0)\, L(z, \partial_z)\, (\exp(\lambda z_0)\, v(z, \lambda)) \\ &= \sum_{i=0}^{k} \lambda^i L_i(z, \partial_z)\, v(z, \lambda), \end{aligned} \tag{5.3}$$

$$\Delta(\lambda, L) = \{(i - s_i, s_i);\ (i, s_i) \in \Delta\}, \tag{5.4}$$

$$\begin{cases} \nu_i = \gamma_i = \sigma_{\ell+1-i}/(\sigma_{\ell+1-i} - 1), \quad 1 \leqslant i \leqslant \ell, \\ \gamma_1 > \gamma_2 > \ldots > \gamma_\ell > 1 = \gamma_{\ell+1}, \end{cases} \tag{5.5}$$

$$\text{P.S.}(L_{i-s_i})\,(z, \xi) = a_{i,s_i}(z, \xi') \quad \text{for} \quad (i, s_i) \in \Delta \tag{5.6}$$

and Condition I implies

$$a_{i,s_i}(0, \xi') \not\equiv 0 \quad \text{for} \quad (i, s_i) \in \Delta. \tag{5.7}$$

$g_1(z)$ is represented in the form

$$g_1(z) = \int_{C(\theta)} \exp(\lambda z_0)\, G_1(z, \lambda)\, d\lambda, \tag{5.8}$$

where $G_1(z, \lambda) \in \widetilde{\mathcal{O}}(\Omega_1 \times \{|\lambda| > 1\})$ and for $(z, \lambda) \in \Omega_1 \times \{|\lambda| > 1\}$

$$|G_1(z, \lambda)| \leqslant A \exp(c'|\lambda|^{1/\gamma_1}) \tag{5.9}$$

and if $|\arg\lambda| \leqslant \theta_1$,

$$|G_1(z,\lambda)| \leqslant A \exp(-c|\lambda|^{1/\gamma_1}) \tag{5.10}$$

for positive constants A, c' and c.

Now we construct $\overline{u}(z)$ in the form $\overline{u}(z) = \sum_{i=1}^{\ell} u_i(z)$, where

$$u_i(z) = \int_{C(\theta)} \exp(\lambda z_0)\, V_i(z,\lambda)\, d\lambda, \quad 1 \leqslant i \leqslant \ell - 1, \tag{5.11}$$

and the form $u_\ell(z)$ will be given after construction of $u_i(z)$ $(1 \leqslant i \leqslant \ell - 1)$. By applying Theorem 1.13 which was discussed in detail in § 4, we have

PROPOSITION 5.1. – *Suppose that* $\ell \geqslant 2$. *There are functions* $V_1(z,\lambda), G_2(z,\lambda) \in \widetilde{\mathcal{O}}(\Omega_2 \times \{|\lambda| > 1\})$, $\Omega_1 \supset \Omega_2$, *such that*

$$L(\lambda; z, \partial_z)\, V_1(z,\lambda) = G_1(z,\lambda) - G_2(z,\lambda), \tag{5.12}$$

where for $(z,\lambda) \in \Omega_2 \times \{|\lambda| > 1\}$

$$\left.\begin{array}{l} |V_1(z,\lambda)| \\ |G_2(z,\lambda)| \end{array}\right\} \leqslant A \exp(b'|\lambda|^{1/\gamma_2}) \tag{5.13}$$

and if $|\arg\lambda| \leqslant \theta_2$,

$$|V_1(z,\lambda)| \leqslant A \exp(-b|\lambda|^{1/\gamma_1}) \tag{5.14}$$

and

$$|G_2(z,\lambda)| \leqslant A \exp(-b|\lambda|^{1/\gamma_2}). \tag{5.15}$$

Here A, b' *and* b *are positive constants.*

Proof. – Set $M(\lambda) = \sup_{z\in\Omega_1} |G_1(z,\lambda)|$. By Theorem 1.13 there are functions $V_1(\tau; z,\lambda)$, $H_1(\tau; z,\lambda) \in \widetilde{\mathcal{O}}(\Omega_2 \times \{|\lambda| > 1\})$ such that

$$L(\lambda; z, \partial_z)\, V_1(\tau; z,\lambda) = G_1(z,\lambda) + \exp(-\tau\lambda^{1/\gamma_2})\, H_1(\tau; z,\lambda) \tag{5.16}$$

and the following estimates hold:

For $(z,\lambda) \in \Omega_2 \times \{|\lambda| > 1\}$,

$$\left.\begin{array}{l} |V_1(\tau; z,\lambda)| \\ |H_1(\tau; z,\lambda)| \end{array}\right\} \leqslant AM(\lambda) \exp(a|\lambda|^{1/\gamma_2}) \tag{5.17}$$

and if $|\arg\lambda| \leqslant \theta_2$,

$$|V_1(\tau; z,\lambda)| \leqslant AM(\lambda) \exp(d|z|\,|\lambda|^{1/\gamma_1}) \tag{5.18}$$

and

$$|H_1(\tau; z, \lambda)| \leqslant \begin{cases} AM(\lambda)\ \exp(\kappa\tau^{q_2/(q_2-1)}\, |\lambda|^{1/\gamma_2}),\ q_2 > 1\,, \\ AM(\lambda)\ (1 + |\lambda|)^N\,,\ q_2 = 1\,. \end{cases} \tag{5.19}$$

Put $G_2(\tau; z, \lambda) = -\exp(-\tau\lambda^{1/\gamma_2})\, H_1(\tau; z, \lambda)$. In view of (5.10) and (5.19), if $|\arg\lambda| \leqslant \theta_2$, there exist $\tau = \hat{\tau}_2$ and a constant $c_2 > 0$ such that

$$|G_2(\hat{\tau}_2; z, \lambda)| \leqslant A\exp(-c_2|\lambda|^{1/\gamma_2})\,. \tag{5.20}$$

From (5.10) and (5.18), there is a small neighbourhood Ω_2 of $z = 0$ such that if $|\arg\lambda| \leqslant \theta_2$

$$|V_1(\hat{\tau}_2; z, \lambda)| \leqslant A\exp(-c_2|\lambda|^{1/\gamma_1})\,. \tag{5.21}$$

Hence, by putting

$$V_1(z, \lambda) = V_1(\hat{\tau}_2; z, \lambda) \quad \text{and} \quad G_2(z, \lambda) = G_2(\hat{\tau}_2; z, \lambda)\,,$$

we have (5.12), (5.14) and (5.15). (5.13) follows from (5.9) and (5.17).

By repeting above arguments we get

PROPOSITION 5.2. – *Suppose that* $\ell \geqslant 2$. *There exist functions* $V_i(z, \lambda)$ $(1 \leqslant i \leqslant \ell - 1)$ *and* $G_i(z, \lambda)$ $(1 \leqslant i \leqslant \ell) \in \widetilde{\mathcal{O}}(\Omega_1 \times \{|\lambda| > 1\})$ *such that*

$$L(\lambda; z, \partial_z)\, V_i(z, \lambda) = G_i(z, \lambda) - G_{i+1}(z, \lambda)\,,\ 1 \leqslant i \leqslant \ell - 1\,, \tag{5.22}$$

where for $(z, \lambda) \in \Omega_1 \times \{|\lambda| > 1\}$

$$\left.\begin{array}{l} |V_i(z, \lambda)| \\ |G_{i+1}(z, \lambda)| \end{array}\right\} \leqslant A\exp(b'|\lambda|^{1/\gamma_{i+1}}) \tag{5.23}$$

and if $|\arg\lambda| \leqslant \theta_{i+1}$,

$$|V_i(z, \lambda)| \leqslant A\exp(-b|\lambda|^{1/\gamma_i}) \tag{5.24}$$

and

$$|G_{i+1}(z, \lambda)| \leqslant A\exp(-b|\lambda|^{1/\gamma_{i+1}})\,. \tag{5.25}$$

Here A, b' *and* b *are positive constants.*

Now by using $V_i(z, \lambda)$ in Proposition 5.1 and 5.2, we define

$$u_i(z) = \int_{C(\theta)} \exp(\lambda z_0)\, V_i(z, \lambda)\, d\lambda\,,\ i = 1, 2, \ldots, \ell - 1\,. \tag{5.26}$$

Then $\overline{v}(z) = \sum_{i=1}^{\ell-1} u_i(z)$ satisfies

$$L(z, \partial_z)\,\overline{v}(z) = \sum_{i=1}^{\ell-1} \int_{C(\theta)} \exp(\lambda z_0)\, L(\lambda; z, \partial_z)\, V_i(z, \lambda)\, d\lambda\,.$$

Hence we obtain

$$L(z, \partial_z)\overline{v}(z) = -g_\ell(z) + g_1(z), \; g_i(z) = \int_{C(\theta)} \exp(\lambda z_0)\, G_i(z, \lambda)\, d\lambda . \tag{5.27}$$

Finally we have to find out $u_\ell(z)$ so as to satisfy

$$L(z, \partial_z)\, u_\ell(z) = g_\ell(z) \text{ and } u_\ell(z) \sim 0 \text{ as } z_0 \longrightarrow 0 \text{ in } S(\omega).$$

From Theorem 4.7, we have

PROPOSITION 5.3. – *There is a function* $u_\ell(z) \in \widetilde{\mathcal{O}}(\Omega_0 - \{z_0 = 0\})$ *for a neighbourhood* Ω_0 *of* $z = 0$ *such that*

$$L(z, \partial_z)\, u_\ell(z) = g_\ell(z) \tag{5.28}$$

and $u_\ell(z)$ *is expressed in the form*

$$u_\ell(z) = \int_{C(\theta)} \exp(\lambda z_0)\, V_\ell^+(z, \lambda)\, \lambda^{-r}\, d\lambda + \sum_{s=1}^{\infty} \int_{C(\theta)} \exp(\lambda z_0)\, v_{-s}(z, \lambda)\, \lambda^{-r-s}\, d\lambda, \tag{5.29}$$

where $V_\ell^+(z, \lambda)$, $v_{-s}(z, \lambda)$ $(s = 1, 2, \ldots) \in \widetilde{\mathcal{O}}(\Omega_0 \times \{|\lambda| > 1\})$ *and for* $(z, \lambda) \in \Omega_0 \times \{|\lambda| > 1\}$ *there are positive constants* A, B_1, b' *and* b *such that*

$$|V_\ell^+(z, \lambda)| \leqslant A \exp(b' |\lambda|^{1/\gamma_\ell}) \tag{5.30}$$

and

$$|v_{-s}(z, \lambda)| \leqslant AB_1^s(s!) \exp(b' |\lambda|^{1/\gamma_\ell}) \tag{5.31}$$

and if $|\arg \lambda| \leqslant \theta_\ell$,

$$|V_\ell^+(z, \lambda)| \leqslant A \exp(-b\, |\lambda|^{1/\gamma_\ell}) \tag{5.32}$$

and

$$|v_{-s}(z, \lambda)| \leqslant AB_1^s(s!) \exp(-b\, |\lambda|^{1/\gamma_\ell}). \tag{5.33}$$

Proof. – $G_\ell(z, \lambda)$ satisfies the condition of Theorem 4.7. So, putting $k_h(z'') = 0$ $(0 \leqslant h \leqslant m - s_m - 1)$ in (4.43), we can get $u_\ell(z)$ in the form of (5.29). In view of (4.35) and (4.36), we have (5.30) ~ (5.33).

Thus $\overline{u}(z) = \sum_{i=1}^{\ell} u_i(z)$ satisfies $L(z, \partial_z)\, \overline{u}(z) = g_1(z)$. The asymptotic behaviour of $\overline{u}(z)$ will be investigated together with $u_0(z)$ in § 7. Estimates (5.24), (5.32) and (5.33) are useful to study asymtotic behaviour of $u_i(z)$ $(1 \leqslant i \leqslant \ell)$ as $z_0 \longrightarrow 0$.

6. Estimates.

In § 6 we shall prove Proposition 3.5 and 4.1. We employ the method used in Hamada [2], Hamada, Leray and Wagschal [3] and Wagschal [10]. Several propositions will be given without proofs. We refer the details of this method and proofs of the propositions to these papers or Komatsu [4].

Let $a(z)$ and $b(z)$ be formal power series. $a(z) \ll b(z)$ means that each Taylor coefficient of $b(z)$ bounds the absolute value of the corresponding coefficient of $a(z)$. In the following of this section we assume that $0 < r < R' < R$.

PROPOSITION 6.1 (Wagschal). – *Let* $\Theta(t)$ *be a formal power series in one variable* t *such that* $\Theta(t) \gg 0$ *and* $(R' - t)\,\Theta(t) \gg 0$. *Then for the derivatives* $\Theta^{(j)}(t)$ $(j = 0, 1, \ldots)$ *we have*

$$\Theta^{(j)}(t) \ll R'\Theta^{(j+1)}(t) \tag{6.1}$$

and

$$(R - t)^{-1}\,\Theta^{(j)}(t) \ll (R - R')^{-1}\,\Theta^{(j)}(t). \tag{6.2}$$

In the sequel let us put

$$t = \rho z_0 + z_1 + \ldots + z_n \tag{6.3}$$

with a constant $\rho \geqslant 1$ to be determined later and assume that $\Theta(t)$ satisfies the conditions in Proposition 6.1.

PROPOSITION 6.2 (Wagschal). – *Let*

$$B(z, \partial_z) = \sum_{|\alpha| \leqslant m, \alpha_0 \leqslant m_0} b_\alpha(z)\,(\partial_z)^\alpha \tag{6.4}$$

be a linear partial differential operator with coefficients $b_\alpha(z)$ *holomorphic on* $\{z \in C^{n+1}\,;\ |z_i| \leqslant R\}$. *Then there is a constant* B *independent of* $\Theta(t)$ *and* $\rho \geqslant 1$ *such that if*

$$u(z) \ll \Theta^{(j)}(t), \tag{6.5}$$

then

$$B(z, \partial_z)\,u(z) \ll B\rho^{m_0}\,\Theta^{(j+m)}(t). \tag{6.6}$$

PROPOSITION 6.3 (De Paris). – *Let*

$$C(z, \partial_z) = \sum_{|\alpha| \leqslant d, \alpha_0 < d} c_\alpha(z)\,(\partial_z)^\alpha \tag{6.7}$$

be a linear partial differential operator with coefficients $c_\alpha(z)$ *holomorphic on* $\{z \in \mathbf{C}^{n+1}\,;\, |z_i| \leqslant \mathrm{R}\}$. *Then there are constants* $\rho \geqslant 1$ *and* B_1 *independent of* $\Theta(t)$ *such that if*

$$\left\{\begin{array}{l} v(z) \ll \Theta^{(j+d)}(t) \\ u_h(z') \ll \Theta^{(j+h)}(t)|_{z_0=0}\,,\ 0 \leqslant h \leqslant d-1\,, \end{array}\right. \tag{6.8}$$

then the solution $u(z)$ *of the initial value problem*

$$\left\{\begin{array}{l} (\partial_{z_0})^d\, u(z) = \mathrm{C}(z\,,\partial_z)\, u(z) + v(z) \\ (\partial_{z_0})^h\, u(0\,,z') = u_h(z')\,,\ \ 0 \leqslant h \leqslant d-1\,, \end{array}\right. \tag{6.9}$$

satisfies

$$u(z) \ll \mathrm{B}_1\, \Theta^{(j)}(t)\,. \tag{6.10}$$

Set

$$\left\{\begin{array}{l} \theta^{(k)}(t) = \dfrac{k\,!}{(r-t)^{k+1}}\,,\quad k \geqslant 0\,, \\[2ex] \theta^{(k)}(t) = \dfrac{1}{(-k-1)\,!}\displaystyle\int_0^t (t-s)^{-k-1}\,\theta^{(0)}(s)\,ds\,,\ \ k \leqslant 0\,. \end{array}\right. \tag{6.11}$$

If $k \geqslant 0$, $\theta^{(k)}(t)$ satisfies the conditions in Proposition 6.1. We have

PROPOSITION 6.4. –

(i) $\left(\dfrac{d}{dt}\right)^h \theta^{(k)}(t) = \theta^{(h+k)}(t)\,.$

(ii) *If* $0 \leqslant t \leqslant r/2$, *then*

$$\left\{\begin{array}{l} |\theta^{(k)}(t)| \leqslant (2/r)^{k+1}\, k\,!\,,\ \ k \geqslant 0\,; \\ |\theta^{(-k)}(t)| \leqslant 2t^k/r(k\,!)\,,\ k > 0\,. \end{array}\right. \tag{6.12}$$

(iii) *If* $\mathrm{R}' > 2r$ *and* $k < 0$, *then*

$$(\mathrm{R}-t)^{-1}\,\theta^{(k)}(t) \ll \frac{2^{|k|}}{(\mathrm{R}'-2r)}\,\theta^{(k)}(t)\,. \tag{6.13}$$

(iv) *Let* $c \geqslant 1$ *and* s *and* j *be nonnegative integers, then*

$$n^j\,\theta^{([cn]+s)}(t) \ll r^j\,\theta^{([cn]+s+j)}(t)\,. \tag{6.14}$$

Since we do not find the proof of (iv) anywhere, we prove it. It follows from

$$n^j \theta^{([cn]+s)}(t) = n^j([cn]+s)!/(r-t)^{([cn]+s+1)}$$
$$\ll \frac{r^j}{(r-t)^j} \frac{n^j([cn]+s)!}{(r-t)^{[cn]+s+1}} \ll r^j \theta^{([cn]+s+j)}(t).$$

Since $\theta^{(k)}(t)$, $k<0$, does not satisfy the conditions in Proposition 6.1, we employ

$$\Theta_k(t) = \frac{R'}{(R'-t)} \theta^{(k)}(t) \quad (k=0, \pm 1, \pm 2, \dots). \tag{6.15}$$

For $\Theta_k(t)$, we have

PROPOSITION 6.5. –

(i) *If* $k<h$,
$$\Theta_h^{(j)}(t) \ll \Theta_k^{(j-k+h)}(t) \tag{6.16}$$

(ii) *If* $k \geqslant 0$,
$$\theta^{(j+k)}(t) \ll \Theta_k^{(j)}(t) \ll \frac{R'}{(R'-r)} \theta^{(j+k)}(t); \tag{6.17}$$

(iii) *If* $k<0$ *and* $R'>2r$,
$$\theta^{(j+k)}(t) \ll \Theta_k^{(j)}(t) \ll \frac{2^{|k|}}{(R'-2r)} \theta^{(j+k)}(t). \tag{6.18}$$

Now we show Proposition 3.5 and 4.1.

Proof of Proposition 3.5. – First let us recall the equations which $w_n(z)$ $(n=1,2,\dots)$ satisfy;

$$w_{n+k}(z) = -\sum_{s=0}^{k-1} \frac{k!}{(k-s)!\,s!} (\partial_{z_0})^{k-s} w_{n+s}(z)$$
$$+ \sum_{i=0}^{k-1} \sum_{\substack{0 \leqslant s \leqslant i \\ 0 < j \leqslant J}} \frac{n!}{(n-j)!\,j!} \frac{i!}{(i-s)!\,s!} A_{i,j}(z, \partial_{z'}) (\partial_{z_0})^{i-s} w_{n+s-j}(z)$$
$$+ \delta_{n,0} f(z). \tag{6.19}$$

We show by induction on n that there are constants M and A such that

$$w_n(z) \ll MA^n \theta^{([n\beta])}(t). \tag{6.20}$$

We note that $w_n(z)=0$ for $0 \leqslant n \leqslant k-1$. Now let us assume that (6.20) is valid for $0 \leqslant n \leqslant N+k-1$. Hence we obtain, by Proposition 6.2,

$$(\partial_{z_0})^{k-s} w_{N+s}(z) \ll MA^{N+s} B\, \theta^{([(N+s)\beta]+k-s)}(t)$$
$$\ll MA^{N+k-1} B\, \theta^{([(n+k)\beta])}(t), \quad (0 \leqslant s \leqslant k-1),$$

and

$$A_{i,j}(z, \partial_{z'})(\partial_{z_0})^{i-s} w_{N+s-j}(z) \ll MA^{N+s-j} B\, \theta^{([(N+s-j)\beta]+M_{i,j}+i-s)}(t),$$

where $M_{i,j} = \text{ord } A_{i,j}(z, \partial_{z'})$. So it follows from (6.14) that

$$\frac{N!}{(N-j)!\,j!} A_{i,j}(z, \partial_{z'})(\partial_{z_0})^{i-s} w_{N+s-j} \ll MA^{N+s-j} C\, \theta^{(h_N)}(t),$$

where $h_N = [(N+s-j)\beta] + M_{i,j} + i - s + j$. In view of the definition of β (see 1.12) we have $h_N \leqslant [(N+s-j)\beta] + i - s + \beta(k-i+j)$ and $h_N \leqslant [(N+k)\beta]$. Hence we have (6.20) for $n = N + k$. Thus it follows from (6.12) that there are constants M and C and a neighbourhood Ω_0 of $z = 0$ such that $|w_n(z)| \leqslant MC^n \Gamma(n\beta + 1)$ for $z \in \Omega_0$.

Proof of Proposition 4.1. – Let us recall that $v_{n,s}(z, \lambda)$ ($n \geqslant 0$, $s \geqslant -n$) satisfy

$$\begin{cases} M_r(z, \partial_z) v_{n,s}(z, \lambda) + \sum_{j>r} M_j(z, \partial_z) v_{n,s-j+r}(z, \lambda) \\ \qquad + \sum_{j<r} M_j(z, \partial_z) v_{n-r+j,s-j+r}(z, \lambda) = \delta_{n,0}\, \delta_{s,0} G(z, \lambda), \\ (\partial_{z_n})^h v_{n,s}(z'', 0, \lambda) = \delta_{n,0}\, \delta_{s,0}\, k_h(z''), \quad 0 \leqslant h \leqslant t_r - 1, \end{cases} \tag{6.21}$$

and $\sup_{z\in\Omega} |G(z, \lambda)| \leqslant M(\lambda)$ and $\text{ord } M_j(z, \partial_z) = t_j$. We show by induction on n and s that

$$v_{n,s}(z, \lambda) \ll AM(\lambda) B^{n+s} C^n\, \Theta^{([n\nu_{l+1}])}_{-[(n+s)\nu_l]}(t), \tag{6.22}$$

where $t = z_0 + z_1 + \ldots + \rho z_n$. Obviously

$$v_{0,0}(z, \lambda) \ll AM(\lambda)\, \Theta^{(0)}_0(t).$$

Assume that (6.22) is valid when $0 \leqslant n \leqslant N - 1$ and when $n = N$ and $-N \leqslant s \leqslant S - 1$. It follows from Proposition 6.2 and (6.16) that

$$\begin{aligned} &\sum_{j>r} M_j(z, \partial_z) v_{N,S-j+r}(z, \lambda) \\ &\ll \sum_{j>r} AM(\lambda) B^{N+S-j+r} C^N D\, \Theta^{([N\nu_{l+1}]+t_j)}_{-[(N+S-j+r)\nu_l]}(t) \\ &\ll \sum_{j>r} AM(\lambda) B^{N+S-j+r} C^N D\, \Theta^{([N\nu_{l+1}]+t_j+[(N+S)\nu_l]-[(N+S-j+r)\nu_l])}_{-[(N+S)\nu_l]}(t). \end{aligned} \tag{6.23}$$

From Lemma 1.12, if $j > r$,

$$[(N + S - j + r)\nu_l] \geqslant [(N + S)\nu_l] + t_j - t_r.$$

Thus we get

$$\sum_{j>r} \mathrm{M}_j(z, \partial_z)\, v_{\mathrm{N,S}-j+r}(z, \lambda) \ll \mathrm{AM}(\lambda)\, \mathrm{B}^{\mathrm{N+S}-1} \mathrm{C}^{\mathrm{N}}\, \mathrm{E}\, \Theta^{([\mathrm{N}\nu_{i+1}]+t_r)}_{-[(\mathrm{N+S})\nu_i]}(t). \tag{6.24}$$

On the other hand

$$\sum_{j<r} \mathrm{M}_j(z, \partial_z)\, v_{\mathrm{N}-r+j,\mathrm{S}-j+r}(z, \lambda) \tag{6.25}$$
$$\ll \mathrm{AM}(\lambda)\, \mathrm{B}^{\mathrm{N+S}} \sum_{j<r} \mathrm{C}^{\mathrm{N}-r+j} \mathrm{D}\, \Theta^{([(\mathrm{N}-r+j)\nu_{i+1}]+t_j)}_{-[(\mathrm{N+S})\nu_i]}(t).$$

From Lemma 1.12, if $j < r$,

$$[(\mathrm{N} - r + j)\,\nu_{i+1}] \leqslant [\mathrm{N}\nu_{i+1}] + t_r - t_j.$$

Thus we also have

$$\sum_{j<r} \mathrm{M}_j(z, \partial_z)\, v_{\mathrm{N}-r+j,\mathrm{S}-j+r}(z, \lambda) \ll \mathrm{AM}(\lambda)\, \mathrm{B}^{\mathrm{N+S}} \mathrm{C}^{\mathrm{N}-1} \mathrm{E}\, \Theta^{([\mathrm{N}\nu_{i+1}]+t_r)}_{-[(\mathrm{N+S})\nu_i]}(t) \tag{6.26}$$

Hence it follows from Proposition 6.3 that (6.22) is valid for $n = \mathrm{N}$ and $s = \mathrm{S}$. We have from (6.18)

$$v_{n,s}(z, \lambda) \ll \mathrm{A}_1 \mathrm{M}(\lambda)\, \mathrm{B}_1^{n+s} \mathrm{C}^n\, \theta^{([n\nu_{i+1}]-[(n+s)\nu_i])}(t). \tag{6.27}$$

So if $[n\nu_{i+1}] \geqslant [(n+s)\nu_i]$,

$$|v_{n,s}(z, \lambda)| \leqslant \mathrm{A}_1 \mathrm{M}(\lambda)\, \mathrm{B}_1^{n+s} \mathrm{C}_1^n\, \Gamma([n\nu_{i+1}] - [(n+s)\nu_i] + 1) \tag{6.28}$$

and if $[n\nu_{i+1}] \leqslant [(n+s)\nu_i]$,

$$|v_{n,s}(z, \lambda)| \leqslant \mathrm{A}_1 \mathrm{M}(\lambda)\, \mathrm{B}_1^{n+s} \mathrm{C}_1^n\, |z|^{[(n+s)\nu_i]-[n\nu_{i+1}]}$$
$$\times\, \Gamma([(n+s)\nu_i] - [n\nu_{i+1}] + 1)^{-1}. \tag{6.29}$$

We can easily obtain (4.6) in Proposition 4.1 from (6.28) and (6.29). We can also have (4.7) by the same way. This completes the proof of Proposition 4.1.

7. Asymptotic behaviour of functions defined by integrals.

In § 7 we study asymptotic behaviour of functions which appeared in the previous sections. We shall complete the proofs of Theorem 1.7, 1.9 and 1.10. Let us recall that $\mathrm{S}(\omega)$ denotes a sector $\{z_0 \in \mathrm{C};\, |\arg z_0| < \omega\}$ and the path $\mathrm{C}(d, \theta)$, simply $\mathrm{C}(\theta)$, is defined by (1.20). We denote by Ω_ω a domain $\{z \in \Omega;\, |\arg z_0| < \omega\}$. We shall first study the functions $u_i(z)$ $(1 \leqslant i \leqslant \ell - 1)$ and next $u_\ell(z)$ and finally $u_0(z)$.

Now set

$$h_m(z) = \int_{C(\theta)} \exp(\lambda z_0)\, \lambda^{m-1} H(z,\lambda)\, d\lambda \quad (m \in Z), \tag{7.1}$$

where $H(z,\lambda) \in \tilde{\mathcal{O}}(\Omega \times \{|\lambda| > \Lambda\})$, $\Omega = \{z \in C^{n+1} ;\, |z_i| \leqslant R\}$, and satisfies the following conditions:

(i) For any a, b $(b > a)$ and any $\epsilon > 0$, there is a constant $C(\epsilon, a, b)$ such that for $(z,\lambda) \in \Omega \times \{\lambda ;\, |\lambda| > \Lambda,\, a < \arg < b\}$

$$|H(z,\lambda)| \leqslant C(\epsilon, a, b)\, \exp(\epsilon|\lambda|). \tag{7.2}$$

(ii) There are constants H, $c > 0$, $\gamma > 1$ and $\overline{\theta}$ with $\pi\gamma/2 > \overline{\theta} > \pi/2$ such that for $\{\lambda ;\, |\lambda| > \Lambda,\, |\arg \lambda| \leqslant \overline{\theta}\}$

$$\sup_{z\in\Omega} |H(z,\lambda)| \leqslant H \exp(-c|\lambda|^{1/\gamma}). \tag{7.3}$$

Now we define a path $\hat{C}(d,\theta)$ as follows: Put for $\pi/2 < \theta < \pi$

$$\left.\begin{aligned} \hat{C}^-(d,\theta) &= \{\lambda = s \exp(-i\theta) ;\, d \leqslant s < \infty\} \\ \hat{C}^0(d,\theta) &= \{\lambda = d \exp(i\rho) ;\, -\theta \leqslant \rho \leqslant \theta\} \\ \hat{C}^+(d,\theta) &= \{\lambda = s \exp(i\theta) ;\, d \leqslant s < \infty\} \end{aligned}\right\} \tag{7.4}$$

and $\hat{C}(d,\theta) = \hat{C}^-(d,\theta) \cup \hat{C}^0(d,\theta) \cup \hat{C}^+(d,\theta)$. $\hat{C}(d,\theta)$ starts at $\infty \exp(-i\theta)$ on $\hat{C}^-(d,\theta)$, passes on $\hat{C}^0(d,\theta)$ and ends at $\infty\exp(i\theta)$ on $\hat{C}^+(d,\theta)$ (see fig. 7.1). $\hat{C}(d,\theta)$ is a deformation of $C(d,0)$.

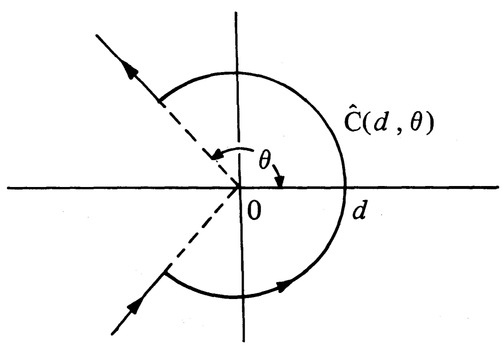

Fig. 7.1.

Under the condition (i) and (ii), we have

PROPOSITION 7.1. –

(i) $h_m(z) \in \tilde{\mathcal{O}}(\Omega - \{z_0 = 0\})$.

Suppose $0 < \omega < \overline{\theta} - \pi/2$. *Then*

(ii) $h_m(z) \sim 0$ *as* $z_0 \longrightarrow 0$ *in* Ω_ω,

(iii) *there are positive constants* A, B *and* C *which depend on* ω *such that for* $z \in \Omega_\omega$, *if* $m \geqslant 0$,

$$|(\partial_z)^\alpha h_m(z)| \leqslant HA^{|\alpha|}B^{m+1}\Gamma((m+\alpha_0)\gamma+1)\Gamma(|\alpha'|+1) \tag{7.5}$$

and if $m \geqslant \alpha_0$,

$$|(\partial_z)^\alpha h_{-m}(z)| \leqslant HA^{|\alpha|}B^{m+1}|z_0|^{m-\alpha_0}\frac{|\alpha'|!}{(m-\alpha_0)!}\exp(-C|z_0|^{-1/(\gamma-1)}), \tag{7.6}$$

where H *in (7.5) and (7.6) is the same in (7.3).*

Proof. – By varying θ in the path $C(\theta)$, we have (i). Let us show (ii) and (iii). Suppose that $\pi > \overline{\theta} > \pi/2$. Put $\theta = 0$ in $C(\theta)$. By deforming $C(0)$ to $\hat{C}(L,\overline{\theta})$, where L is a constant such that $L > \Lambda$, we have

$$\lim_{\substack{z_0\to 0\\ z_0\in S(\omega)}} h_m(z) = \int_{\hat{C}(L,\overline{\theta})} \lambda^{m-1}H(z,\lambda)\,d\lambda \tag{7.7}$$

uniformly in $z' \in \Omega'$, $\Omega' = \{z' \in C^n ; |z_i| \leqslant R\}$. By deformation of the path $\hat{C}(L,\overline{\theta})$ to a path lying in the domain $\{\mathrm{Re}\lambda > 0\}$, (7.7) is zero.

Let us show (iii). We have

$$(\partial_z)^\alpha h_m(z) = \int_{\hat{C}(L,\overline{\theta})} \exp(\lambda z_0)\left\{\sum_{\ell=0}^{\alpha_0}\binom{\alpha_0}{\ell}\lambda^{m+\ell-1}(\partial_{z_0})^{\alpha_0-\ell}(\partial_{z'})^{\alpha'}H(z,\lambda)\right\}d\lambda. \tag{7.8}$$

Let $m \geqslant 0$. Then we have for $z \in \Omega_\omega$

$$\begin{aligned}|(\partial_z)^\alpha h_m(z)| &\leqslant H\left\{\sum_{\ell=0}^{\alpha_0} B_1^{|\alpha|-\ell+1}\binom{\alpha_0}{\ell}\Gamma(|\alpha|-\ell+1)\right\}\\ &\qquad \int_{\hat{C}(L,\overline{\theta})}\exp(-c|\lambda|^{1/\gamma})|\lambda|^{m+\ell-1}|d\lambda|\\ &\leqslant HA^{m+1}B^{|\alpha|}\Gamma((m+\alpha_0)\gamma+1)\Gamma(|\alpha'|+1).\end{aligned} \tag{7.9}$$

Let $m \geqslant \alpha_0$. Put $L(\ell) = (m-\ell+1)|z_0|^{-1} + d|z_0|^{-\gamma/(\gamma-1)}$, where $d > 0$ will be determined later. We have

$$\begin{aligned}|(\partial_z)^\alpha h_{-m}(z)| &\leqslant H\sum_{\ell=0}^{\alpha_0}\binom{\alpha_0}{\ell}B_1^{|\alpha|-\ell+1}\Gamma(|\alpha|-\ell+1)\\ &\quad\times\int_{\hat{C}(L(\ell),\overline{\theta})}L(\ell)^{-(m-\ell+1)}\exp(|\lambda z_0|-c|\lambda|^{1/\gamma})|d\lambda|.\end{aligned} \tag{7.10}$$

Let us estimate a function $H_1(z, \lambda) = \exp(|\lambda z_0| - c|\lambda|^{1/\gamma})$. We have for $\lambda \in \hat{C}^0(L(\ell), \overline{\theta})$

$$|H_1(z, \lambda)| \leqslant e^{m-\ell+1} \exp(d|z_0|^{-1/(\gamma-1)} - cL(\ell)^{1/\gamma}) \leqslant e^{m-\ell+1} \exp(d|z_0|^{-1/(\gamma-1)} - cd^{1/\gamma}|z_0|^{-1/(\gamma-1)}). \tag{7.11}$$

So we choose d so small that it satisfies $d - cd^{1/\gamma} \leqslant -C < 0$. Thus we have for $\lambda \in \hat{C}^0(L(\ell), \overline{\theta})$

$$|H_1(z, \lambda)| \leqslant e^{m-\ell+1} \exp(-C|z_0|^{-1/(\gamma-1)}). \tag{7.12}$$

For $\lambda \in \hat{C}^{\pm}(L(\ell), \overline{\theta})$ and $z \in \Omega_\omega$, we have $\operatorname{Re} \lambda z_0 \leqslant -a|\lambda|\,|z_0|$ and

$$|H_1(z, \lambda)| \leqslant \exp(-a|\lambda z_0| - b|z_0|^{-1/(\gamma-1)}), \quad b > 0. \tag{7.13}$$

Therefore, it follows from (7.10), (7.12) and (7.13) that there are $A = A(\omega)$, $B = B(\omega)$ and $C = C(\omega)$ such that

$$|(\partial_z)^\alpha h_{-m}(z)| \leqslant HA^{|\alpha|} B^{m+1} |z_0|^{m-\alpha_0} \exp(-C|z_0|^{-1/(\gamma-1)}) \frac{|\alpha'|!}{(m-\alpha_0)!}. \tag{7.14}$$

Next suppose $\overline{\theta} \geqslant \pi$. If $|\arg z_0 + \theta| < \pi/2$, the expression (7.1) holds. By using it and choosing $L(\ell)$ as in the above arguments, we have (ii) and (iii) in Proposition 7.1.

Let us apply Proposition 7.1 to the functions $u_i(z)$ $(1 \leqslant i \leqslant \ell)$ constructed in § 6. Recall that

$$u_i(z) = \int_{C(\theta)} \exp(\lambda z_0) V_i(z, \lambda)\, d\lambda, \quad (1 \leqslant i \leqslant \ell - 1), \tag{7.15}$$

where

$$|V_i(z, \lambda)| \leqslant A \exp(c'|\lambda|^{1/\gamma_{i+1}}), \quad \gamma_i = \nu_i, \tag{7.16}$$

and if $|\arg \lambda| \leqslant \theta_{i+1}$, $\pi/2 < \theta_{i+1} < \pi\gamma_{i+1}/2$,

$$|V_i(z, \lambda)| \leqslant A \exp(-c|\lambda|^{1/\gamma_i}) \tag{7.17}$$

and

$$u_\ell(z) = \int_{C(\theta)} \exp(\lambda z_0) V_\ell^+(z, \lambda) \lambda^{-r}\, d\lambda + \sum_{s=1}^{\infty} \int_{C(\theta)} \exp(\lambda z_0) \hat{v}_{-s}(z, \lambda) \lambda^{-s-r}\, d\lambda, \tag{7.18}$$

where

$$|V_\ell^+(z, \lambda)| \leqslant A \exp(c'|\lambda|^{1/\gamma_\ell}), \tag{7.19}$$

$$|v_{-s}(z, \lambda)| \leqslant AB^s s!\, \exp(c'|\lambda|^{1/\gamma_\ell}) \quad s > 0, \tag{7.20}$$

and if $|\arg\lambda| \leqslant \theta_\ell$, $\pi/2 < \theta_\ell < \pi\gamma_\ell/2$,

$$|V_\ell^+(z,\lambda)| \leqslant A\exp(-c|\lambda|^{1/\gamma_\ell}), \tag{7.21}$$

$$|\hat{v}_{-s}(z,\lambda)| \leqslant AB^s s!\exp(-c|\lambda|^{1/\gamma_\ell}). \tag{7.22}$$

Here $z \in \Omega$, $|\lambda| > 1$, $\theta_1 > \theta_2 > \ldots > \theta_\ell > \pi/2$ and c' and c are positive constants.

For $u_i(z)$, $1 \leqslant i \leqslant \ell - 1$, we have

PROPOSITION 7.2. – *Let* ω_{i+1} *be a number with*

$$0 < \omega_{i+1} < \theta_{i+1} - \pi/2.$$

Then

(i) $u_i(z) \sim 0$ *and* $g_{i+1}(z) \sim 0$ *as* $z_0 \longrightarrow 0$ *in* $z \in \Omega_{\omega_{i+1}}$,

(ii) *for* $z \in \Omega_{\omega_{i+1}}$,

$$|(\partial_z)^\alpha u_i(z)| \leqslant AB^{|\alpha|}\Gamma(\alpha_0\gamma_i + 1)\Gamma(|\alpha'| + 1). \tag{7.23}$$

Proof. – Proposition 7.2 follows from Proposition 7.1 (see (5.25) and (5.27)).

For $u_\ell(z)$, which belongs to $\widetilde{\mathcal{O}}(\Omega - \{z_0 = 0\})$, we have

PROPOSITION 7.3. – *There are constants* r_0, A, B *and* C *such that for* $z \in \Omega_{\omega_\ell}$ *with* $0 < \omega_\ell < \theta_\ell - \pi/2$ *and* $|z_0| \leqslant r_0$

$$|u_\ell(z)| \leqslant A\exp(-C|z_0|^{-1/(\gamma_\ell - 1)}) \tag{7.24}$$

and

$$|(\partial_z)^\alpha u_\ell(z)| \leqslant AB^{|\alpha|}\Gamma(\alpha_0\gamma_\ell + 1)\Gamma(|\alpha'| + 1). \tag{7.25}$$

Proof. – *Put*

$$u_\ell^+(z) = \int_{C(\theta)} \exp(\lambda z_0) V_\ell^+(z,\lambda)\lambda^{-r}d\lambda \tag{7.26}$$

and

$$u_{\ell,s}(z) = \int_{C(\theta)} \exp(\lambda z_0) v_{-s}(z,\lambda)\lambda^{-r-s}d\lambda. \tag{7.27}$$

It follows from Proposition 7.1 that (7.24) and (7.25) hold, if we replace $u_\ell(z)$ by $u_\ell^+(z)$. So we have only to consider $\sum_{s=1}^{\infty} u_{\ell,s}(z)$. We have, from Proposition 7.1, for $z \in \Omega_{\omega_\ell}$

$$|u_{\ell,s}(z)| \leqslant AB^{s+1}|z_0|^{s-1}\exp(-C|z_0|^{-1/(\gamma_\ell - 1)}). \tag{7.28}$$

Hence if $|z_0| \leqslant r_1 = 1/2B$, $\sum_{s=1}^{\infty} |u_{\ell,s}(z)|$ converges. Thus we have (7.24). Let us show (7.25). In view of (iii) in Proposition 7.1 we have

$$|(\partial_z)^\alpha u_{\ell,s}(z)| \leqslant \begin{cases} A_1 B_1^s C(|\alpha'|!)(s!)\Gamma((\alpha_0 - s)\gamma_\ell + 1), & \alpha_0 \geqslant s, \\ A_1 B_1^s C^{|\alpha|} |z_0|^{s-\alpha_0} (|\alpha'|!)(s!)/\Gamma((s-\alpha_0)+1), & \alpha_0 \leqslant s. \end{cases} \tag{7.29}$$

Hence, there is an r_2 such that for $z \in \Omega_{\omega_{\ell+1}} \cap \{|z_0| \leqslant r_2\}$

$$\begin{aligned} \sum_{s=1}^{\infty} |(\partial_z)^\alpha u_{\ell,s}(z)| &\leqslant \sum_{s=1}^{\alpha_0} A_1 B_1^s C^{|\alpha|}(|\alpha'|!)(s!)\Gamma((\alpha_0 - s)\gamma_\ell + 1) \\ &\quad + \sum_{s>\alpha_0} A_1 B_1^s C^{|\alpha|} |z_0|^{s-\alpha_0} (|\alpha'|!)(s!)/\Gamma((s-\alpha_0)+1) \\ &\leqslant AB^{|\alpha|}\Gamma(\alpha_0\gamma_\ell + 1)\Gamma(|\alpha'| + 1). \end{aligned} \tag{7.30}$$

Thus we have (7.25).

Next we investigate the function $u_0(z)$. To do so we study asymptotic behaviour of functions defined as follows. Put

$$\psi(\tau; z) = \frac{1}{2\pi i}\int_{C(\theta)} \exp(\lambda z_0)\lambda^{1/p-1}\Psi(\tau; z, \lambda)\,d\lambda, \tag{7.31}$$

where

$$\Psi(\tau; z, \lambda) = \int_0^{\tau\lambda^\delta} \exp(-\lambda^{1/p}\zeta)\Phi(z, \zeta)\,d\zeta. \tag{7.32}$$

Here $\delta = (q-1)/p$, $p, q \in N$, $p > q$, $(p, q) = 1$, $0 < \tau < \hat{\tau}$, and

$$\Phi(z, \zeta) = \sum_{n=0}^{\infty} \varphi_n(z)\zeta^{np}/(np)!, \tag{7.33}$$

where $\varphi_n(z)$ is holomorphic in Ω such that for $z \in \Omega$

$$|(\partial_z)^\alpha \varphi_n(z)| \leqslant MA^n B^{|\alpha|}\Gamma(n\beta + |\alpha| + 1), \quad \beta = p/q > 1. \tag{7.34}$$

We choose $\hat{\tau}$ in order that $\Phi(z, \zeta)$ converges. Put

$$\Phi_m^1(z, \zeta) = \sum_{n=0}^{m} \varphi_n(z)\zeta^{np}/(np)! \tag{7.35}$$

and

$$\Phi_m^2(z, \zeta) = \sum_{n=m+1}^{\infty} \varphi_n(z)\zeta^{np}/(np)!. \tag{7.36}$$

LEMMA 7.4. – *There are constants* M, C, c *and* $\hat{\zeta}$ *such that*

$$|(\partial_z)^\alpha \Phi_m^2(z,\zeta)| \leqslant \begin{cases} MB^{|\alpha|}C^{m+1}\,\Gamma(m\beta+|\alpha|+1)\,|\zeta|^{(m+1)p} & (7.37) \\ \qquad \exp(c|\zeta|^{q/(q-1)})/\Gamma(mp+1),\ q>1, \\ MB^{|\alpha|}C^{m+1}\,\Gamma(m\beta+|\alpha|+1)\,|\zeta|^{(m+1)p}/\Gamma(mp+1), \\ \qquad |\zeta|<\hat{\zeta},\ q=1. \end{cases}$$

Proof. – We have from (7.34)

$$|(\partial_z)^\alpha \Phi_m^2(z,\zeta)| \leqslant MB^{|\alpha|}\sum_{n=m+1}^{\infty} A^n\,\Gamma(n\beta+|\alpha|+1)\,|\zeta|^{np}/\Gamma(np+1)$$

$$\leqslant MB^{|\alpha|}A^{m+1}|\zeta|^{(m+1)p}\sum_{n=0}^{\infty} A_1^{n+m+1}\,\Gamma((n+1)\beta)$$

$$\Gamma((m\beta+|\alpha|+1)\,|\zeta|^{np}/\Gamma((n+1)p)\,\Gamma(mp+1).$$

So, there are constants c and $\hat{\zeta}$ such that (7.37) is valid.

LEMMA 7.5. – *For any* ω *with* $0<\omega<\pi(\beta-1)/2$, *if* $z_0 \longrightarrow 0$ *in* Ω_ω,

$$\frac{1}{2\pi i}\int_{C(\theta)} \exp(\lambda z_0)\,\lambda^{1/p-1}\,d\lambda \int_{\tau\lambda^\delta}^{\infty} \exp(-\lambda^{1/p}\zeta)\,\Phi_m^1(z,\zeta)\,d\zeta \longrightarrow 0. \tag{7.38}$$

Proof. – We have

$$\int_{\tau\lambda^\delta}^{\infty} \exp(-\lambda^{1/p}\zeta)\,\Phi_m^1(z,\zeta)\,d\zeta = \exp(-\tau\lambda^{1/\beta})\,\hat{\Phi}_m^1(z,\lambda), \tag{7.39}$$

where $\hat{\Phi}_m^1(z,\lambda)$ is a polynomial of $\lambda^{-1/p}$. By varying θ in C(θ) or deforming C(θ) to $\hat{C}(\hat{\theta})$ with $\omega+\pi/2<\hat{\theta}<\pi\beta/2$, we have (7.38).

Put

$$\psi_m^1(\tau;z) = \frac{1}{2\pi i}\int_{C(\theta)} \exp(\lambda z_0)\,\lambda^{1/p-1}\,d\lambda \int_0^{\tau\lambda^\delta} \exp(-\lambda^{1/p}\zeta)\,\Phi_m^1(z,\zeta)\,d\zeta \tag{7.40}$$

and

$$\psi_m^2(\tau;z) = \frac{1}{2\pi i}\int_{C(\theta)} \exp(\lambda z_0)\,\lambda^{1/p-1}\,d\lambda \int_0^{\tau\lambda^\delta} \exp(-\lambda^{1/p}\zeta)\,\Phi_m^2(z,\zeta)\,d\zeta \tag{7.41}$$

We have $\psi(\tau;z) = \psi_m^1(\tau;z) + \psi_m^2(\tau;z)$ (see (7.31)).

LEMMA 7.6. – *For any* ω *with* $0<\omega<\pi(\beta-1)/2$,

$$\psi_m^1(\tau;z) \sim \sum_{n=0}^{m} \varphi_n(z)\,(z_0)^n/n! \text{ as } z_0 \longrightarrow 0 \text{ in } z\in\Omega_\omega. \tag{7.42}$$

Proof. – We have

$$\psi_m^1(\tau;z) = \frac{1}{2\pi i}\int_{C(\theta)} \exp(\lambda z_0)\,\lambda^{1/p-1}\,d\lambda \left\{ \int_0^\infty \exp(-\lambda^{1/p}\zeta)\,\Phi_m^1(z,\zeta)\,d\zeta - \int_{\tau\lambda^\delta}^\infty \exp(-\lambda^{1/p}\zeta)\,\Phi_m^1(z,\zeta)\,d\zeta \right\}$$

$$= \frac{1}{2\pi i}\int_{C(\theta)} \exp(\lambda z_0) \left\{ \left(\sum_{n=0}^m \varphi_n(z)\,\lambda^{-(n+1)}\right) - \lambda^{1/p-1}\int_{\tau\lambda^\delta}^\infty \exp(-\lambda^{1/p}\zeta)\,\Phi_m^1(z,\zeta)\,d\zeta \right\} d\lambda$$

$$= \sum_{n=0}^m \varphi_n(z)\,(z_0)^n/n\,! - \frac{1}{2\pi i}\int_{C(\theta)} \exp(\lambda z_0)\,\lambda^{1/p-1}\,d\lambda \left(\int_{\tau\lambda^\delta}^\infty \exp(-\lambda^{1/p}\zeta)\,\Phi_m^1(z,\zeta)\,d\zeta\right).$$

Hence from Lemma 7.5 we have (7.42).

LEMMA 7.7. – *Suppose that* $|\arg\lambda| \leqslant \hat\theta$ *with* $\pi/2 < \hat\theta < \pi\beta/2$ *and* $q > 1$. *Then there is a* $\tau_0 = \tau_0(\hat\theta)$ *such that*

$$\left| \int_0^{\tau_0\lambda^\delta} \exp(-\lambda^{1/p}\zeta + c\,|\zeta|^{q/(q-1)})\,|\zeta|^{(m+1)p}\,|d\zeta| \right| \leqslant K\Gamma(mp+1)\,C^m\,|\lambda|^{-(m+1+1/p)},\quad m \in N,$$

holds for some constants K *and* C *dependent on* τ_0.

Proof. – Put $\zeta(\tau;t) = t\tau\lambda^\delta$ $(0 \leqslant t \leqslant 1)$ and

$$h(\zeta) = -\operatorname{Re}\lambda^{1/p}\zeta + c\,|\zeta|^{q/(q-1)}.$$

We have $h(\zeta(\tau;t)) = -t\tau\operatorname{Re}\lambda^{1/\beta} + c(t\tau)^{q/(q-1)}|\lambda|^{1/\beta}$. Since $|\arg\lambda| \leqslant \hat\theta < \pi\beta/2$, there are $\tau_0 = \tau_0(\hat\theta)$ and $d > 0$ such that $h(\zeta(\tau_0;t)) \leqslant -d\,|\lambda|^{1/\beta}t$ for $0 \leqslant t \leqslant 1$. Hence there are constants K and C such that

$$\int_0^1 \exp(h(\zeta(\tau_0;t))\,(t\tau_0)^{(m+1)p}\,\tau_0\,|\lambda|^{(q-1)(m+1)+\delta}\,dt \leqslant (\tau_0)^{(m+1)p+1}\,|\lambda|^{(q-1)(m+1)+\delta}\int_0^1 \exp(-d\,|\lambda|^{1/\beta}t)\,t^{(m+1)p}\,dt \leqslant K\,|\lambda|^{-(m+1+1/p)}\,C^m\,\Gamma(mp+1).$$

LEMMA 7.8. – *For any* ω *with* $0 < \omega < \pi(\beta-1)/2$, *there are constants* $\tau_1 = \tau_1(\omega)$, M *and* A *such that for* $z \in \Omega_\omega$ *and* $m \geqslant \alpha_0$

$$|(\partial_z)^\alpha\,\psi_m^2(\tau_1;z)| \leqslant MA^{m+1}B^{|\alpha|}\Gamma(m\beta + |\alpha| + 1)\,|z_0|^{m-\alpha_0+1}. \quad (7.43)$$

Proof. – Take $\hat{\theta}$ such that $\pi/2 + \omega < \hat{\theta} < \pi\beta/2$. If $q > 1$, by putting $\tau = \tau_1 = \tau_0(\hat{\theta})$, we have from Lemma 7.4 and 7.7

$$|(\partial_z)^\alpha \psi_m^2(\tau_1, z)| \leqslant KA^{m+1} B^{|\alpha|} |z_0|^{m-\alpha_0+1} \Gamma(m\beta + |\alpha| + 1) \int_C |\lambda|^{-(m-\alpha_0+2)} |d\lambda|, \tag{7.44}$$

where C is a deformation of $C(\theta)$ in $\{|\arg \lambda| \leqslant \hat{\theta}\}$. Hence we have (7.43). If $q = 1$, we also have (7.43).

In view of Lemma 7.6 and 7.8, we have

PROPOSITION 7.9. – *For any* ω *with* $0 < \omega < \pi(\beta - 1)/2$, *there is a* $\tau_1 = \tau_1(\omega)$ *such that*

$$\psi(\tau_1; z) \sim \sum_{n=0}^{\infty} \varphi_n(z) (z_0)^n/n! \quad as \quad z_0 \longrightarrow 0 \quad in \quad z \in \Omega_\omega. \tag{7.45}$$

In the following we fix $\tau = \tau_1$. Now we show

PROPOSITION 7.10. – *There are constants* M *and* C *such that for* $z \in \Omega_\omega$ *with* $0 < \omega < \pi(\beta - 1)/2$,

$$|(\partial_z)^\alpha \psi(\tau_1; z)| \leqslant MC^{|\alpha|} \Gamma(\alpha_0 \beta + 1) \Gamma(|\alpha'| + 1). \tag{7.46}$$

Proposition 7.10 is used to show Theorem 1.11. To show proposition 7.10 we give lemmas. Put

$$I_{\ell,n}(z_0) = \frac{1}{2\pi i} \int_C \exp(\lambda z_0) \lambda^{\ell-1+1/p} d\lambda \int_{\tau\lambda^\delta}^{\infty} \exp(-\lambda^{1/p}\zeta) \zeta^{np}/(np)!\, d\zeta. \tag{7.47}$$

LEMMA 7.11. –

$$\int_{\tau\lambda^\delta}^{\infty} \lambda^{n+1/p} (\exp(-\lambda^{1/p}\zeta) \zeta^{np}/(np)!) d\zeta = \exp(-\tau\lambda^{1/\beta}) \left\{ \sum_{s=0}^{np} (\tau\lambda^{1/\beta})^{np-s}/(np-s)! \right\}. \tag{7.48}$$

Proof. – By integration by parts, we have

$$\begin{aligned}
\int_{\tau\lambda^\delta}^{\infty} \lambda^{n+1/p} \exp(-\lambda^{1/p}\zeta) (\zeta^{np}/(np)!) d\zeta
&= \int_{\tau\lambda^\delta}^{\infty} \{(-\partial_\zeta)^{np+1} \exp(-\lambda^{1/p}\zeta)\} (\zeta^{np}/(np)!) d\zeta \\
&= \sum_{s=0}^{np} \{(-\partial_\zeta)^{np-s} \exp(-\lambda^{1/p}\zeta)\} \zeta^{np-s}/(np-s)! \Big|_{\zeta=\tau\lambda^\delta} \\
&= \sum_{s=0}^{np} ((\tau\lambda^{1/\beta})^{np-s}/(np-s)!) \exp(-\tau\lambda^{1/\beta}).
\end{aligned}$$

LEMMA 7.12. – *For any* ω *with* $0 < \omega < \pi(\beta - 1)/2$, *there are constants* $A = A(\omega)$ *and* $B = B(\omega)$ *such that for* $z_0 \in S(\omega) \cap \{|z_0| \leqslant 1\}$

$$|I_{\ell,n}(z_0)| \leqslant \begin{cases} AB^{\ell+n}\,\Gamma(\beta(\ell - n) + 1) & (\ell \geqslant n), \\ AB^{\ell+n}\,\Gamma(\beta(n - \ell) + 1)^{-1} & (\ell \leqslant n). \end{cases} \tag{7.49}$$

Proof. – Choose the path C so that $\operatorname{Re}\lambda z_0 < 0$ and $\operatorname{Re}\lambda^{1/\beta} > 0$. By taking $C = C(\theta)$ or $\hat{C}(\theta)$ for suitable θ, we have from Lemma 7.11

$$|I_{\ell,n}(z_0)| \leqslant M \sum_{s=0}^{np} \int_C \exp(-\tau c |\lambda|^{1/\beta}) \, (|\lambda|^{\ell-n-1+(np-s)/\beta}/(np-s)!) \, |d\lambda| .$$

Hence we have (7.49).

Put for a nonnegative integer ℓ

$$\psi^i_{m,\ell}(\tau_1, z) = \frac{1}{2\pi i} \int_{C(\theta)} \lambda^{\ell+1/p-1} \exp(\lambda z_0) \, d\lambda \int_0^{\tau_1 \lambda^\delta} \exp(-\lambda^{1/p}\zeta) \, \Phi^i_m(z, \zeta) \, d\zeta \tag{7.50}$$

$(i = 1, 2)$.

LEMMA 7.13. – *Let* $\ell \leqslant m$. *Then for*

$$z \in \Omega_\omega, \quad 0 < \omega < \pi(\beta - 1)/2,$$

$$|\psi^1_{m,\ell}(\tau_1, z)| \leqslant AB^{m+\ell}\,\Gamma(m\beta + \ell - m + 1) \tag{7.51}$$

holds for some constants $A = A(\omega)$ *and* $B = B(\omega)$.

Proof. – We have

$$\begin{aligned} \psi^1_{m,\ell}(\tau_1, z) &= \frac{1}{2\pi i} \int_{C(\theta)} \exp(\lambda z_0) \lambda^{\ell+1/p-1} \, d\lambda \left\{ \int_0^\infty \exp(-\lambda^{1/p}\zeta) \, \Phi^1_m(z, \zeta) \, d\zeta \right. \\ &\qquad \left. - \int_{\tau_1\lambda^\delta}^\infty \exp(-\lambda^{1/p}\zeta) \, \Phi^1_m(z, \zeta) \, d\zeta \right\} \\ &= \frac{1}{2\pi i} \int_{C(\theta)} \exp(\lambda z_0) \Big(\sum_{n=0}^m \varphi_n(z) \lambda^{\ell-n-1} \Big) d\lambda - \sum_{n=0}^m \varphi_n(z) \, I_{\ell,n}(z_0) \\ &= \sum_{s=\ell}^m \varphi_s(z)/(s-\ell)! - \sum_{n=0}^m \varphi_n(z) \, I_{\ell,n}(z_0). \end{aligned} \tag{7.52}$$

Therefore, it follows from Lemma 7.12 and (7.34) that

$$
\begin{aligned}
|\psi^1_{m,\ell}(\tau_1, z)| \leqslant & \sum_{s=\ell}^{m} \mathrm{MA}_1^s\,\Gamma(s\beta+1)/(s-\ell)! \\
& + \sum_{n=0}^{\ell} \mathrm{MA}_1^n\,\Gamma(n\beta+1)\,\mathrm{C}_1^{\ell+n}\,\Gamma((\ell-n)\beta+1) \\
& + \sum_{n=\ell+1}^{m} \mathrm{MA}_1^n\,\Gamma(n\beta+1)\,\mathrm{C}_1^{\ell+n}\,\Gamma((n-\ell)\beta+1)^{-1} \\
& \leqslant \mathrm{AB}^{m+\ell}\,\Gamma(m\beta+\ell-m+1).
\end{aligned}
$$

LEMMA 7.14. – *Let* $\ell \leqslant m$. *Then for*

$$z \in \Omega_\omega\,,\quad 0 < \omega < \pi(\beta-1)/2\,,$$

$$|\psi^2_{m,\ell}(\tau_1, z)| \leqslant \mathrm{AB}^{m+\ell}\,\Gamma(m\beta+\ell-m+1) \tag{7.53}$$

holds for some constants $\mathrm{A} = \mathrm{A}(\omega)$ *and* $\mathrm{B} = \mathrm{B}(\omega)$.

Proof. – We have, from Lemma 7.4 and 7.7, for a suitable deformation C of $\mathrm{C}(\theta)$

$$
\begin{aligned}
|\psi^2_{m,\ell}(\tau_1, z)| \leqslant \mathrm{KD}^m \int_{\mathrm{C}} |\exp(\lambda z_0)|\ |\lambda|^{\ell-m-2}\,d\lambda\,\Gamma(m\beta+1) \\
\leqslant \mathrm{AB}^{m+\ell}\,\Gamma(m\beta+\ell-m+1).
\end{aligned}
$$

Proof of Proposition 7.10. – First we note that

$$\psi(\tau_1, z) = \psi^1_m(\tau_1\,; z) + \psi^2_m(\tau_1, z).$$

Put $m = \alpha_0$. We have

$$
\begin{aligned}
(\partial_z)^\alpha\,\psi^i_m(\tau_1\,; z) = \frac{1}{2\pi i}\sum_{\ell=0}^{m} \binom{m}{\ell} \int_{\mathrm{C}(\theta)} \lambda^{\ell+1/p-1}\exp(\lambda z_0)\,d\lambda \\
\int_0^{\tau_1\lambda^\delta} (\partial_{z_0})^{m-\ell}(\partial_{z'})^{\alpha'}\,\Phi^i_m(z,\zeta)\,d\zeta\,.
\end{aligned}
\tag{7.54}
$$

In view of Lemma 7.4, 7.13 and 7.14 it follows that

$$|(\partial_z)^\alpha\,\psi^i_m(\tau_1\,; z)| \leqslant \mathrm{AB}^{|\alpha|}\,\Gamma(\alpha_0\beta+1)\,\Gamma(|\alpha'|+1) \ \text{ for } \ z \in \Omega_\omega\,.$$

Now we apply Proposition 7.9 to $u_0(z)$:

$$u_0(z) = \frac{1}{2\pi i}\int_{\mathrm{C}(\theta)} \exp(\lambda z_0)\,\lambda^{1/p-1}\,d\lambda \int_0^{\tau_1\lambda^\delta} \exp(-\lambda^{1/p}\zeta)\,w(z,\zeta)\,d\zeta\,, \tag{7.55}$$

where

$$w(z,\zeta) = \sum_{n=k}^{\infty} w_n(z)\,\zeta^{np}/(np)! \tag{7.56}$$

and for z in a neighbourhood U of $z = 0$

$$|(\partial_z)^\alpha w_n(z)| \leqslant AB^n C\Gamma(n\beta + |\alpha| + 1). \tag{7.57}$$

Hence we have

$$u_0(z) \sim \sum_{n=k}^{\infty} w_n(z)\,(z_0)^n/n! \text{ as } z_0 \longrightarrow 0 \text{ in } U_S. \tag{7.58}$$

Since $L(z, \partial_z)\,u_0(z) - f(z) \sim 0$ and $u_0(z) = O((z_0)^k)$ as $z_0 \longrightarrow 0$ in U_S, it follows from uniqueness of solutions of formal power series that $u_0(z) \sim \hat{u}(z)$. Thus this completes the proof of Theorem 1.10.

Finally we show Theorem 1.11. Put

$$S_+(\omega) = \{z \in C^{n+1}\,;\, |\arg z_0| < \omega\}$$

and

$$S_-(\omega) = \{z \in C^{n+1}\,;\, |\arg z_0 - \pi| < \omega\} \text{ with } 0 < \omega < \pi(\gamma_\ell - 1)/2.$$

Then it follows from Theorem 1.10 that there are functions $u_+(z)$ and $u_-(z)$ such that $u_+(z),\ u_-(z) \in \widetilde{\mathfrak{O}}(U - \{z_0 = 0\})$,

$$L(z, \partial z)\,u_\pm(z) = f(z) \tag{7.60}$$

and

$$\begin{cases} u_+(z) \sim \hat{u}(z) \text{ as } z_0 \longrightarrow 0 \text{ in } U_{S_+(\omega)}, \\ u_-(z) \sim \hat{u}(z) \text{ as } z_0 \longrightarrow 0 \text{ in } U_{S_-(\omega)}. \end{cases} \tag{7.61}$$

Define $u(x)$ as follows

$$u(x) = \begin{cases} u_+(z)|_{R^{n+1}} \text{ for } x_0 > 0, \\ u_-(z)|_{R^{n+1}} \text{ for } x_0 < 0. \end{cases} \tag{7.62}$$

$u(x)$ defined by (7.62) can be extended as a C^∞ function up to $\{x_0 = 0\}$. Thus we have for $x \in V = U \cap \{\operatorname{Im} z = 0\}$,

$$\begin{cases} L(x, \partial_x)\,u(x) = f(x) \\ (\partial x_0)^\ell\, u(0, x') = u_\ell(x'),\ 0 \leqslant \ell \leqslant k-1. \end{cases} \tag{7.63}$$

In view of Proposition 7.11, we have the estimate (1.28) of $u(x)$. This completes the proof of Theorem 1.11.

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Suano ŌUCHI,
Department of Mathematics
Faculty of Science and Technology
Jōchi (Sophia) University
Chiyoda-ku, Tokyo 102 (Japan).