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IMPROVEMENT OF GRAUERT-RIEMENSCHNEIDER'S THEOREM FOR A NORMAL SURFACE

by Jean GIRAUD

1. Vanishing theorem.

1.1. A *surface* is a noetherian, excellent, normal scheme of dimension 2. A desingularization of X is a proper and birational map $f: \tilde{X} \rightarrow X$ such that \tilde{X} is regular. The set

$$(1) \quad \text{Sing}(f) = \{x \in X, \dim(f^{-1}(x)) > 0\}$$

is made up of finitely many closed points and f is an isomorphism above

$$(2) \quad X_f = X - \text{Sing}(f) \subset X_{\text{reg}} = \{x \in X, 0_{x,x} \text{ is regular}\}.$$

We usually denote by E_i the irreducible components of

$$(3) \quad E(f) = f^{-1}(\text{Sing}(f))$$

and for $A = \mathbf{N}, \mathbf{Z}$ or \mathbf{Q} , we let

$$(4) \quad \text{NS}(f, A) = \bigoplus A E_i.$$

We do not assume that $X_f = X_{\text{reg}}$, hence X itself may be regular. For any $V = \sum V_i \cdot E_i \in \text{NS}(f, \mathbf{Q})$, we write

$$(5) \quad V \geq 0 \text{ when all } V_i \text{ are } \geq 0$$

$$(6) \quad V \geq 0 \text{ when all } -V \cdot E_i \text{ are } \geq 0.$$

Note that the minus sign is justified by

$$(7) \quad V \geq 0 \Rightarrow V \geq 0.$$

To prove (7) we let $V = V_+ - V_-$; since $V \geq 0$, we have

$0 \leq -V \cdot V = -V_+ \cdot V_- + V_-^2 \leq V_-^2$, hence $V_- = 0$, since the intersection matrix is negative definite. We introduce the dual basis of $\text{NS}(f, \mathbf{Q})$

$$(8) \quad E_i^* \text{ defined by } E_i^* \cdot E_j = -\delta_{ij}$$

and we observe that

$$(9) \quad E_i^* \geq 0, \quad dE_i^* \in \text{NS}(f, \mathbf{N})$$

where d is the absolute value of the determinant of the intersection matrix.

LEMMA 1.2. — *For any $V \in \text{NS}(f, \mathbf{Q})$ there exists a unique $[V] \in \text{NS}(f, \mathbf{Z})$ such that*

- (i) $V \leq [V]$,
- (ii) if $W \in \text{NS}(f, \mathbf{Z})$ and if $V \leq W$ then $[V] \leq W$.

We will prove that $[V]$ is the infimum for the usual order relation of $E(V) = \{W \in \text{NS}(f, \mathbf{Z}), V \leq W\}$. Let $N \in \mathbf{Z}$ be such that $dN \leq \inf(V \cdot E_i)$; we have $-dN \sum E_i^* \in E(V)$, hence $E(V)$ is non empty. For $i = 1, 2$, let $W_i = \sum W_{i,j} E_j \in E(V)$ and let $Z = \sum Z_j E_j$ with $Z_j = \inf(W_{1,j}, W_{2,j})$. By Artin's trick we prove that $Z \in E(V)$ as follows. For any j , we have $Z_j = W_{1,j}$ or $Z_j = W_{2,j}$. By symmetry we can assume that $Z_j = W_{1,j}$ and we get

$$Z \cdot E_j = W_{1,j} E_j^2 + \sum_{k \neq j} Z_k E_k \cdot E_j \leq W_{1,j} \cdot E_j \leq V \cdot E_j$$

hence $Z \geq V$. To conclude, we note that the coordinates of any $W = \sum W_i E_i \in E(V)$ are bounded from below since $W_i = -W \cdot E_i^* \geq -V \cdot E_i^*$ since E_i^* is ≥ 0 . Observe the obvious

$$(1) \quad [V+W] \leq [V] + [W]; \quad [V+E] = [V] + E \text{ if } E \in \text{NS}(f, \mathbf{Z}).$$

We also let

$$(2) \quad [V] = -[-V] \text{ in such a way that } [V] \leq V \leq [V].$$

1.3. Let L be an invertible sheaf on \tilde{X} . We define

$$(1) \quad e_f(L) \in \text{NS}(f, \mathbf{Q}) \quad \text{by} \quad e_f(L) \cdot E_i = \deg(L|E_i) \text{ for any } i.$$

We also write $L \geq 0$ instead of $e_f(L) \geq 0$ and this means $L.E_i \leq 0$ for all i . We will often drop the subscript f . Sending V to $0_{\tilde{X}}(V)$ we identify $NS(f, Z)$ to a subgroup of $Pic(\tilde{X})$ and since $V = e_f(0_{\tilde{X}}(V))$, $0_{\tilde{X}}(V) \geq 0$ is equivalent to $V \geq 0$. Hence, when we write $Pic(\tilde{X})$ additively, we can safely write V in place of $0_{\tilde{X}}(V)$ and $L + V$ in place of $L(V) = L \otimes 0_{\tilde{X}}(V)$. We will sometimes write $[L]$ instead of $[e_f(L)]$.

1.4. We can also give an algorithmic description of $[V]$ as follows. Start with $Z \in NS(f, Z)$ such that $Z \leq [V]$. For instance, if $V = \sum V_i E_i$ let $Z = \sum V'_i E_i$ where V'_i is the smallest integer $\geq V_i$. If $Z \neq [V]$ there must exist a i such that $Z.E_i > V.E_i$ and we still have $Z + E_i \leq [V]$. In fact, since $V \leq [V]$, we have $([V] - Z).E_i \leq (V - Z).E_i < 0$, hence $([V] - Z) \geq E_i$ since $[V] - Z$ is effective with integral coefficients. We now replace Z by $Z + E_i$ and reach $[V]$ in a finite number of steps.

VANISHING THEOREM 1.5. — *Let $f: \tilde{X} \rightarrow X$ be a desingularization of a normal surface X , let $E = f^{-1}(\text{Singl}(f))$ and let L be an invertible sheaf on \tilde{X} .*

(i) *If $[L] \geq 0$ then $H_E^1(\tilde{X}, L) = 0$.*

(ii) *If $[L] \geq 0$ then $f_*(L)$ is reflexive.*

(iii) *Let K be the dualizing sheaf of \tilde{X} . If $[K - L] \geq 0$ then $R^1 f_*(L) = 0$.*

1.5.1. To prove (i) we let $M = [L]$ and $L' = L(-M)$ in such a way that $[L'] = 0$ and $M \geq 0$, $M \in NS(f, N)$. For any $V \in NS(f, N)$, $V \neq 0$, there exists an E_i such that $(L' + V).E_i < 0$. Otherwise we would have $L' + V \leq 0$ hence $L' \leq -V$, hence $0 = [L'] \leq -V < 0$ which is impossible. We observe that E_i must be contained in the support of V , otherwise we would have $V.E_i \geq 0$, hence

$$(L' + V).E_i \geq ([L'] + V).E_i = V.E_i \geq 0.$$

Furthermore, since $M \geq 0$, we have

$$(L + V).E_i = (L' + M + V).E_i \leq (L' + V).E_i < 0.$$

As a consequence we get

$$(1) \quad V - E_i \in NS(f, N) \quad \text{and} \quad (L + V).E_i < 0.$$

As a consequence we get $H^0(E_i, L(V)|E_i) = 0$ hence the map

$$(2) \quad H^0(V - E_i; L(V - E_i)|(V - E_i)) \rightarrow H^0(V; L(V)|V)$$

is surjective. By induction on V , we conclude that, if $[L] \geq 0$, we have

$$(3) \quad H^0(V, L(V)|V) = 0 \quad \text{for any } V \in \text{NS}(f; \mathbf{N})$$

hence $H_E^1(\tilde{X}; L) = \varinjlim H^0(V; L(V)|V) = 0$. This proves (i) and we get (iii) by duality.

1.5.2. To prove (ii), we can assume that $[L] = 0$ since $f_*(L)$ reflexive implies that, for any $V \in \text{NS}(f; \mathbf{N})$, the map $f_*(L) \rightarrow f_*(L(V))$ is an isomorphism. Let $u: f_*(L) \rightarrow f_*(L)^{vv}$ be the map from $f_*(L)$ to its bidual. Since L is invertible, we know that u is an isomorphism over the open subset X_f of X . Since X is normal, we know that $\text{coker}(u)$ is finite and since f is proper, this implies the existence of some $V \in \text{NS}(f; \mathbf{N})$ such that $f_*(L)^{vv} = f_*(L(V))$. Since $[L] = 0$, we know that $H^0(V, L(V)|V) = 0$ hence $f_*(L) \rightarrow f_*(L(V))$ is an isomorphism and this concludes the proof.

1.5.3. We do not really need duality for surfaces to state and prove (iii). In fact, we can define

$$(1) \quad K_f \in \text{NS}(f; \mathbf{Q}) \quad \text{by} \quad (K_f + E_i) \cdot E_i = -2\chi(0_{E_i}) \quad \text{for all } i,$$

and write the hypothesis $[K_f - e_f(L)] \geq 0$. As for the proof it runs parallel to the proof of (i) and uses the fact that $H^1(E_i, M) = 0$ if M is an invertible sheaf on the reduced and irreducible Gorenstein curve E_i with $\text{deg}(M) > -2\chi(0_{E_i})$; details are left to the reader. We define $C(f)$ and C_+ in $\text{NS}(f; \mathbf{N})$ by

$$(2) \quad [K_f] = C_+ - C(f).$$

Observe that if we denote by $K_{\tilde{X}}$ and K_X the dualizing sheaves of \tilde{X} and X we have

$$(3) \quad K_f = e_f(K_{\tilde{X}}) \quad \text{and} \quad K_X = f_*(K_{\tilde{X}}(C(f))).$$

The first formula comes from (1). For the second observe that $[K_{\tilde{X}}(C(f))] = K_{\tilde{X}} + C(f) = C_+ \geq 0$ hence its direct image is reflexive by (1.5(ii)) and coincide with K_X over X_f , hence it must be K_X .

COROLLARY 1.6. — *Under the hypothesis of (1.5), let L be an invertible sheaf on \tilde{X} such that $[L] = 0$. Then $f_*(L)$ is reflexive and the map $u: \mathbf{R}^1 f_*(L) \rightarrow H^1(C(f), L|C(f))$ is an isomorphism.*

We know that u is surjective. Let us introduce $V \in \text{NS}(f, \mathbb{Z})$ such that $[K_f + C(f) - e_f(L) - V] = 0$. We claim that $V \geq 0$. In fact $0 = [K_f + C(f) - e_f(L) - V] \geq K_f + C(f) - e_f(L) - V$ hence $e_f(L) + V \geq K_f + C(f)$ hence $V = [e_f(L) + V] \geq [K_f + C(f)] = C_+ \geq 0$. We have a diagram

$$\begin{array}{ccc}
 \mathbf{R}^1 f_* (L) & \xrightarrow{u} & \mathbf{R}^1 f_* (L|C(f)) \\
 \downarrow v & & \downarrow \\
 \mathbf{R}^1 f_* (L(V)) & \xrightarrow{w} & \mathbf{R}^1 f_* (L(V)|C(f)).
 \end{array}$$

By (1.5.1(3)), the morphism v is injective hence it is enough to show that w is injective. This follows from $\mathbf{R}^1 f_* (L(V - C(f))) = 0$ which comes from (1.5 (iii)) since $[K_f - e_f(L) - V + C(f)] = 0$.

COROLLARY 1.7. — *We have $\mathbf{R}^1 f_* (0_{\tilde{X}}) \simeq H^1(C(f); 0_{C(f)})$ and $\mathbf{R}^1 f_* (0_{\tilde{X}}) = 0$ is equivalent to $C(f) = 0$.*

We get the isomorphism by (1.6) applied to $L = 0_{\tilde{X}}$. Hence $C(f) = 0$ implies $\mathbf{R}^1 f_* (0_{\tilde{X}}) = 0$. Conversely, if $\mathbf{R}^1 f_* (0_{\tilde{X}}) = 0$ and $C(f) \neq 0$, we have $\chi(0_{C(f)}) > 0$ which means

$$\begin{aligned}
 0 > (K_{\tilde{X}} + C(f)) \cdot C(f) &= (K_f + C(f)) \cdot C(f) \\
 &\geq ([K_f] + C(f)) \cdot C(f) = C_+ \cdot C(f) \geq 0
 \end{aligned}$$

a contradiction.

PROPOSITION 1.8. — *Let $f: \tilde{X} \rightarrow X$ be a desingularization of a normal surface X and let M be a reflexive sheaf of rank one on X . There exists a pair (L, u) where L is an invertible sheaf on \tilde{X} such that $[e_f(L)] = 0$ and $u: f_*(L)|X_f \simeq M|X_f$ is an isomorphism. The pair (L, u) is unique up to a unique isomorphism. Furthermore $M = f_*(L)$.*

1.8.1. It is clear that there exists a pair (L', u') , where L' is invertible on \tilde{X} and $u': f_*(L')|X_f \simeq M|X_f$ is an isomorphism. If (L'', u'') is another solution, we canonically have $L'' = L'(V)$, $V \in \text{NS}(f, \mathbb{Z})$, hence we get existence and uniqueness since $[e_f(L'(V))] = [e_f(L')] + V$. By (1.5(ii)), $f_*(L)$ is reflexive since $[e_f(L)] = 0$, hence $f_*(L) \simeq M$ since both are reflexive and coincide over X_f .

1.8.2. We denote by $f^v(M)$ the invertible sheaf on \tilde{X} characterized by $[f^v(M)] = 0$ and $f_*(f^v(M)) = M$. We observe that we have

$$(1) \quad e_f(f^v(M)) \in \text{NS}(f, \mathbf{Q}), \quad e_f(f^v(M)) \leq 0,$$

but this element is not necessarily zero. However, if M is *invertible*, we obviously have $f^v(M) = f^*(M)$ since $e_f(f^*(M)) = 0$. More generally, it is useful to compare $f^v(M)$ with another lifting \tilde{M} defined as follows

$$(2) \quad M' = f^*(M)/\text{torsion} \quad \tilde{M} = M'^{vv} = \text{bidual of } M'.$$

COROLLARY 1.8.3. — *Let M be a reflexive sheaf of rank one on X . Then $M \leq 0$ and $[\tilde{M}] \leq 0$. We have $f^v(M) = \tilde{M}(-[\tilde{M}])$.*

Since M' is torsion free of rank one it is invertible except at finitely many closed points; hence \tilde{M} is invertible. To prove that $\tilde{M} \leq 0$, assume that there exists E_i such that $\tilde{M}.E_i < 0$. Then $f_*(\tilde{M}(-E_i)) = f_*(\tilde{M}) = M$. In a neighborhood U of the generic point of E_i , we have $M' = \tilde{M}$, hence \tilde{M} is generated on a possibly smaller neighborhood U' by sections of M , hence we cannot have $f_*(\tilde{M}(-E_i)) = f_*(\tilde{M})$. By definition of $[\tilde{M}]$, we get $[\tilde{M}] \leq 0$ out of $\tilde{M} \leq 0$. We deduce $f^v(M) = \tilde{M}(-[\tilde{M}])$ from $[\tilde{M}(-[\tilde{M}])] = 0$.

COROLLARY 1.8.4. — *Assume that \tilde{X} dominates some desingularization X' of X . We have $f = gh$ with $\tilde{X} \xrightarrow{h} X' \xrightarrow{g} X$. For any reflexive sheaf of rank one M on X we have $f^v(M) = h^*(g^v(M))$.*

Since \tilde{X} and X' are regular and h proper and birational, we have $h_*h^*(g^v(M)) = g^v(M)$ hence $f_*h^*(g^v(M)) = M$, hence we only have to prove that $[e_f(h^*(g^v(M)))] = 0$. We use the map

$$(1) \quad h^* : \text{NS}(g, \mathbf{Q}) \rightarrow \text{NS}(f, \mathbf{Q})$$

which preserves integrality, positivity and the intersection numbers. We still have to prove that we have, for any $V \in \text{NS}(g, \mathbf{Q})$

$$(2) \quad h^*([V]) = [h^*(V)].$$

For any $E \in \text{NS}(f, \mathbf{N})$, we have $h^*(V).E = V.h_*(E) \geq [V].h_*(E) = h^*([V]).E$, hence $h^*(V) \leq h^*([V])$, hence $[h^*(V)] \leq h^*([V])$, in other words $h^*([V]) = [h^*(V)] + A$, $A \in \text{NS}(f, \mathbf{N})$.

From $h^*(V) \leq [h^*(V)]$, we deduce $V \leq h_*([h^*(V)]) = h_*h^*([V]) - h_*(A) = [V] - h_*(A)$. By definition of $[V]$, we deduce that $[V] \leq [V] - h_*(A)$, hence $h_*(A) = 0$, hence $A \in \text{NS}(h, \mathbf{N})$. We get $0 = h^*(V)$. $A \geq [h^*(V)]$. $A = h^*([V])$. $A - A^2 = -A^2$, hence $A = 0$.

PROPOSITION 1.9. — *Let $f: \tilde{X} \rightarrow X$ and assume that $\mathbf{R}^1 f_*(\mathcal{O}_{\tilde{X}}) = 0$.*

(i) *Let M be a reflexive sheaf of rank one on X . We have $f^v(M) = f^*(M)/\text{torsion}$ and $\mathbf{R}^1 f_*(f^v(L)) = 0$.*

(ii) *Let L be an invertible sheaf on \tilde{X} such that $L \leq 0$. The map $f^*f_*(L) \rightarrow L$ is surjective and $\mathbf{R}^1 f_*(L) = 0$.*

We first prove (ii). We let $M = f_*(L)$, $L_0 = \text{Im}(f^*(M) \rightarrow L)$, $L_1 = \text{bidual of } L_0$ and we get $L_0 \subset L_1 \subset L$ and $M \subset f_*(L_0) \subset f_*(L_1) \subset f_*(L) = M$. Since $\mathbf{R}^1 f_*(L_0) = 0$, we get $f_*(L_1/L_0) = 0$ and this implies $L_1/L_0 = 0$ since L_1/L_0 has finite support. Let us define $V \in \text{NS}(f, \mathbf{N})$ by $L = L_0(V)$. We have $f_*(L|V) = 0$, hence $\chi(K_V) = \chi(L|V) - L.V = -h^1(L|V) - L.V \leq -L.V$. Since $L \leq 0$, we get $-L.V \leq 0$ hence $\chi(\mathcal{O}_V) \leq 0$, hence $V = 0$ since $h^1(\mathcal{O}_V) = 0$. This means that $L_0 = L$, from which $\mathbf{R}^1 f_*(L) = 0$ follows.

To prove (i) we let $L = f^v(M)$ and apply (ii) to L (see (1.8.3)); recall that $M = f_*f^v(L)$ by (1.8).

As an exercise, we now deduce some well known facts about rational singularities.

PROPOSITION 1.10. — *Let $f: \tilde{X} \rightarrow X$ be a desingularization and assume that $\mathbf{R}^1 f_*(\mathcal{O}_{\tilde{X}}) = 0$. Let I be an ideal of \mathcal{O}_X . The following conditions are equivalent*

- (i) *I is integrally closed and $\text{IO}_{\tilde{X}}$ is invertible,*
- (ii) *$I = f_*((\text{IO}_{\tilde{X}})^{vv})$,*
- (iii) *There exists an effective divisor D on \tilde{X} , with $\mathcal{O}_{\tilde{X}}(-D) \geq 0$ such that $I = f_*(\mathcal{O}_{\tilde{X}}(-D))$.*

Furthermore, if we have (iii), we necessarily have $\text{IO}_{\tilde{X}} = \mathbf{M}_{\tilde{X}}(-D)$.

If $\text{IO}_{\tilde{X}}$ is invertible, then \tilde{X} dominates the normalized blowing up of I , hence $f_*(\text{IO}_{\tilde{X}})$ is the integral closure of I . Hence (i) \Rightarrow (ii), since in that case $\text{IO}_{\tilde{X}} = \text{IO}_{\tilde{X}}^{vv}$. Since $(\text{IO}_{\tilde{X}})^{vv} \leq 0$, we have $\text{IO}_{\tilde{X}}^{vv} = \mathcal{O}_{\tilde{X}}(-D)$, with D effective (not necessarily vertical) and $D \geq 0$; hence (ii) \Rightarrow (iii). If we assume (iii), then I is integrally closed and (1.9 (ii)) implies that

$IO_{\tilde{X}} = O_{\tilde{X}}(-D)$, hence (iii) \Rightarrow (i) and we have also proven the last assertion.

It follows that we have a 1-1-correspondance between ideals I of O_X which satisfy the above conditions and effective divisors D on X with $D \geq 0$. We have that I is primary if and only if D is vertical ($\dim f(D) = 0$) and I is reflexive (i.e. the ideal of a Weil divisor) if and only if $[D] = 0$. Observe that (1.9(i)) tells us that a reflexive I satisfy (i). Observe that if I is the maximal ideal of some closed point x , then we must have (ii), hence the corresponding D must be the connected component of the fundamental cycle corresponding to x . To complete the picture, recall Lipman's result saying that the set of ideals satisfying (i) is stable by multiplication, which means that $f_*(O_{\tilde{X}}(-D-E)) = f_*(O_{\tilde{X}}(-D))f_*(O_{\tilde{X}}(-E))$ if D and E are effective and $D \geq 0, E \geq 0$.

Example 1.11. — We now assume that $f: \tilde{X} \rightarrow X$ is the *minimal* desingularization and that X is the spectrum of a local ring R with algebraically closed residue field, in such a way that $K_{\tilde{X}} \leq 0$; this implies $[K_f] = -C(f)$. Assume that K_X is *invertible* which means that R is a *Gorenstein ring*. Since $f^*(K_X) = K_{\tilde{X}}(V)$ for some vertical V and $e_f(f^*(K_X)) = 0$, we conclude that $V = K_f$, hence K_f has integral coefficients, hence $K_f = -C(f)$ and $K_{\tilde{X}}(C(f)) = f^*(K_X) \approx O_{\tilde{X}}$.

If we have rational singularity, we know that $C(f) = 0$, hence $K_f = 0$, hence we get the well known result that $E_i^2 = -2$ for all i . If $C(f) \neq 0$, we still have that the dualizing sheaf $K_{C(f)} = K_{\tilde{X}}(C(f)) \otimes O_{C(f)}$ is isomorphic to $O_{C(f)}$. The converse is also true, see for instance [2].

2. Genus formula.

2.1. Let k be a field and X be a proper k -scheme of dimension 2 which is normal. We want to study Weil divisors of X , or equivalently reflexive sheaves of rank one on X . Such a sheaf M is determined by the invertible sheaf $i^*(M)$ since $M \rightarrow i_*i^*(M)$ is an isomorphism where $i: X_{\text{reg}} \rightarrow X$ is the inclusion of the open set X_{reg} made up of regular points of X . In other words, we study $\text{Pic}(X_{\text{reg}})$. Let $f: \tilde{X} \rightarrow X$ be a desingularization of X , we have an exact sequence

$$(1) \quad 0 \rightarrow \text{NS}(f, Z) \xrightarrow{a} \text{Pic}(\tilde{X}) \xrightarrow{b} \text{Pic}(X_{\text{reg}}) \rightarrow 0$$

where $a(D)$ is the class of $\mathcal{O}_{\tilde{X}}(D)$ and b is induced by the inclusion $j: X_{\text{reg}} \rightarrow X$. The canonical lifting $f^v(M)$ of a reflexive sheaf of rank one M on X defined in (1.8.2) gives us a *non-linear* section of b . By composition with the usual map

$$(2) \quad e_f: \text{Pic}(\tilde{X}) \rightarrow \text{NS}(f, \mathbf{Z})^* \subset \frac{1}{d} \text{NS}(f, \mathbf{Z}) \subset \text{NS}(f, \mathbf{Q}), \quad (1.3)$$

we get a class

$$(3) \quad e_f(f^v(M)) \in \frac{1}{d} \text{NS}(f, \mathbf{Z})$$

which can only take a *finite number of values* since $[e_f(f^v(M))] = 0$. Of course, this is still non linear. To recover the classical linear theory of [6], we recall that, for $A = \mathbf{Z}$ or \mathbf{Q} , the quadratic module $\text{NS}(f, A)$ lies inside the Néron-Severi group $\text{NS}(\tilde{X}, A)$ and we define

$$(4) \quad \text{NS}(X, A) = \text{orthogonal of } \text{NS}(f, A) \text{ inside } \text{NS}(\tilde{X}, A)$$

which gives an orthogonal decomposition

$$(5) \quad \text{cl}(f^v(M)) = \text{cl}(M) + e_f(f^v(M))$$

inside $\text{NS}(\tilde{X}, \mathbf{Q}) = \text{NS}(X, \mathbf{Q}) \oplus \text{NS}(f, \mathbf{Q})$. We also have another linear invariant

$$(6) \quad d_f(M) = \text{class of } e_f(f^v(M)) \text{ in } \text{NS}(f, \mathbf{Z})^*/\text{NS}(f, \mathbf{Z}).$$

It is clear that the two linear invariants $\text{cl}(M)$ and $d_f(M)$ can be computed with any lifting L of M , namely $\text{cl}(M)$ is the orthogonal projection on $\text{NS}(X, \mathbf{Q})$ of $\text{cl}(L)$ and $d_f(M)$ is the image of $e_f(L)$; proof: $L = f^v(M)(D)$ for some $D \in \text{NS}(f, \mathbf{Z})$. For instance, if K_X and $K_{\tilde{X}}$ are the dualizing sheaves of X and \tilde{X} we have an orthogonal decomposition

$$(7) \quad \text{cl}(K_{\tilde{X}}) = \text{cl}(K_X) + K_f \quad (1.5.3)$$

and

$$(8) \quad e_f(K_X) = K_f - [K_f].$$

If we introduce the effective divisor $C(f) = [K_f]_-$ as in (1.5.3) we know that the multi-degree of $f^v(M)|_{C(f)}$ can only take a finite number of

values, hence the same holds for the length of

$$(9) \quad \mathbf{R}^1 f_* (f^v(\mathbf{M})) = \mathbf{H}^1(\mathbf{C}(f); f^v(\mathbf{M})|\mathbf{C}(f)), \quad (1.6).$$

THEOREM 2.2. — *Let \mathbf{M} be a reflexive sheaf of rank one on \mathbf{X} . We have*

$$(1) \quad \chi(\mathbf{M}) = \frac{1}{2}(\text{cl}(\mathbf{M}), \text{cl}(\mathbf{M}) - \text{cl}(\mathbf{K}_{\mathbf{X}})) + \chi(\mathbf{O}_{\mathbf{X}}) + \frac{1}{2} e(\mathbf{M}) d(\mathbf{M})$$

where the scalar product is computed in $\text{NS}(\mathbf{X}, \mathbf{Q})$ and for any desingularization $f: \tilde{\mathbf{X}} \rightarrow \mathbf{X}$ of \mathbf{X} we have

$$(2) \quad e(\mathbf{M}) = (e_f(f^v(\mathbf{M})), e_f(f^v(\mathbf{M})) - \mathbf{K}_f)$$

$$(3) \quad d(\mathbf{M}) = \lg \mathbf{R}^1 f_* (f^v(\mathbf{M})) - \lg \mathbf{R}^1 f_* (\mathbf{O}_{\tilde{\mathbf{X}}}) \\ = h^1(\mathbf{C}(f); f^v(\mathbf{M})|\mathbf{C}(f)) - h^1(\mathbf{C}(f); \mathbf{O}_{\mathbf{C}(f)}) \quad (1.5.3).$$

Proof. — Apply the usual Riemann-Roch formula to $f^v(\mathbf{M}) = \mathbf{L}$. Since $\mathbf{M} = f_* (f^v(\mathbf{M}))$, we get

$$\chi(\mathbf{M}) = \chi(\mathbf{L}) + \lg \mathbf{R}^1 f_* (\mathbf{L}) = (\mathbf{L}, \mathbf{L} - \mathbf{K}_{\tilde{\mathbf{X}}})/2 + \chi(\mathbf{O}_{\tilde{\mathbf{X}}}) + \lg \mathbf{R}^1 f_* (\mathbf{L}) \\ = \chi(\mathbf{L}, \mathbf{L} - \mathbf{K}_{\tilde{\mathbf{X}}})/2 + \chi(\mathbf{O}_{\mathbf{X}}) + \lg \mathbf{R}^1 f_* (\mathbf{L}) - \lg \mathbf{R}^1 f_* (\mathbf{O}_{\tilde{\mathbf{X}}})$$

and split the scalar product $(\mathbf{L}, \mathbf{L} - \mathbf{K}_{\tilde{\mathbf{X}}})$ according to the orthogonal decomposition $\text{NS}(\tilde{\mathbf{X}}, \mathbf{Q}) = \text{NS}(\mathbf{X}, \mathbf{Q}) + \text{NS}(f, \mathbf{Q})$.

According to (1.8.4), the terms $e(\mathbf{M})$ and $d(\mathbf{M})$ do not depend on the choice of the desingularization. Furthermore we have

$$(4) \quad e(\mathbf{M}) = \sum_{x \in \text{Sing}(\mathbf{X})} e(\mathbf{M}, x), \quad d(\mathbf{M}) = \sum_{x \in \text{Sing}(\mathbf{X})} d(\mathbf{M}, x)$$

where $e(\mathbf{M}, x)$ and $d(\mathbf{M}, x)$ are defined by replacing \mathbf{X} by $\text{Spec}(\mathbf{O}_{\mathbf{X}, x})$, or even by $\text{Spec}(\hat{\mathbf{O}}_{\mathbf{X}, x})$ as is easily seen. Furthermore $e(\mathbf{M}, x) = d(\mathbf{M}, x) = 0$ if \mathbf{M} is invertible in a neighborhood of x . Furthermore $d(\mathbf{M}, x) = 0$ if $\mathbf{O}_{\mathbf{X}, x}$ is a rational singularity (1.7). We also know that $e(\mathbf{M})$ and $d(\mathbf{M})$ can only take a finite number of values.

For $n \in \mathbf{Z}$, we let $\mathbf{M}^n = i_* (i^*(\mathbf{M})^n) = \text{bidual of } \mathbf{M}^{\otimes n}$ and we have

$$(5) \quad \chi(\mathbf{M}^n) = \frac{n^2}{2} (\text{cl}(\mathbf{M}), \text{cl}(\mathbf{M})) - \frac{n}{2} (\text{cl}(\mathbf{M}), \text{cl}(\mathbf{K}_{\mathbf{X}})) \\ + \chi(\mathbf{O}_{\mathbf{X}}) + e(\mathbf{M}^n)/2 + d(\mathbf{M}^n).$$

Observe that $e(M^n) = 0$ if the determinant of the intersection matrix divides n . In fact, in that case, we have $d_f(M^n) = 0$ hence $e_f(f^v(M)) = [e_f(f^v(M))] = 0$. For instance, if X is the Satake compactification of some Hilbert-Blumenthal surface and $M = K_X$, we can get an a priori proof of the formula for the rank of the vector spaces $H^0(X, K_X^n)$ of automorphic forms [3].

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