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ON THE GREATEST PRIME FACTOR OF $n^2 + 1$

by **J.-M. DESHOUILLERS** and **H. IWANIEC**

1. Introduction.

In 1967 C. Hooley [2] (see also [3]) showed that if D is not a perfect square then the greatest prime factor of $n^2 - D$ exceeds $n^{11/10}$ infinitely often. In fact Hooley's arguments yield a slightly better result with the exponent 11/10 replaced by any θ less than $\theta_0 = 1.100\,148\,3 \ldots$ the solution of

$$(1) \quad \frac{14}{3}\left(\theta - \frac{12}{11}\right) + \frac{28}{9}\log\left(1 + \frac{33}{14}\left(\theta - \frac{12}{11}\right)\right) - \frac{41}{33} + \frac{32}{9}\log\frac{11}{8} = 0.$$

Among several innovative ideas in Hooley's proof one finds a very interesting application of A. Weil's estimate for Kloosterman sums

$$(2) \qquad \mathrm{S}(n\,m; c) = \sum_{\substack{d (\mathrm{mod}\, c) \\ (d,c)=1}} e\left(n\frac{\bar{d}}{c} + m\frac{d}{c}\right) \ll (n, m, c)^{\frac{1}{2}} c^{\frac{1}{2}+\varepsilon}$$

where the symbol $\bar{d}$ stands for a solution of $d\bar{d} \equiv 1 \,(\mathrm{mod}\, c)$. Recently the authors [1] investigated linear forms in Kloosterman sums $\mathrm{S}(n\mathrm{Q}, m; c)$ with the variables of the summation n, m and c counted with a smooth weight function, showing (see Lemma 3) that there exists a considerable cancellation of terms.

In the paper we inject this result into the Chebyshev-Hooley method to prove the following

THEOREM. — *For any $\varepsilon > 0$ there exist infinitely many integers n such that $n^2 + 1$ has a prime factor greater than $n^{\theta - \varepsilon}$, where θ satisfies*

$$2 - \theta - 2\,\mathrm{Log}\,(2 - \theta) = \frac{5}{4} \qquad (\theta = 1.202\,468 \ldots).$$

Our result can be generalized to $n^2 - \mathrm{D}$ by using Hooley's arguments from [3].

The authors express their thanks to Prof. C. Hooley for interesting comments and corrections.

2. Chebyshev's method.

Let $x \geqslant 2$ and let b be a non-negative function of C^∞-class with support in $[x,2x]$ and the derivates of which satisfy

$$b^{(l)}(\xi) \ll x^{-l}, \qquad l = 0, 1, 2, \ldots,$$

the implied constant in $\ll$ depending on l alone. Denote

$$\mathrm{X} = \int b(\xi)\,\mathrm{d}\xi \qquad \text{and} \qquad |\mathcal{A}_d| = \sum_{n^2+1 \,\equiv\, 0 \,(\mathrm{mod}\, d)} b(n).$$

We begin with applying Chebyshev's idea to calculate

$$\text{(1)} \quad \mathrm{T}(x) = \sum_p |\mathcal{A}_p| \log p = \sum_d |A_d| \,\Lambda\,(d) + \mathrm{O}(x)$$

$$= \sum_n b(n) \sum_{d \mid n^2+1} \Lambda\,(d) + \mathrm{O}(x) = \sum_n b(n) \log (n^2+1) + \mathrm{O}(x)$$

$$= 2\,(\log x) \int b(\xi)\,\mathrm{d}\xi + \mathrm{O}(x) = 2\mathrm{X} \log x + \mathrm{O}(x).$$

The partial sum

$$\mathrm{T}_0(x) = \sum_{p \leqslant x} |\mathcal{A}_p| \log p$$

$$= \sum_{p \leqslant x} \sum_{\nu^2+1 \equiv 0 \,(\mathrm{mod}\, p)} (\log p) \sum_{n \equiv \nu \,(\mathrm{mod}\, p)} b(n)$$

can be evaluated easily by the Poisson summation formula.

Lemma 1. — *For any* $f(\xi)$ *of* C^1 *class with compact support in* $(0, \infty)$ *we have*

$$\sum_{n \equiv a \,(\mathrm{mod}\, q)} f(n) = \frac{1}{q} \sum_h e\left(-\frac{ah}{q}\right) \hat{f}\left(\frac{h}{q}\right), \qquad h \in \mathbf{Z}$$

where $\hat{f}(t)$ *is the Fourier transform of* $f(\xi)$.

By Lemma 1

$$\sum_{n \equiv \nu \,(\mathrm{mod}\, p)} b(n) = \frac{1}{p} \sum_h e\left(-\frac{\nu h}{p}\right) \hat{b}\left(\frac{h}{p}\right).$$

For $h = 0$ we have $\hat{b}(0) = \mathrm{X}$. If $h \neq 0$ by partial integration two times we get

$$\hat{b}\left(\frac{h}{p}\right) = \int b(\xi) e\left(\frac{h}{p}\xi\right) \mathrm{d}\xi$$
$$= \left(\frac{p}{2\pi i h}\right)^2 \int b''(\xi) e\left(\frac{h}{p}\xi\right) \mathrm{d}\xi \ll h^{-2} p^2 x^{-1}.$$

This yields

$$\sum_{n \equiv \nu \,(\mathrm{mod}\, p)} b(n) = \frac{\mathrm{X}}{p} + \mathrm{O}\left(\frac{p}{x}\right)$$

whence

$$\mathrm{T}_0(x) = \mathrm{X} \log x + \mathrm{O}(x). \tag{2}$$

Letting P_x be the greatest prime factor of $\prod_{x<n<2x} (n^2+1)$ by (1) and (2) it follows that

$$\mathrm{S}(x) = \sum_{x<p\leqslant \mathrm{P}_x} |\mathscr{A}_p| \log p = \mathrm{X} \log x + \mathrm{O}(x). \tag{3}$$

Our aim is to estimate $\mathrm{S}(x)$ from above and deduce from it a lower estimate for P_x.

3. Splitting up of $\mathrm{S}(x)$.

In what follows it will be convenient to have p counted with a smooth weight function. Therefore we arrange the sum $\mathrm{S}(x)$ as

$$\mathrm{S}(x) = \sum_{1 \leqslant j \leqslant \mathrm{J}} \mathrm{S}(x, \mathrm{P}_j) + \mathrm{O}(x) \tag{4}$$

with $\mathrm{P}_j = 2^j x$, $0 \leqslant j \leqslant \mathrm{J} \leqslant 2 \log x$ and

$$\mathrm{S}(x, \mathrm{P}_j) = \sum_{\mathrm{P}_j < p \leqslant 4\mathrm{P}_j} |\mathscr{A}_p| \mathrm{C}_j(p) \log p \tag{5}$$

where $C_j(\xi)$ are non-negative functions of C^∞-class which satisfy the three following conditions

$$\operatorname{Supp} C_j \subset [P_j, 4P_j]$$

$$\sum_{0\leqslant j\leqslant J} C_j(\xi) = \begin{cases} 1 & \text{if } \ 2x < \xi \leqslant P_x \\ O(1) & \text{if } \ x < \xi \leqslant 2x \quad \text{or} \quad P_x < \xi \leqslant 2P_x \\ 0 & \text{ortherwise} \end{cases} \tag{6}$$

$C_j^{(l)}(\xi) \ll P_j^{-l}$, with the implied constant depending on l alone. The error term $O(x)$ in (4) comes from a trivial estimate for the contribution of primes p in the interval $(x,2x]$ which is not completely covered.

4. Application of the sieve method.

A typical sum to be considered is

$$S(x,P) = \sum_{P<p\leqslant 4P} |\mathscr{A}_p| C(p) \log p$$

with $x < P \leqslant 2P_x$. Let $x \geqslant D \geqslant 1$ and let $\{\lambda_d\}_{d\leqslant D}$ be an upper bound sieve of level D, i.e. a sequence of real numbers such that

$$\lambda * 1 \geqslant \mu * 1, \qquad \lambda_1 = 1, \qquad \lambda_d = 0 \quad \text{for} \quad d \geqslant D.$$

We also assume that $|\lambda_d| \leqslant 1$ for all d and that $\lambda_d = 0$ when d is not square-free.

Thus

$$S(x,P) \leqslant \sum_{d\leqslant D} \lambda_d \sum_{m\equiv 0 \,(\mathrm{mod}\, d)} |\mathscr{A}_m| C(m) \log m.$$

By the Poisson summation formula we write

$$|\mathscr{A}_m| = \sum_{\nu^2+1\equiv 0\,(\mathrm{mod}\, m)} \ \sum_{n\equiv \nu\,(\mathrm{mod}\, m)} b(n)$$

$$= \frac{\omega(m)}{m} X + r(\mathscr{A},m)$$

where $\omega(m)$ is the number of incongruent solutions of $\nu^2 + 1 \equiv 0 \,(\mathrm{mod}\, m)$ and

$$r(\mathscr{A},m) = \frac{1}{m} \sum_{h\neq 0} \ \sum_{\nu^2+1\equiv 0\,(\mathrm{mod}\, m)} e\left(-\frac{\nu h}{m}\right) \hat{b}\left(\frac{h}{m}\right). \tag{7}$$

According to the above we write

$$(8)\qquad S(x,P) \leqslant XV(x,P) + R(x,P)$$

where $XV(x,P)$ is considered as a main term

$$V(x,P) = \sum_{d<D} \lambda_d \sum_{m\equiv 0\,(\mathrm{mod}\,d)} \frac{\omega(m)}{m} C(m)\log m$$

and $R(x,P)$ is the total error term

$$R(x,P) = \sum_{d<D} \lambda_d R(x,d,P)$$

with

$$(9)\quad R(x,d,P) = \sum_{h\neq 0}\ \sum_{m\equiv 0\,(\mathrm{mod}\,d)} \frac{C(m)}{m}\log m \sum_{\nu^2+1\equiv 0\,(\mathrm{mod}\,m)} \hat{b}\left(\frac{h}{m}\right) e\left(-\frac{\nu h}{m}\right).$$

5. Transformation of R(x,d,P).

We are searching for D as large as possible for which the estimate

$$(10)\qquad R(x,P) \ll x^{1-\varepsilon}$$

is available. By partial integration $l = [4\varepsilon^{-1}]$ times we get

$$\hat{b}\left(\frac{h}{m}\right) = \left(-2\pi i\frac{h}{m}\right)^{-l} \int b^{(l)}(\xi) e\left(\frac{h}{m}\xi\right) d\xi \ll x\left(\frac{P}{|h|x}\right)^{l} \ll h^{-2}$$

for $|h| \geqslant Px^{\varepsilon-1} = H$, say. Hence truncating the series (9) at $h = H$ we make an error $0(\tau(d)/d)$ which contributes to $R(x,P)$ an admissible amount

$$\sum_{d<D} \frac{\tau(d)}{d} \ll (\log D)^2 \ll (\log x)^2.$$

For the remaining terms we need an explicit formula for the solutions of

$$(11)\qquad \nu^2 + 1 \equiv 0 \pmod{m}.$$

LEMMA 2. (*Gauss*). – *Let* $m > 1$. *If* (11) *is soluble then* m *is represented properly as a sum of two squares*

$$(12)\qquad m = r^2 + s^2, \qquad (r,s) = 1, \qquad r,s > 0.$$

There is a one to one correspondence between the incongruent solutions $\nu (\mathrm{mod}\, m)$ *of* (11) *and the solutions* (r,s) *of* (12) *given by*

$$\frac{\nu}{m} = \frac{\bar{r}}{s} - \frac{r}{s(r^2+s^2)}.$$

Proof. – See [5] and [3], p. 34, eq. (68).

By Lemma 2 we get

$$\sum_{\nu^2+1\equiv 0\,(\mathrm{mod}\, m)} e\left(-\frac{\nu h}{m}\right) = \sum_{\substack{r^2+s^2=m\\ r,s>0,\,(r,s)=1}} e\left(-h\frac{\bar{r}}{s}\right)\left\{1+\mathrm{O}\left(\frac{r|h|}{sm}\right)\right\}$$

whence letting $g(m,h) = \dfrac{\mathrm{C}(m)}{m}(\log m)\hat{b}\left(\dfrac{h}{m}\right)$ we obtain

$$\mathrm{R}(x,d,\mathrm{P}) = \sum_{0<|h|\leqslant \mathrm{H}} \sum_{\substack{(r,s)=1,\, r,\, s>0\\ r^2+s^2\equiv 0\,(\mathrm{mod}\, d)}} g(r^2+s^2,h)e\left(-h\frac{\bar{r}}{s}\right) + \mathrm{O}(d^{-1}\,\mathrm{P}x^{3\varepsilon-1}).$$

Here the error $\mathrm{O}(d^{-1}\,\mathrm{P}x^{3\varepsilon-1})$ contributes to $\mathrm{R}(x;\mathrm{P})$ less than $\mathrm{P}x^{3\varepsilon-1}\log x \ll x^{1-\varepsilon}$ provided $\mathrm{P} \leqslant x^{2-5\varepsilon}$ which we henceforth assume.

For sum over r we apply Poisson's summation formula giving

$$\sum_{\substack{(r,s)=1\\ r^2+s^2\equiv 0\,(\mathrm{mod}\, d)}} g(r^2+s^2,h)e\left(-h\frac{\bar{r}}{s}\right)$$

$$\sum_{\substack{u(\mathrm{mod}\, ds)\\ (u,s)=1\\ u^2+s^2\equiv 0\,(\mathrm{mod}\, d)}} e\left(-h\frac{\bar{u}}{s}\right) \sum_{r\equiv u\,(\mathrm{mod}\, ds)} g(r^2+s^2,h)$$

$$= \frac{1}{ds}\sum_{k} \sum_{\substack{u(\mathrm{mod}\, ds)\\ (u,s)=1\\ u^2+s^2\equiv 0\,(\mathrm{mod}\, d)}} e\left(-h\frac{\bar{u}}{s} - k\frac{u}{ds}\right)\mathrm{G}(h,k;s)$$

where $\mathrm{G}(h,k;s) = \displaystyle\int g(\xi^2+s^2,h)e(k\xi/ds)\,d\xi$. Writing $u = \alpha s + \beta d$ with $\alpha^2+1\equiv 0\,(\mathrm{mod}\, d)$ it becomes

$$\frac{1}{ds}\sum_{\alpha^2+1\equiv 0\,(\mathrm{mod}\, u)} \sum_{k} e\left(-\frac{\alpha k}{d}\right)\mathrm{S}(-h\bar{d},-k;s)\mathrm{G}(h,k;s).$$

For $k = 0$ the Kloosterman sum $\mathrm{S}(-h\bar{d},-k;s)$ reduces to a Ramanujan sum for which we have

$$|\mathrm{S}(-h\bar{d},0;s)| \leqslant (h,s).$$

Therefore the terms with $k = 0$ contribute less than

$$\frac{\tau(d)}{d} \sum_{0<|h|\leqslant \mathrm{H}} \sum_{s\leqslant 2\sqrt{\mathrm{P}}} \frac{(h,s)}{s} \frac{x \operatorname{Log} \mathrm{P}}{\mathrm{P}} \ll \frac{\sqrt{\mathrm{P}}}{d} x^{\varepsilon} \ll \frac{x^{1-\varepsilon}}{d}.$$

Finally

$$\text{(13)} \quad \mathrm{R}(x,d,\mathrm{P}) = \frac{1}{d} \sum_{\alpha^2+1\equiv 0 \,(\mathrm{mod}\, d)} \sum_{0<|h|\leqslant \mathrm{H}} \sum_{k\neq 0} e\left(-\frac{\alpha k}{d}\right)$$

$$\sum_{\substack{s>0\\(s,d)=1}} \frac{1}{s} \mathrm{S}(-h\bar{d},-k;s)\mathrm{G}(h,k;s) + \mathrm{O}\left(\frac{x^{1-\varepsilon}}{d}\right).$$

6. Linear forms in Kloosterman sums.

Let N, M, C $\geqslant 1$ and $f(n,m,c)$ be a function of C^6 class with compact support in [C,2C] with respect to c and satisfying

$$\text{(14)} \quad \left|\frac{\partial^{l_1+l_2+l_3}}{\partial_n^{l_1}\partial_m^{l_2}\partial_c^{l_3}} f(n,m,c)\right| \leqslant \mathrm{N}^{-l_1}\mathrm{M}^{-l_2}\mathrm{C}^{-l_3}, \qquad 0 \leqslant l_1, l_2, l_3 \leqslant 2.$$

In this section we borrow from [1] an estimate for the average of trilinear forms (cf. *Theorem* 11)

$$\mathrm{B}_d^{\pm}(\mathrm{N},\mathrm{M},\mathrm{C}) = \sum_{0<n\leqslant \mathrm{N}} \sum_{0<m\leqslant \mathrm{M}} \sum_{(c,d)=1} b_{m,d}\mathrm{S}(n\bar{d},\pm m;c)f(n,m,c)$$

where $b_{m,d}$ are arbitrary complex numbers.

LEMMA 3. – *If* $f(n,m,c)$ *satisfies* (14) *then for any* $\varepsilon > 0$ *we have*

$$\left(\sum_{\mathrm{D}<d\leqslant 2\mathrm{D}} |\mathrm{B}_d^{+}(\mathrm{N},\mathrm{M},\mathrm{C})|\right)^2 \ll (\mathrm{CDMN})^{\varepsilon}\mathrm{N}\left(\sum_{\substack{0<m\leqslant \mathrm{M}\\ \mathrm{D}<d\leqslant 2\mathrm{D}}} |b_{m,d}|^2\right)$$

$$\times \left\{\frac{\mathrm{D}(\mathrm{DC}^2+\mathrm{MN}+\mathrm{NC}^2)(\mathrm{DC}^2+\mathrm{MN}+\mathrm{MC}^2)}{\mathrm{DC}^2+\mathrm{MN}} + \leqslant \sqrt{\mathrm{D}(\mathrm{D}+\mathrm{M})}\,.\mathrm{C}^3\right\}$$

and the same upper bound holds for $(\Sigma|\mathrm{B}_d^{-}|)^2$, *the constant implied in* $\ll$ *depending on* ε *at most.*

7. Estimation of the error.

In order to make Lemma 3 applicable we first split up the sum over s in $\mathrm{R}(x,d,\mathrm{P})$ into $\ll \log \mathrm{P}$ sums of the type

$$\sum_{(s,d)=1} \frac{a(s)}{s} \mathrm{S}(-h\bar{d}, \pm k; s)\mathrm{G}(h, \mp k; s) \tag{14}$$

where $a(s)$ is a function of C^2 class with support $[\mathrm{S},2\mathrm{S}]$, $\mathrm{S} \leqslant 2\sqrt{\mathrm{P}}$ and satisfying $a^{(l)}(s) \ll \mathrm{S}^{-l}$ for $l = 0,1,2$. The terms with $|k| \geqslant \mathrm{DSP}^{-1/2}x^{3\varepsilon} = \mathrm{K}$, say, can be eliminated trivially : integrate by parts $l = [4\varepsilon^{-1}]$ times with respect to ξ in $\mathrm{G}(h, \pm k, s)$, getting

$$\mathrm{G}(h, \pm k, s) = \left(\frac{-ds}{2\pi i k}\right)^l \int \frac{\partial^l}{\partial \xi^l} g(\xi^2 + s^2, h) e\left(\frac{\xi k}{ds}\right) d\xi$$
$$\ll \left(\frac{ds}{|k|\sqrt{\mathrm{P}}} x^{2\varepsilon}\right)^l \sqrt{\mathrm{P}} \ll k^{-2}x^{-1}.$$

Therefore such terms contribute to $\mathrm{R}(x,d,\mathrm{P})$ less than

$$\frac{\tau(d)}{d} \sum_{0<|h|\leqslant \mathrm{H}} \sum_{k\geqslant 1} \frac{1}{k^2 x} \sum_{0<s\leqslant 2\sqrt{\mathrm{P}}} 1 \ll \frac{\mathrm{P}^{3/2}}{dx^2} x^{\varepsilon} \ll \frac{x^{1-\varepsilon}}{d}.$$

For $0 \leqslant |h| \leqslant \mathrm{H}$, $0 < |k| \leqslant \mathrm{K}$ and $\mathrm{S} < s \leqslant 2\mathrm{S}$ we trivially have

$$\frac{\partial^{l_1+l_2+l_3}}{\partial h^{l_1} \partial k^{l_2} \partial s^{l_3}} \mathrm{G}(h, \mp k, s) \frac{a(s)}{s} \ll |h|^{-l_1} |k|^{-l_2} |s|^{-l_3} \frac{x^{1+12\varepsilon}}{\mathrm{S}\sqrt{\mathrm{P}}} \tag{15}$$

for $0 \leqslant l_1, l_2, l_3 \leqslant 2$. This shows that Lemma 3 is applicable with

$$f(h,k,s) = \frac{\mathrm{S}\sqrt{\mathrm{P}}}{x^{1+13\varepsilon}} \frac{a(s)}{s} \mathrm{G}(h, \mp k, s) \text{ giving}$$

$$\left(\sum_{\mathrm{D}<d\leqslant 2\mathrm{D}} |\mathrm{R}(x,d,\mathrm{P})|\right)^2 \ll x^{2-2\varepsilon} + \frac{x^{2+40\varepsilon}\mathrm{HK}}{\mathrm{D}^2\mathrm{P}}$$
$$\times \sup_{1\leqslant \mathrm{S}\leqslant 2\leqslant\sqrt{\mathrm{P}}} \left\{\frac{\mathrm{D}^2(\mathrm{DS}^2+\mathrm{HK}+\mathrm{HS}^2)(\mathrm{DS}^2+\mathrm{HK}+\mathrm{KS}^2)}{\mathrm{S}^2(\mathrm{DS}^2+\mathrm{HK})} + \mathrm{DS}\sqrt{\mathrm{D}(\mathrm{D}+\mathrm{K})}\right\}$$
$$\ll x^{2-2\varepsilon} + (\mathrm{D}^2 x + \mathrm{DP} + \mathrm{D}x\mathrm{P}^{\frac{1}{2}})x^{48\varepsilon}.$$

Therefore (10) holds if

(16)

$$D \leqslant x^{\frac{1}{2} - 25\varepsilon}, \qquad D \leqslant x^{2-10\varepsilon} P^{-1} \qquad \text{and} \qquad D \leqslant x^{1-10\varepsilon} P^{-\frac{1}{2}}.$$

This result can be compared with Hooley's $D = x^{1-\varepsilon} P^{-3/4}$. .

8. Evaluation of the main term.

For d square-free with $\omega(d) \neq 0$ consider

$$L(s,d) = \sum_{m=1}^{\infty} \frac{\omega(dm)}{\omega(d)} m^{-s}.$$

LEMMA 4. – *We have*

$$L(s,d) = \frac{\zeta(s) L(s,\chi_4)}{\zeta(2s)} \prod_{p|d} \left(1 + \frac{1}{p^s}\right)^{-1}. \tag{17}$$

Proof. – Follow the arguments of [3] on pp. 31-32 and equation (6.1). Writing

$$C(m) \frac{\log m}{m} = \frac{1}{2\pi i} \int_{(\sigma)} R(s) m^{-s}\, ds, \qquad \sigma > 0$$

by Mellin's inversion formula and partial integration two times

$$R(s) = \int C(\xi) \frac{\log}{\xi} \xi^{s-1}\, d\xi \ll (|s|+1)^{-2} P^{\sigma-1} \log P.$$

Therefore

$$\begin{aligned}
\sum_{m \equiv 0\,(\mathrm{mod})} \frac{\omega(m)}{m} c(m) \log m &= \frac{1}{2\pi i} \int_{(\sigma)} R(s) \frac{\omega(d)}{d^s} L(s,d)\, ds \\
&= R(1) \frac{\omega(d)}{d} \frac{L(1,\chi_4)}{\zeta(2)} \prod_{p|d} \left(1 + \frac{1}{p}\right)^{-1} + \frac{1}{2\pi i} \int_{\left(\frac{1}{2}\right)} R(s) \frac{\omega(d)}{d^s} L(s,d)\, ds \\
&= \frac{\omega(d)}{d} \prod_{p|d} \left(1 + \frac{1}{p}\right)^{-1} \frac{L(1,\chi_4)}{\zeta(2)} \int C(\xi) \frac{\log \xi}{\xi} d\xi + O\left(\frac{\tau^2(d)}{\sqrt{dP}} \log P\right)\cdot
\end{aligned}$$

This yields

$$\mathrm{V}(x,\mathrm{P}) = \left(\sum_{d<\mathrm{D}} \frac{\lambda_d}{d}\rho(d)\right)\frac{\mathrm{L}(1,\chi_4)}{\zeta(2)}\int \mathrm{C}(\xi)\frac{\log\xi}{\xi}\,d\xi + \mathrm{O}\left(\sqrt{\frac{\mathrm{D}}{\mathrm{P}}}(\log x)^4\right)$$

where $\rho(d) = \omega(d)\prod_{p|d}\left(1+\frac{1}{p}\right)^{-1}$. Now we specify λ_d to be those of the Rosser sieve giving (see [4])

$$\begin{aligned}\sum_{d<\mathrm{D}} \lambda_d \frac{\rho(d)}{d} &= \prod_{p<\mathrm{D}}\left(1-\frac{\rho(p)}{p}\right)\left(2e^{\gamma}+\mathrm{O}\left(\frac{1}{\log \mathrm{D}}\right)\right)\\ &= \prod_{p<\mathrm{D}}\left(1-\frac{1}{p}\right)\frac{\zeta(2)}{\mathrm{L}(1,\chi_4)}\left(2e^{\gamma}+\mathrm{O}\left(\frac{1}{\log \mathrm{D}}\right)\right)\\ &= \frac{2\zeta(2)}{\mathrm{L}(1,\chi_4)}\frac{1}{\log \mathrm{D}}\left(1+\mathrm{O}\left(\frac{1}{\log \mathrm{D}}\right)\right)\end{aligned}$$

by the Mertens prime number theorem. Hence we conclude that

$$\mathrm{V}(x,\mathrm{P}) = \frac{2}{\log \mathrm{D}}\int \mathrm{C}(\xi)\frac{\log\xi}{\xi}\,d\xi\left(1+\mathrm{O}\left(\frac{1}{\log \mathrm{D}}\right)\right).$$

We choose D equal to $x^{1-10\varepsilon}\mathrm{P}^{-\frac{1}{2}}$ thus by (6) the total main term is equal to

$$\begin{aligned}\mathrm{X}\sum_{0\leqslant j\leqslant \mathrm{J}} \mathrm{V}(x,\mathrm{P}_j) &= 2(1+\mathrm{O}(\varepsilon))\mathrm{X}\int_x^{\mathrm{P}_x} \frac{\operatorname{Log}\xi}{\xi \operatorname{Log}(x/\sqrt{\xi})}\,d\xi\\ &= 2(1+\mathrm{O}(\varepsilon))\mathrm{X}\int_1^{\theta}\frac{t\,dt}{1-t/2}\operatorname{Log} x\\ &= (1+\mathrm{O}(\varepsilon))f(\theta)\mathrm{X}\operatorname{Log} x\end{aligned}$$

where $f(\theta) = 4(1-\theta-2\operatorname{Log}(2-\theta))$ and is less than 1 for $\theta = 1.202\,468\,87$. The proof the Theorem follows from this and (3).

One may note that the truth of Selberg's eigenvalue conjecture leads to the lower bound $x^{\sqrt{3/2}-\varepsilon}$ for P_x.

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