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CONFORMAL CURVATURE FOR THE NORMAL BUNDLE OF A CONFORMAL FOLIATION

by Angel MONTESINOS

1. Introduction.

In a previous paper [4], the present author proved the following theorem: if $\mathcal{V}$ is a conformal foliation of codimension q, then $\mathrm{Pont}^k(\nu\,;\mathbf{R}) = 0$ for $k > q$, being ν the normal bundle of $\mathcal{V}$. This result generalizes Pasternack's theorem [6] for riemannian foliations, and was previously proved for $q \geqslant 3$ by S. Nishikawa and H. Sato [5].

Our technique suggested the existence of a conformal curvature tensor for the normal bundle. We shall define it and derive some properties generalizing the usual ones. For instance, the normal bundle is conformally flat if and only if its conformal curvature vanishes. We also prove that the Pontrjagin ring of the normal bundle can be expressed in terms of the conformal curvature. This provides an alternative proof for the quoted theorem.

2. Double forms.

We will use the language of double forms, following a line that generalizes the one introduced by G. de Rham [7], A. Gray [2] and R.S. Kulkarni [3].

Let $\mathcal{V}$ be a distribution of codimension q on the riemannian manifold (M, g) (all objects are C^∞), and $\mathcal{H}$ its orthogonal complement. We will always denote by A, B, ... (resp. X, Y, Z, W, ...) the vector fields belonging to $\mathcal{V}$ (resp. $\mathcal{H}$). Unrestricted vector fields are written Q, S, T, ...

Let $\mathcal{D} = \oplus \mathcal{D}^{r,s}$ be the exterior associative algebra of $(TM, \mathcal{H})$-double forms, that is, $\alpha \in \mathcal{D}^{r,s}$ is an r-form on M with values in horizontal (namely, they annihilate vectors of $\mathcal{V}$) s-forms. If $\alpha \in \mathcal{D}^{r,s}$ and $\beta \in \mathcal{D}^{a,b}$, then $\alpha \wedge \beta = (-1)^{ra+sb} \beta \wedge \alpha$. We shall put $\alpha^m = \alpha \wedge \ldots \wedge \alpha$ (m times). The projectors that correspond to the decomposition $TM = \mathcal{V} \oplus \mathcal{H}$ are denoted by v, h. Let ∇ be a metric linear connection on $\mathcal{H}$, i.e.

$$Q(g(X, Y)) = g(\nabla_Q X, Y) + g(X, \nabla_Q Y),$$

and $\{e_u\}$ $(u = 1, \ldots, q)$ a local frame of orthonormal vector fields of $\mathcal{H}$.

2.1. DEFINITIONS. – *Let* $\omega \in \mathcal{D}^{k,\ell}$

i) *covariant differential* $D : \mathcal{D}^{k,\ell} \longrightarrow \mathcal{D}^{k+1,\ell}$:

$$D\omega(Q_1, \ldots, Q_{k+1};) = \sum_i (-1)^{i+1} \nabla_{Q_i} \omega(Q_1, \ldots, \hat{Q}_i, \ldots, Q_{k+1};)$$
$$+ \sum_{i<j} (-1)^{i+j} \omega([Q_i, Q_j], Q_1, \ldots, \hat{Q}_i, \ldots, \hat{Q}_j, \ldots, Q_{k+1};);$$

ii) *contraction* $c : \mathcal{D}^{k,\ell} \longrightarrow \mathcal{D}^{k-1,\ell-1}$

$$c\omega(Q_1, \ldots, Q_{k-1}; X_1, \ldots, X_{\ell-1})$$
$$= \omega(e_u, Q_1, \ldots, Q_{k-1}; e_u, X_1, \ldots, X_{\ell-1})$$

(we sum over repeated indices);

iii) *if* $\alpha \in \mathcal{D}$, *then* $\omega \,\overline{\wedge}\, \alpha = \omega(\ ; e_u,\ ) \wedge \alpha(\ ; e_u,\ )$.

Let R be the curvature of ∇, and put

$$K(Q, S; X, Y) = g(R(Q, S) X, Y).$$

Then, the Bianchi identity reads $DK = 0$. Also, if

$$N(Q, S; X) = g(\nabla_Q hS - \nabla_S hQ - h[Q, S], X)$$

stands for the torsion of ∇, we have $Dg = N$, where $g \in \mathcal{D}^{1,1}$ is defined by $g(Q; X) = g(Q, X) = g(hQ, X)$.

2.2. PROPOSITION. – *If* $\omega \in \mathcal{D}$, *then* $D^2\omega = K \,\overline{\wedge}\, \omega$.

Proof. – Since $D \,|\mathcal{D}^{k,0} = d$, we have for $\omega \in \mathcal{D}^{k,0}$: $D^2\omega = 0 = K \,\overline{\wedge}\, \omega$. Assume that the formula holds in $\mathcal{D}^{k,\ell}$ and let $\omega \in \mathcal{D}^{k,\ell}$, $\alpha \in \mathcal{D}^{0,1}$. Then

$$D^2(\omega \wedge \alpha) = D^2 \omega \wedge \alpha + \omega \wedge D^2 \alpha = (K \overline{\wedge} \omega) \wedge \alpha + \omega \wedge D^2 \alpha ,$$

because $D(\omega \wedge \alpha) = D\omega \wedge \alpha + (-1)^k \omega \wedge D\alpha$ for every $\alpha \in \mathcal{O}$. Now $(D\alpha)(Q\,;\;) = \nabla_Q \alpha$, whence

$$(D^2\alpha)(Q, S\,;\;) = R(Q, S)\alpha = K \overline{\wedge} \alpha ,$$

and our claim follows by substitution.

For each integer $m \geqslant 0$, we define the operators $h_m, v_m : \mathcal{O} \to \mathcal{O}$ as follows. If $m = 0$, they are the identity. If $\alpha \in \mathcal{O}^{r,s}$ and $m > r$, then $h_m \alpha = v_m \alpha = 0$. If $0 < m \leqslant r$, then:

$$v_m \alpha(Q_1, \ldots, Q_r\,;\;) = \sum_{1 \leqslant i_1 < \cdots < i_m \leqslant r} \alpha(Q_1, \ldots, vQ_{i_1}, \ldots, vQ_{i_m}, \ldots, Q_r\,;\;)$$

$$h_m \alpha(Q_1, \ldots, Q_r\,;\;) = \sum_{1 \leqslant i_1 < \cdots < i_m \leqslant r} \alpha(Q_1, \ldots, hQ_{i_1}, \ldots, hQ_{i_m}, \ldots, Q_r\,;\;) .$$

By induction it is straightforward to prove

2.3. PROPOSITION.

i) $v_m h_n = h_n v_m$, $v_m v_n = v_n v_m$, $h_m h_n = h_n h_m$;

ii) if $\omega \in \mathcal{O}^{k,\ell}$, *then* $(h + v)\,\omega = k\omega$ (*where* $h = h_1, v = v_1$), and $\omega = \sum_{m=0}^{k} v_m h_{k-m}\, \omega$;

iii) $v_m(\alpha \wedge \beta) = \sum_{r=0}^{m} v_r \alpha \wedge v_{m-r} \beta$, $h_m(\alpha \wedge \beta) = \sum_{r=0}^{m} h_r \alpha \wedge h_{m-r} \beta$;

iv) $vv_m = mv_m + (m+1)\, v_{m+1}$, $hh_m = mh_m + (m+1)\, h_{m+1}$; *more generally,* $v_m = \dfrac{1}{m\,!} \sum_{i=1}^{m} S^i_m v^i$, $h_m = \dfrac{1}{m\,!} \sum_{i=1}^{m} S^i_m h^i$, *where* S^i_m *are the Stirling numbers of the first kind* [1], *and* $v^i = v \ldots v$, $h^i = h \ldots h$ (i times);

v) $v_m g = 0$ *if* $m \geqslant 1$; $cv_m = v_m c$, $ch_m = h_m c + h_{m-1} c$;

vi) $h_r(g^s \wedge c^s \omega) = g^s \wedge c^s h_r \omega$.

3. The conformal operator.

The main purpose of this section is to generalize the conformal map defined by Kulkarni [3].

In the following, if $\omega \in \mathcal{D}^{k+1,\ell+1}$, we put $b = q - k - \ell$.

3.1. LEMMA.

i) $c^r(g \wedge \omega) = g \wedge c^r \omega + r(b + r - 3)\, c^{r-1}\omega + rc^{r-1} v\omega$;

ii) *if* $c(g \wedge \omega) = 0$, *then* $v_s\omega = 0$ *for* $s > 2 - b$.

Proof. – i) By direct computation one gets

$$c(g \wedge \omega) = g \wedge c\omega + (b - 2)\,\omega + v\omega .$$

The general result follows by induction.

ii) Since $c(g \wedge \omega) = 0$, we have $c(g \wedge v_s\omega) = 0$ for $s \geqslant 0$. On the other hand, it is clear that $c^r v_s \omega = 0$ if $r + s > k + 1$. Assume that $c^r v_s \omega = 0$ for $r + s > t \leqslant k + 1$, and let $r + s = t$. Then i) gives:

$$\begin{aligned} 0 &= c^{r+1}(g \wedge v_s\omega) \\ &= g \wedge c^{r+1} v_s\omega + (r + 1)(b + r - 2)\, c^r v_s \omega + (r + 1)\, c^r v v_s \omega \\ &= (r + 1)(b + t - 2)\, c^r v_s \omega , \end{aligned}$$

by 2.3.iv). Thus, by induction we have $c^r v_s \omega = 0$ if $b + r + s - 2 > 0$; our claim follows by taking $r = 0$.

By induction one can also prove

3.2. LEMMA. – $c(g^{r+1} \wedge c^{r+1}\omega) = g^{r+1} \wedge c^{r+2}\omega$

$$+ (r + 1)(b + r) g^r \wedge c^{r+1}\omega + (r + 1) g^r \wedge c^{r+1} v\omega .$$

3.3. THEOREM. – *Let* $\omega \in \mathcal{D}^{k+1,\ell+1}$ *with* $b = q - k - \ell > 0$. *Then, there is a unique element* $\operatorname{dev}\omega \in \mathcal{D}^{k,\ell}$ *satisfying*

$$c(\omega - g \wedge \operatorname{dev}\omega) = 0 .$$

It is given by

$$\operatorname{dev}\omega = \sum_{r,s=0}^{k} \frac{(-1)^{r+s}(b - 1)!\,(r + s)!}{(b + r + s)!\,(r + 1)!\,r!}\, g^r \wedge c^{r+1} v_s \omega .$$

Proof. – The uniqueness proceeds from 3.1.ii). It is enough to prove that $c(g \wedge \operatorname{dev}\omega) = c\omega$. By 3.2 and the relation

$$vv_s = sv_s + (s + 1)\, v_{s+1}$$

(2.3.iv), we have:

$$c(g \wedge \operatorname{dev} \omega) = \sum_{r,s=0}^{k} \frac{(-1)^{r+s}(b-1)!\,(r+s)!}{(b+r+s)!\,(r+1)!\,r!} (g^{r+1} \wedge c^{r+2} v_s \omega$$

$$+ (r+1)(b+r) g^r \wedge c^{r+1} v_s \omega + s(r+1) g^r \wedge c^{r+1} v_s \omega$$

$$+ (s+1)(r+1) g^r \wedge c^{r+1} v_{s+1} \omega)$$

$$= \left\{ \sum_{r,s=0}^{k} \frac{(-1)^{r+s}(b-1)!\,(r+s)!}{(b+r+s-1)!\,r!\,r!} \right.$$

$$- \sum_{r=1}^{k} \sum_{s=0}^{k} \frac{(-1)^{r+s}(b-1)!\,(r+s-1)!}{(b+r+s-1)!\,r!\,(r-1)!}$$

$$\left. - \sum_{r=0}^{k} \sum_{s=1}^{k} \frac{(-1)^{r+s}(b-1)!\,(r+s-1)!\,s}{(b+r+s-1)!\,r!\,r!} \right\} g^r \wedge c^{r+1} v_s \omega$$

$$= c\omega .$$

The *conformal operator* "con" is defined by

$$\operatorname{con} \omega = \omega - g \wedge \operatorname{dev} \omega .$$

3.4. PROPOSITION. – *Let* $\omega \in \mathcal{D}^{k+1,\ell+1}$, *and* $b = q - k - \ell = 1$. *Then* $h_{k+1} \operatorname{con} \omega = 0$.

Proof. – Is a direct consequence of the corresponding result of Kulkarni [3], having in mind the formula 3.3 and 2.3.vi).

4. Conformal change of the metric.

We emphasize that in the previous process, the riemannian metric on M has been only used to define the normal bundle of $\mathcal{V}$ as a subbundle of TM. All our results on double forms depend only on the riemannian metric on $\mathcal{H}$, namely on $g \in \mathcal{D}^{1,1}$.

Classically, the operator con, acting upon curvatures, associates a tensor field to each class of conformally equivalent riemannian connections. Thus, in order to generalize this concept, we need to define the corresponding classes of connections on $\mathcal{H}$.

4.1. DEFINITION. – *Let* g *be a riemannian metric on* $\mathcal{H}$, *and* ∇ *a linear connection on* $\mathcal{H}$. *We say that* ∇ *is g-riemannian if it is g-metric and* $h_2 \mathrm{N} = 0$.

Let g be a riemannian metric on M such that $\mathcal{V} \perp \mathcal{H}$. If $\hat{\nabla}$ is its Levi-Civita connection on TM and we put $\nabla_Q X = h\hat{\nabla}_Q X$, then ∇ is g-riemannian. However, not all the riemannian connections on $\mathcal{H}$ come in this way from connections on TM. Also, in general there are many g-riemannian connections on $\mathcal{H}$.

Nevertheless, if ∇ is a g-riemannian connection on $\mathcal{H}$ and $\tilde{g} = e^{2\sigma} g$, where σ is a function on M, there is a unique $\tilde{g}$-riemannian connection $\tilde{\nabla}$ on $\mathcal{H}$ whose torsion is given by $\tilde{\mathrm{N}} = e^{2\sigma}(\mathrm{N} - g \wedge v d\sigma)$, with $d\sigma \in \mathcal{D}^{1,0}$. We have

$$\tilde{\nabla}_Q X = \nabla_Q X + Q(\sigma) X + X(\sigma) hQ - g(Q, X) Z, \tag{1}$$

where $Z = g^{-1}(d\sigma h, \ )$. We call $\tilde{\nabla}$ the *connection conformally associated* to ∇ by the conformal transformation $\tilde{g} = e^{2\sigma} g$. This defines the equivalence classes on which we want to define the conformal curvature.

Let $\tilde{\mathrm{K}} \in \mathcal{D}^{2,2}$ be the curvature of (1). After calculation we get

$$\tilde{\mathrm{K}} = e^{2\sigma}\left\{ \mathrm{K} - \mathrm{N} \wedge d\sigma h + g \wedge \left(\mathrm{D} d\sigma h + \frac{1}{2} g(Z, Z) g - h d\sigma \wedge d\sigma h\right) \right\}, \tag{2}$$

where $d\sigma h \in \mathcal{D}^{0,1}$ and $h d\sigma \in \mathcal{D}^{1,0}$. Since con is a linear map, we have for $q \geqslant 3$: $\operatorname{con} \tilde{\mathrm{K}} = e^{2\sigma} \operatorname{con}(\mathrm{K} - \mathrm{N} \wedge d\sigma h)$.

In fact, note that $\tilde{g}^r \wedge \tilde{c}^r = g^r \wedge c^r$, whence $\widetilde{\operatorname{con}} = \operatorname{con}$. Also, $\operatorname{con}(g \wedge \eta) = 0$.

We define the tensor C by $g(\mathrm{C}(Q, S)X, Y) = \operatorname{con} \mathrm{K}(Q, S; X, Y)$.

We recall that a conformal foliation can be characterized in the following manner [4]. Let $\mathcal{V}$ be a foliation on M. It is conformal iff there exist some riemannian metric g on M (it defines $\mathcal{H} = \mathcal{V}^{\perp}$) and vertical 1-form λ such that $(\mathrm{L}_A g)(X, X) = g(X, X)\lambda(A)$.

Then, one can define a g-riemannian connection ∇ on $\mathcal{H}$ by

$$\nabla_Q X = h\hat{\nabla}_Q X - h\hat{\nabla}_X vQ + \frac{1}{2}\lambda(Q) X, \tag{3}$$

where $\hat{\nabla}$ is the Levi-Civita connection of g on TM. That is the Bott connection except the term $\frac{1}{2}\lambda(Q)X$. Some properties derived in [4] for (3) are written in the language of double forms as follows:

$$Dg = N = \frac{1}{2}\lambda \wedge g \tag{4}$$

$$vK = -\frac{1}{2}\, v d\lambda \wedge g \tag{5}$$

$$v_2 K = 0$$

$$K(X,Y;Z,W) - K(Z,W;X,Y) = -\frac{1}{2}(d\lambda \wedge g)(X,Y;Z,W), \tag{6}$$

where $d\lambda \in \mathcal{D}^{1,1}$. The last formula derives from

$$\text{cycl}\, K(Q, S; hT, \ ) = \frac{1}{2}\,\text{cycl}\, d\lambda(Q,S) g(hT, \ ).$$

The following theorem shows that the existence of a conformally invariant conformal curvature for the normal bundle of a distribution is an exclusive property of conformal foliations.

4.2. THEOREM. – *Let* $q \geqslant 3$. *If* C *is a conformal invariant, then* $\mathcal{V}$ *is a conformal foliation. Conversely, if* $(\mathcal{V}, g, \lambda)$ *is a conformal foliation,* $\mathcal{H}$ *admits a unique g-riemannian connection such that* C *is a conformal invariant.*

Proof. – C is a conformal invariant iff $\text{con}(N \wedge d\sigma h) = 0$ for every function σ on M. By using the formula 3.3 we have then:

$$N \wedge d\sigma h = \frac{1}{q-1}\, g \wedge (cN \wedge d\sigma h - N(Z, \ ; \ )). \tag{7}$$

Since $cN(Z, \ ; \ ) = 0$ because $h_2 N = 0$, left contraction of (7) with Z gives $N(Z, \ ; \ ) \wedge d\sigma h = \frac{1}{q-1} N(Z, \ ; \ ) \wedge d\sigma h$.

Thus $N(Z, \ ; \ ) \wedge d\sigma h = 0$ and $N(Z, \ ; \ )$ must be a multiple of $d\sigma h$. Therefore there must be some 1-form λ such that $N(Z, \ ; \ ) = \frac{1}{2}\lambda \wedge d\sigma h$, and λ is vertical because $h_2 N = 0$. By substitution in (7) we get $N = \frac{1}{q-1}\, g \wedge \left(cN - \frac{1}{2}\lambda\right)$.

Hence, $cN = \frac{q}{2}\lambda$ and $N = \frac{1}{2}\lambda \wedge g$. Now

$$N(A, X; Y) = g(\nabla_A X - [A, X], Y) = \frac{1}{2}\lambda(A) g(X, Y).$$

Therefore $\nabla_A X = h[A, X] + \frac{1}{2}\lambda(A) X$, and

$$(L_A g)(X, Y) = g(X, Y)\lambda(A).$$

Also

$$0 = \nabla_Q hS - \nabla_S hQ - h[Q, S] - \frac{1}{2}\lambda(Q) hS + \frac{1}{2}\lambda(S) hQ = h[vQ, vS],$$

because $h_2 N = 0$. Hence $\mathcal{V}$ is integrable, and therefore a conformal foliation.

Conversely, if $(\mathcal{V}, g, \lambda)$ is a conformal foliation, the connection (3) solves the problem. The uniqueness is obvious because ∇ must be g-riemannian and with torsion $N = \frac{1}{2}\lambda \wedge g$.

In the next, $(\mathcal{V}, g, \lambda)$ will be a conformal foliation.

For the study of the case $q = 3$, we shall need the element $\mathrm{con}_0 K = D\,\mathrm{dev}\,K$.

4.3. THEOREM. – *If* $q > 3$ *and* $\mathrm{con}\,K = 0$, *then* $h_2\,\mathrm{con}_0 K = 0$. *If* $q = 3$, *then* $h_2\,\mathrm{con}_0 K$ *is a conformal invariant.*

Proof. – If $q > 3$ we have

$$D\,\mathrm{con}\,K = -\frac{1}{2}\lambda \wedge g \wedge \mathrm{dev}\,K + g \wedge \mathrm{con}_0 K.$$

If $\mathrm{con}\,K = 0$, then $g \wedge \left(\mathrm{con}_0 K + \frac{1}{2}\lambda \wedge \mathrm{dev}\,K\right) = 0$. Since $\mathrm{con}_0 K \in \mathcal{D}^{2,1}$, $b = q - 1$, $2 - b = 3 - q < 0$; thus, 3.1.ii) says that $\mathrm{con}_0 K + \frac{1}{2}\lambda \wedge \mathrm{dev}\,K = 0$. Therefore $h_2\,\mathrm{con}_0 K = 0$ because $h\lambda = 0$.

Let $q \geqslant 3$. If $\tilde{g} = e^{2\sigma} g$, we have: $\widetilde{\mathrm{dev}}\,\tilde{K} = \mathrm{dev}\,K + \eta$ where $\eta = \frac{1}{2}\lambda \wedge d\sigma h + D d\sigma h + \frac{1}{2} g(Z, Z) g - h d\sigma \wedge d\sigma h$.

If $\mu \in \mathcal{D}^{1,1}$, we have

$$\tilde{D}\mu = D\mu - (g \wedge \mu)(\ ; Z,\ ) - v d\sigma \wedge \mu + (\mathcal{C}\mu) \wedge d\sigma h,$$

where $\mathcal{C}\mu(Q, S) = \mu(Q; hS) - \mu(S; hQ)$. Hence

$$\widetilde{\mathrm{con}}_0\, \widetilde{\mathrm{K}} = \mathrm{con}_0\, \mathrm{K} + \mathrm{D}\eta - (g \wedge \mathrm{dev}\, \mathrm{K})(\ ;\mathrm{Z},\ ) - (g \wedge \eta)(\ ;\mathrm{Z},\ )$$
$$- v d\sigma \wedge \mathrm{dev}\, \mathrm{K} - v d\sigma \wedge \eta + \mathcal{C}\, \mathrm{dev}\, \mathrm{K} \wedge d\sigma h + \mathcal{C}\, \eta \wedge d\sigma h .$$

But

$$\mathrm{D}\eta = \mathrm{K}(\ ;\mathrm{Z},\ ) + (g \wedge \eta)(\ ;\mathrm{Z},\ ) + \frac{1}{2} \lambda \wedge h d\sigma \wedge d\sigma h$$
$$- \frac{1}{4} g(\mathrm{Z},\mathrm{Z}) \lambda \wedge g + \frac{1}{2} d\lambda \wedge d\sigma h - d(h d\sigma) \wedge d\sigma h - \frac{1}{2} \lambda \wedge \mathrm{D} d\sigma h ,$$
$$\mathcal{C}\eta = d(h d\sigma) ,$$

and $\mathcal{C}\, \mathrm{dev}\, \mathrm{K} = - \frac{1}{2} d\lambda$, as it is easily derived from (6). Hence

$$\widetilde{\mathrm{con}}_0\, \widetilde{\mathrm{K}} = \mathrm{con}_0\, \mathrm{K} + \mathrm{con}\, \mathrm{K}(\ ;\mathrm{Z},\ ) + \frac{1}{2} \lambda \wedge h d\sigma \wedge d\sigma h$$
$$- \frac{1}{4} g(\mathrm{Z},\mathrm{Z}) \lambda \wedge g - \frac{1}{2} \lambda \wedge \mathrm{D} d\sigma h - v d\sigma \wedge \mathrm{dev}\, \mathrm{K} - v d\sigma \wedge \eta .$$

Therefore $h_2\, \widetilde{\mathrm{con}}_0\, \widetilde{\mathrm{K}} = h_2\, \mathrm{con}_0\, \mathrm{K} + h_2\, \mathrm{con}\, \mathrm{K}(\ ;\mathrm{Z},\ )$.

If $q = 3$, $h_2\, \mathrm{con}\, \mathrm{K} = 0$ by 3.4. Then $h_2\, \widetilde{\mathrm{con}}_0\, \widetilde{\mathrm{K}} = h_2\, \mathrm{con}_0\, \mathrm{K}$.

5. Conformally flat normal bundle.

The purpose of this section is to generalize the Weyl-Schouten theorem.

5.1. DEFINITION. – *Let* $(\mathcal{V}, g, \lambda)$ *be a conformal foliation and* ∇ *the connection defined in* (3). *We say that* $\mathcal{H}$ *is conformally flat if for each* $m \in \mathrm{M}$ *there is a neighborhood* U *of* m *and a function* σ *on* U *such that* $\widetilde{\mathrm{K}} = 0$, *where* $\widetilde{\mathrm{K}}$ *is the curvature corresponding to* $\widetilde{g} = e^{2\sigma} g$.

5.2. THEOREM. – *Let* $(\mathcal{V}, g, \lambda)$ *be a conformal foliation. If* $q \leqslant 2$, *then* $\mathcal{H}$ *is conformally flat. For* $q = 3$, $\mathcal{H}$ *is conformally flat iff* $h_2\, \mathrm{con}_0\, \mathrm{K} = 0$. *For* $q > 3$, $\mathcal{H}$ *is conformally flat iff* $\mathrm{con}\, \mathrm{K} = 0$.

Proof. – A conformal foliation is locally conformally riemannian, that is, we can assume that $\lambda = 0$ in a neighborhood of m [4]. Then, one can choose local coordinates such that $\mathcal{V}$ is the vertical foliation of a riemannian submersion $\pi : U \longrightarrow B$. We consider three connections: the Levi-Civita connection ∇^* on B; the Levi-Civita connection $\hat{\nabla}$ on U, and the connection on $\mathcal{H}$: $\nabla_Q X = h\hat{\nabla}_Q X - \hat{\nabla}_X vQ$.

If X_* is a vector field on B, we denote by X its (horizontal) lift to U. Then, we have $g_*(X_*, Y_*) = g(X, Y)$. As it is well known, $h\hat{\nabla}_X Y$ and $h[X, Y]$ are the lifts of $\nabla^*_{X_*} Y_*$ and $[X_*, Y_*]$, respectively; if A is vertical, then $h[A, X] = 0$ whenever X is an horizontal lift of some vector field on B.

Thus, let X, Y, Z, W be the lifts of X_*, Y_*, Z_*, W_*. Then $K(X, Y; Z, W) = g(R(X, Y)Z, W)$, and

$$g(\nabla_X \nabla_Y Z, W) = g(h\hat{\nabla}_X h\hat{\nabla}_Y Z, W) = g_*(\nabla^*_{X_*} \nabla^*_{Y_*} Z_*, W_*)$$

$$\begin{aligned} g(\nabla_{[X,Y]} Z, W) &= g(\nabla_{h[X,Y]} Z, W) + g(\nabla_{v[X,Y]} Z, W) \\ &= g_*(\nabla^*_{[X_*,Y_*]} Z_*, W_*) + g(\hat{\nabla}_{v[X,Y]} Z - \hat{\nabla}_Z v[X, Y], W). \end{aligned}$$

The last term is zero because

$$h(\hat{\nabla}_{v[X,Y]} Z - \hat{\nabla}_Z v[X, Y]) = h[v[X, Y], Z] = 0.$$

Hence we have proved $K(X, Y; Z, W) = K^*(X_*, Y_*; Z_*, W_*)$.

If $q \geqslant 3$, we have

$$\operatorname{con} K(X, Y; Z, W) = \operatorname{con} K^*(X_*, Y_*; Z_*, W_*).$$

Also $\operatorname{dev} K(X; Y) = \operatorname{dev} K^*(X_*; Y_*)$, and $v \operatorname{dev} K = 0$, because $\lambda = 0$. Therefore $\operatorname{con}_0 K(X, Y; Z) = \operatorname{con}_0 K^*(X_*, Y_*; Z_*)$.

Thus, the conditions of the theorem imply that B is conformally flat. In other words, there is some function σ on B (reduce B if necessary) such that $e^{2\sigma} g_*$ gives a flat connection on B. Since then the pair $e^{2\sigma} g$ and $e^{2\sigma} g_*$ also defines a riemannian submersion, we conclude that $h_2 \widetilde{K} = 0$, being $\widetilde{K}$ the curvature of the connection on $\mathcal{H}|U$ defined by $\widetilde{g} = e^{2\sigma} g$. Now,

$$(L_A g)(X, X) = e^{2\sigma}(L_A g)(X, X) = 0;$$

hence $\widetilde{\lambda} = 0$, $v\widetilde{K} = 0$ and $\widetilde{K} = 0$ on U.

Since the imposed conditions are obviously necessary, the theorem is proved.

6. Pontrjagin classes.

For $q \geqslant 3$ we can express the Pontrjagin classes of $\mathcal{H}$ in terms of the conformal curvature.

6.1. LEMMA. – $v \operatorname{con} K = 0$.

Proof. – We have

$$\operatorname{con} K = K - \frac{1}{q-2} g \wedge cK + \frac{c^2 K}{2(q-1)(q-2)} g^2 - \frac{1}{2(q-2)} g \wedge v d\lambda ,$$

and our claim follows directly having in mind (5) and 2.3.v).

6.2. LEMMA. – $\operatorname{con} K \,\overline{\wedge}\, g = 0$.

Proof. – This formula holds in B (see 5.2) as it is well known; in fact, in B it becomes the first Bianchi identity. Since $\operatorname{con} K = h_2 \operatorname{con} K$ by 6.1., it also holds in M.

Let $\mathcal{T}$ denote the module of differential forms on M with values in horizontal covariant tensor fields. We define the product $\otimes$ in $\mathcal{T}$ as follows: if α, β are forms on M and a, b are horizontal covariant tensor fields, then $\alpha \otimes a$ and $\beta \otimes b$ are in $\mathcal{T}$. Thus, we put $(\alpha \otimes a) \otimes (\beta \otimes b) = (\alpha \wedge \beta) \otimes (a \otimes b)$.

Let $\omega \in \mathcal{D}^{k,1}$, $\theta \in \mathcal{D}^{\ell,1}$. Then

$$\omega \wedge \theta = \omega \otimes \theta + (-1)^{k\ell+1} \theta \otimes \omega .$$

If $\omega, \theta \in \mathcal{T}$, we put $\omega \cdot \theta = \omega(\; ; \; , e_u) \otimes \theta(\; ; e_u , \;)$. And if ω is a form with values in 2-covariant horizontal tensor fields, we put $\operatorname{tr} \omega = \omega(\; ; e_u , e_u)$. Then, as it is well known, the Pontrjagin ring of $\mathcal{H}$ is generated by the elements $\operatorname{tr}(K \cdot \ldots \cdot K)$.

6.3. THEOREM. –

$$\operatorname{tr}(K \cdot \ldots \cdot K) = \operatorname{tr}(\operatorname{con} K \cdot \ldots \cdot \operatorname{con} K) + 2^{-(2r-1)} d\lambda \wedge \ldots \wedge d\lambda$$

($2r$ copies in each term). Hence $\operatorname{tr}(K \cdot \ldots \cdot K) = \operatorname{tr}(\operatorname{con} K \cdot \ldots \cdot \operatorname{con} K)$ *in the de Rham cohomology.*

Proof. – We have

$$\operatorname{con} K = K - g \wedge \operatorname{dev} K = K - g \otimes \operatorname{dev} K - \operatorname{dev} K \otimes g .$$

Also $g \cdot g = 0$, $\operatorname{dev} K \cdot \operatorname{dev} K = 0$, and

$$g \cdot \operatorname{dev} K = - \operatorname{dev} K \cdot g = - \frac{1}{2} d\lambda \in \mathcal{D}^{2,0} .$$

Then,

$$K \cdot \ldots \cdot K = \operatorname{con} K \cdot \ldots \cdot \operatorname{con} K + 2^{-2r} d\lambda \wedge \ldots \wedge d\lambda \otimes (g \otimes \operatorname{dev} K - \operatorname{dev} K \otimes g) + \text{terms of the form}$$

$$d\lambda \wedge \ldots \wedge d\lambda \otimes (g \otimes \operatorname{con} K \cdot \ldots \cdot \operatorname{con} K \cdot \operatorname{dev} K + \operatorname{con} K \cdot \ldots \cdot \operatorname{con} K \cdot \operatorname{dev} K \otimes g).$$

In fact,

$$\begin{aligned} K \cdot K &= (\operatorname{con} K + g \otimes \operatorname{dev} K + \operatorname{dev} K \otimes g) \cdot (\operatorname{con} K + g \otimes \operatorname{dev} K + \operatorname{dev} K \otimes g) \\ &= \operatorname{con} K \cdot \operatorname{con} K + \operatorname{con} K \cdot \operatorname{dev} K \otimes g + g \otimes \operatorname{con} K \cdot \operatorname{dev} K \\ &\quad + \frac{1}{2} d\lambda \otimes g \otimes \operatorname{dev} K - \frac{1}{2} d\lambda \otimes \operatorname{dev} K \otimes g , \end{aligned}$$

because $\operatorname{con} K \cdot g = 0$ by 6.2, and $\operatorname{con} K \cdot \operatorname{dev} K = \operatorname{dev} K \cdot \operatorname{con} K$.

In the same way one gets the general formula by induction. Therefore

$$\operatorname{tr}(K \cdot \ldots \cdot K) = \operatorname{tr}(\operatorname{con} K \cdot \ldots \cdot \operatorname{con} K) + 2^{-(2r-1)} d\lambda \wedge \ldots \wedge d\lambda ,$$

because $\operatorname{tr}(g \otimes \operatorname{dev} K) = g \cdot \operatorname{dev} K$, and

$$\operatorname{tr}(g \otimes \operatorname{con} K \cdot \ldots \cdot \operatorname{con} K \cdot \operatorname{dev} K) = g \cdot \operatorname{con} K \cdot \ldots \cdot \operatorname{con} K \cdot \operatorname{dev} K = 0 .$$

In other words, one can substitute the curvature by the conformal curvature tensor in the computation of $\operatorname{Pont}(\mathcal{H} ; \mathbf{R})$.

6.4. THEOREM. – *Let* $(\mathcal{V}, g, \lambda)$ *be a conformal foliation of codimension* q. *Then* $\operatorname{Pont}^k(\mathcal{H} ; \mathbf{R}) = 0$ *for* $k > q$.

Proof. – If $q = 1$, the result follows from Bott's theorem. If $q \geqslant 3$, it is a direct consequence of 6.3 and 6.1. If $q = 2$, it is enough to prove that in cohomology $\operatorname{tr}(K \cdot K) = 0$. Since $\operatorname{tr}(K \cdot K) \in \mathcal{D}^{4,0}$, we have by 2.3.ii):

$$\operatorname{tr}(\mathrm{K} \cdot \mathrm{K}) = \sum_{m=0}^{4} h_m v_{4-m} \operatorname{tr}(\mathrm{K} \cdot \mathrm{K}) .$$

Now, $h_m = 0$ for $m > 2$ because $q = 2$. Hence

$$\operatorname{tr}(\mathrm{K} \cdot \mathrm{K}) = v_4 \operatorname{tr}(\mathrm{K} \cdot \mathrm{K}) + h v_3 \operatorname{tr}(\mathrm{K} \cdot \mathrm{K}) + h_2 v_2 \operatorname{tr}(\mathrm{K} \cdot \mathrm{K})$$
$$= h_2 \operatorname{tr}(v\mathrm{K} \cdot v\mathrm{K}) = \frac{1}{4} h_2 \operatorname{tr}(vd\lambda \wedge g \cdot vd\lambda \wedge g) = \frac{1}{2} h_2 (vd\lambda \wedge vd\lambda),$$

where $vd\lambda \in \mathcal{O}^{2,0}$. But

$$\frac{1}{2} d\lambda \wedge d\lambda = \frac{1}{2} \sum_{m=0}^{4} h_m v_{4-m}(d\lambda \wedge d\lambda) = \frac{1}{2} h_2 (vd\lambda \wedge vd\lambda) .$$

Hence $\operatorname{tr}(\mathrm{K} \cdot \mathrm{K}) = \frac{1}{2} d(\lambda \wedge d\lambda)$,

and the theorem is proved.

6.5. THEOREM. – If $(\mathcal{V}, g, \lambda)$ *is a conformally flat conformal foliation, then* $\operatorname{Pont}(\mathcal{H}; \mathbf{R}) = 1$.

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