ANNALES DE L'INSTITUT FOURIER

VESELIN PETKOV
GEORGI POPOV

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Annales de l'institut Fourier, tome 32, nº 3 (1982), p. 111-149
<http://www.numdam.org/item?id=AIF_1982__32_3_111_0>

Ann. Inst. Fourier, Grenoble
32, 3 (1982), 111-149

ASYMPTOTIC BEHAVIOUR OF THE SCATTERING PHASE FOR NON-TRAPPING OBSTACLES

by V. PETKOV and G. POPOV

0. Introduction.

This paper is devoted to the asymptotics of the scattering phase $s(\lambda)$ related to the laplacian in the exterior of a bounded domain $\mathcal{O} \subset \mathbf{R}^n$. The first result concerning the asymptotic behavior of $s(\lambda)$ for strictly convex domains and the Dirichlet problem was announced by Buslaev [7]. Recently, the same problem has been studied by Majda and Ralston [24] where the first three terms in the asymptotics of $s(\lambda)$ are found (see also [34], [35]). The techniques, used by Majda and Ralston, are connected with a trace formula, proved by Jensen and Kato [15], as well as with the important progress, made by Melrose [30], in the investigation of the forward scattering amplitude for strictly convex bodies.

For non-convex domains the best known result is due to Jensen and Kato [15] where for starlike domains the first term in the asymptotics of $s(\lambda)$ is given. The approach in [15] is based on a trace formula, discussed below, and on the monotonic property of $s(\lambda)$ which enables one to apply a Tauberian theorem. For domains with more complicated geometry, the monotonicy of $s(\lambda)$ is not known (see [16]). On the other hand, the estimate for the remainder in [15] is not the best possible, since the tools related to the Laplace transform are usually not sufficient to obtain a sharp estimate. Finally, notice that Majda and Ralston [24], [35] have conjectured that the asymptotic expansion, given in [7], [24], holds for every non-trapping obstacle.

The analogue of $s(\lambda)$ for bounded domains is the function $N(\lambda)$ equal to the number of the eigenvalues of the laplacian which are not greater then λ^2. After the classical works of Weyl [41] and Courant [9], the efforts have been concentrated on the proof of a sharp bound for the remainder in the asymptotics of $N(\lambda)$, predicted by the Weyl's conjecture. Recently, Seeley [39], Pham The Lai [33] and Ivrii [13] succeeded to obtain a rigorous proof of this conjecture. In particular, Ivrii [13], under some assumption on the set of the periodic bicharacteristics, established a stronger result, concerning the form of the second term in the asymptotics of $N(\lambda)$. In his work Ivrii developed a new perturbation technique for the investigation of the singularity at $t = 0$ of the Fourier transform of $N(\lambda)$. The knowledge of this singularity combined with a Tauberian theorem leads to an estimate for the remainder. Recently, Ivrii [14] proved a remarkable result, describing the asymptotics of the spectral function for the laplacian under general boundary conditions. Notice that the monotonicity of $N(\lambda)$ is crucial for the application of a Tauberian theorem.

The main purpose of this paper is to prove the conjecture of Majda and Ralston for non-trapping obstacles. Our analysis is based on a precise examination of the Fourier transform $\sigma(t)$ of the scattering phase $s(\lambda)$. To study $\sigma(t)$, as $|t| \longrightarrow \infty$, we apply some facts concerning the kernel of the scattering operator, while for the investigation of $\sigma(t)$ for t close to 0 we use the techniques due to Ivrii. On the other hand, we study the Dirichlet problem as well as the Neumann one with an additional term. In this direction, our results are new even for strictly convex domains.

In order to give a precise statement of our main result, we need to introduce some notations. Let $\Omega \subset \mathbf{R}^n$, $n \geqslant 3$, be an open domain with bounded and connected complement $\mathcal{O} = \mathbf{R}^n \backslash \overline{\Omega}$ and smooth boundary $\partial\Omega$. Denote by H_0 the self-adjoint extension of the laplacian $-\Delta$ in $L^2(\mathbf{R}^n)$. Next, let $H_D\,(H_N)$ be the self-adjoint extension of the laplacian $-\Delta$ in $L^2(\Omega)$ with boundary condition of Dirichlet (Neumann) type on $\partial\Omega$. Throughout this paper the Neumann boundary condition has the form

$$\left(\frac{\partial u}{\partial \nu}(x) + \gamma(x)\, u(x)\right)_{\partial\Omega} = 0,$$

where $\gamma(x) \in C^\infty(\partial\Omega)$, $\gamma(x) \geqslant 0$ and ν is the outward unit normal to $\partial\Omega$, pointing into $\mathcal{O}$. Associated to H_j, $j = 0, D, N$, are the following quadratic forms

$$q_0(f, g) = (\nabla f, \nabla g)_{L^2(\mathbf{R}^n)}, \; q_0(f, g) = (\nabla f, \nabla g)_{L^2(\Omega)},$$

$$q_N(f, g) = (\nabla f, \nabla g)_{L^2(\Omega)} + \int_{\partial\Omega} \gamma(x) f(x) \overline{g(x)}\, dS_x .$$

These forms are positive, closable and the self-adjoint operators, corresponding to the closure of q_j, are just H_j. Consider the operators $B_j = \sqrt{H_j}$ with domains

$$D(B_0) = H_1(\mathbf{R}^n), \quad D(B_N) = H_1(\Omega),$$

$$D(B_D) = \{u \in H_1(\Omega),\; u|_{\partial\Omega} = 0\}.$$

Denote by $[D(B_0)]$ the closure of $D(B_0)$ with respect to the norm $\| B_0 u \|^2_{L^2(\mathbf{R}^n)}$ and introduce the Hilbert space

$$\mathcal{H}_0 = [D(B_0)] \oplus L^2(\mathbf{R}^n).$$

The operator

$$A_0 = i \begin{pmatrix} 0 & I \\ -H_0 & 0 \end{pmatrix}$$

with domain $D(A_0) = D(B_0^2) \oplus D(B_0)$ is self-adjoint in $\mathcal{H}_0$ and generates a group $\mathcal{U}_0(t)$ of unitary operators on $\mathcal{H}_0$. Similarly, let $[D(B_D)]$ denote the closure of $D(B_D)$ in the norm $\| B_D u \|^2_{L^2(\Omega)}$. Consider the Hilbert space $\mathcal{H}_D = [D(B_D)] \oplus L^2(\Omega)$ and the self-adjoint operator

$$A_D = i \begin{pmatrix} 0 & I \\ -H_D & 0 \end{pmatrix}$$

with domain $D(A_D) = D(B_D^2) \oplus D(B_D)$. Let $\mathcal{U}_D(t)$ be the group of unitary operators on $\mathcal{H}_D$ generated by A_D. The wave operators $W_\pm(A_D, A_0; P) = s - \lim_{t \to \mp\infty} \mathcal{U}_D(t)\; P\, \mathcal{U}_0(-t)$ exist and are complete (see [19, 20, 10, 37]). Here $P : \mathcal{H}_0 \longrightarrow \mathcal{H}_D$ denotes the orthogonal projection. Everywhere in what follows we use the notations of Reed and Simon [37] for the wave operators and the associated scattering operators. The scattering operator for the Dirichlet problem becomes $S(A_D, A_0, P) = (W_-(A_D, A_0; P))^{-1}\, W_+(A_D, A_0; P)$.

To handle the Neumann problem, it is convenient to extend H_N on $L^2(\mathbf{R}^n)$, setting $H_N f = f$ for $f \in (L^2(\Omega))^\perp$, where the orthogonal complement is taken in $L^2(\mathbf{R}^n)$. Similarly, we put $B_N f = f$

for $f \in (L^2(\Omega))^\perp$. Denoting by $[D(B_N)]$ the closure of $D(B_N)$ with respect to $(q_N(u, u))^{1/2}$, we introduce the space

$$\mathcal{H}_N = ([D(B_N)] \oplus L^2(\mathcal{O})) \oplus L^2(\mathbf{R}^n)$$

and consider the self-adjoint operator

$$A_N = i \begin{pmatrix} 0 & I \\ -H_N & 0 \end{pmatrix}.$$

Since $\mathcal{H}_0 \subset \mathcal{H}_N$, we can determine the wave operators

$$W_\pm(A_N, A_0) = s - \lim_{t \to \mp\infty} e^{itA_N}\, \mathcal{U}_0(-t)$$

without using a suitable projection. Following the approach, developed in [37], we prove the existence and completeness of $W_\pm(A_N, A_0)$. Therefore, we set $S(A_N, A_0) = (W_-(A_N, A_0))^{-1} W_+(A_N, A_0)$.

In what follows, for brevity of the notations, we use the sign $+$ for the Dirichlet problem and the sign $-$ for the Neumann problem. The scattering operator becomes an operator-valued function $S_\pm(\lambda)$ in the spectral representation of A_0 (see [6, 19, 20]). Moreover, $S_\pm(\lambda)$ has the form

$$S_\pm(\lambda) = I + K_\pm(\lambda) \tag{0.1}$$

where $K_\pm(\lambda)$ is trace class [19, 20]. This important property can be deduced from the one established for the scattering operator $S(B_j \oplus I, B_0)$, $j = D, N$ (see section 2). As it is shown in [11, 38], the representation (0.1) implies the existence of $\det S_\pm(\lambda)$, and we obtain $e^{-2\pi i s_\pm(\lambda)} = \det S_\pm(\lambda)$. The function $s_\pm(\lambda)$, called scattering phase, coincides with the spectral shift, studied by Krein [17, 18] and Krein and Birman [6]. Another important objective is the fact, that we can choose $s_\pm(\lambda)$ to be smooth for $\lambda > 0$. This phenomenon is connected with the Rellich's uniqueness theorem which holds for the problems under consideration (see [19, 20]).

Now we shall precise the non-trapping condition. For this purpose consider the generalized bicharacteristics of the operator $\partial_t^2 - \Delta$, introduced by Melrose and Sjöstrand in [26, 27, 28]. The projections of the generalized bicharacteristics on $\overline{\Omega}$ will be called generalized geodesics.

DEFINITION. – *We say that* $\mathcal{O}$ *is non-trapping if for every* $R > 0$ *with* $\mathcal{O} \subset B_R = \{x ; |x| \leqslant R\}$ *there exists a number* $T(R) > 0$ *such* *that there are no generalized geodesics with length* T(R) *within* $\bar{\Omega} \cap B_R$.

Our main result is the following

THEOREM 1. — *Assume* $\mathcal{O}$ *non-trapping. Then we have*

$$s_\pm(\lambda) = \frac{(4\pi)^{-\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} \operatorname{vol} \mathcal{O} \lambda^n \pm \frac{(4\pi)^{-\left(\frac{n-1}{2}\right)}}{4\Gamma\left(\frac{n-1}{2}+1\right)} \operatorname{vol} \partial \mathcal{O} \lambda^{n-1}$$

$$- \frac{(4\pi)^{-\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \int_{\partial \mathcal{O}} \left(\frac{H(x)}{6} - 2\gamma(x)\right) dS_x \lambda^{n-2} \qquad (0.2)$$

$$+ O(\lambda^{n-3}), \ \lambda \longrightarrow +\infty,$$

where $H(x)$ *denotes the mean curvature at* $x \in \partial \mathcal{O}$, dS_x *is the Lebesgue measure on* $\partial \mathcal{O}$ *and for* $s_+(\lambda)$ *the term, involving* $\gamma(x)$, *must be omitted.*

The proof of theorem 1 is very long. The analysis of the Fourier transform $\sigma_\pm(t)$ of $s_\pm(\lambda)$ is based on a suitable trace formula. In [15] Jensen and Kato proved that

$$\operatorname{tr}(e^{-tH_D} \oplus 0 - e^{-tH_0}) = t \int_0^\infty e^{-\lambda t} s_+(\sqrt{\lambda})\, d\lambda\,, \qquad (0.3)$$

where e^{-tH_D} is extended as 0 on $(L^2(\Omega))^\perp$. This formula actually appears in [7] with heat kernels replaced by powers of resolvent kernels. Working with (0.3), we could not obtain an information about the singularities of $\sigma_+(t)$. It is more convenient to use the formula

$$2\operatorname{tr} \int_{-\infty}^{\infty} \rho(t)\, (\cos B_D t \oplus 0 - \cos B_0 t)\, dt$$

$$= \int_{-\infty}^{\infty} \frac{d\hat{\rho}}{d\lambda}(\lambda)\, s_+(\lambda)\, d\lambda\,, \quad \rho \in C_0^\infty(\mathbf{R}), \ (0.4)$$

where $\cos B_D t$ is extended as 0 on $(L^2(\Omega))^\perp$ and

$$\rho(\lambda) = \int_{-\infty}^{\infty} \rho(t)\, e^{-i\lambda t}\, dt.$$

This result for n odd is given by Lax and Phillips [21] for starlike domains and by Bardos, Guillot and Ralston [2, 3] in the general case. One way to prove (0.4), is to exploit the trace class property of the operator

$$(\mathrm{H_D} + \mathrm{I})^{-\mathrm{K}} \oplus 0 - (\mathrm{H_0} + \mathrm{I})^{-\mathrm{K}}, \tag{0.5}$$

where $\mathrm{K} > \frac{n}{2}$. This assertion for Dirichlet and Neumann problem is established by Birman [4, 5], provided the boundary smooth. For domains with more complicated geometry similar results are given by Deift [10] (see also [37]). Following the ideas, used in [15], we get

THEOREM 2. – *For every* $\rho \in \mathrm{C}_0^\infty(\mathbf{R})$ *we have*

$$2\mathrm{tr} \int_{-\infty}^{\infty} \rho(t)\,(\cos \mathrm{B}_\pm t \oplus 0 - \cos \mathrm{B}_0 t)\, dt = \int_{-\infty}^{\infty} \frac{d\hat{\rho}}{d\lambda}(\lambda)\, s_\pm(\lambda)\, d\lambda \tag{0.6}$$

where $\mathrm{B}_+(\mathrm{B}_-)$ *stands for* $\mathrm{B_D}(\mathrm{B_N})$.

Theorem 2 can be proved, without appealing to the trace class property of (0.5) (see [32]). Notice that a trace formula, involving the scattering phase for the Schrödinger operator, has been obtained by Colin de Verdière [8] and Guillopé [12].

The monotonicity of the scattering phase $s_\pm(\lambda)$ for non-trapping obstacles is an unsolved problem. For this reason we cannot apply an argument, based on a Tauberian type theorem. Nevertheless, the non-trapping hypothesis enables us to overcome this difficulty and to obtain $\sigma_\pm(t) \in \mathrm{C}^\infty(\mathbf{R}\setminus 0)$. To do this, we use essentially the recent results of Melrose and Sjöstrand [26, 27, 28] on the propagation of singularities, involving those about the case when the bicharacteristics are tangent to infinite order to $\mathrm{T}^*(\partial\Omega)$. Moreover, examining the kernel of the scattering operator, we show that $t\sigma_\pm(t)$ coincides, as $|t| \longrightarrow \infty$, with a function whose Fourier transform is rapidly decreasing.

Finally, it is necessary to study the singularity at $t = 0$. It is easy to reduce the situation to that, investigated by Ivrii [13]. Therefore, the arguments of Ivrii lead to the asymptotic expansion $\dfrac{ds_\pm(\lambda)}{d\lambda} \sim \sum\limits_{j=0}^{\infty} c_j^\pm \lambda^{n-1-j}$. With this observation in mind, we can apply the trace formula (0.3) and the result of Mc Kean and Singer [25], to find c_j^+. In the same fashion, one could cover the Neumann problem with $\gamma(x) = 0$, but the case $\gamma(x) \neq 0$ makes some troubles. For this reason we prefer to compute $c_j^\pm$, taking into account the perturbation formula for the kernel of $\cos \mathrm{B}_\pm t \oplus 0$, due to Ivrii [13].

We believe the same approach might be useful for some other scattering problems. To simplify the calculation, we work near the boundary with special local coordinates. In these coordinates the operator $\partial_t^2 - \Delta$, frozen at a point on the boundary, has coefficients which preserve some of the geometrical information connected with $\partial \mathcal{O}$. Furthermore, we study the integrand in the third term in (0.2) at some suitably chosen point $x \in \partial \mathcal{O}$ and reduce the general case to the special one.

The plan of the paper is as follows. In sections 1 and 2 we introduce the scattering phase and prove theorem 2. In section 3 we study the behavior of $\sigma_\pm(t)$ as $|t| \longrightarrow \infty$. In section 4 we expose the modifications to the argument in [1], needed to show $\sigma_\pm(t) \in C^\infty(\mathbf{R} \setminus 0)$. Finally, the sections 5 and 6 are devoted to the computation of $c_j^\pm$, $j = 0, 1, 2$. A part of our results, concerning the case n odd, was announced in [31, 32].

The authors are grateful to James Ralston for helpful comments about the case n even and to Johannes Sjöstrand for the discussion on the propagation of singularities.

1. Scattering phase related to the operators B_1, B_0.

Let H_j, $j = D, N$ be the self-adjoint extension to the laplacian $-\Delta$ in $L^2(\Omega)$, defined in the introduction. Using the functional calculus, consider the operators $C_0 = (I + H_0)^{-K}$ and $C_j = (I + H_j)^{-K}$, $j = D, N$, where $K > \frac{n}{2}$ is an integer, which will be fixed in what follows. It is convenient to extend C_j as 0 on $(L^2(\Omega))^\perp$. The extended operator will be denoted by $C_j \oplus 0$. Next, for simplicity of the notations, we write C_1 for C_D or C_N and B_1 for B_D or B_N if some special choice is not mentioned.

The following proposition plays a crucial role for the existence of the scattering phase.

PROPOSITION 1.1. – *The operator* $C_1 \oplus 0 - C_0$ *is trace class in* $L^2(\mathbf{R}^n)$.

Since we consider domains with smooth boundary, the proposition 1.1 is contained in the results of Birman [4, 5]. A stronger result has been obtained by Deift [10], provided the boundary has the cone property (see also [37], Appendix to XI.10).

The first consequence of the above proposition is the existence of the scattering operator $S(C_1 \oplus 0, C_0)$. In the spectral representation of C_0 this operator becomes an operator-valued function $S(\lambda, C_1 \oplus 0, C_0)$ which differs from the identity by a trace class operator. This enables us to define $\det S(\lambda, C_1 \oplus 0, C_0)$ (see [6, 11, 38]).

The second consequence of the trace class property is the existence of the function

$$\begin{aligned} \xi(\lambda) &= \xi(\lambda, C_1 \oplus 0, C_0) \\ &= \pi^{-1} \lim_{z \to \lambda + i0} \arg \det(I + (C_1 \oplus 0 - C_0)(C_0 - z)^{-1}) \end{aligned}$$

for a.e. $\lambda \in \mathbf{R}$. This follows from the Krein theory of the spectral shift, developed in [17, 18, 6]. Before we proceed with the operators B_1, B_0, let us list some properties of $\xi(\lambda)$:

(a) $\xi(\lambda) \in L^1(\mathbf{R})$, $\int_{-\infty}^{\infty} \xi(\lambda)\, d\lambda = \operatorname{tr}(C_1 \oplus 0 - C_0)$,

(b) $\xi(\lambda) = 0$ for $\lambda \notin [0, 1]$,

(c) given a function $\phi \in C_0^\infty(\mathbf{R})$, the operator $\phi(C_1 \oplus 0) - \phi(C_0)$ is trace class and $\operatorname{tr}(\phi(C_1 \oplus 0) - \phi(C_0)) = \int_{-\infty}^{\infty} \phi'(\lambda)\, \xi(\lambda)\, d\lambda$,

(d) $e^{-2\pi i \xi(\lambda)} = \det S(\lambda, C_1 \oplus 0, C_0)$ for a.e. $\lambda \in (0, 1)$.

The properties (a)-(d) are established in [17, 18, 6]. Another detailed proof of (c) and (d) is given in [12], Chapter II.

In what follows, $\xi(\lambda)$ will be called scattering phase. This notion is motivated by the property (d). The following lemma garantees that we can choose $\xi(\lambda)$ to be smooth for $\lambda \in (0, 1)$.

LEMMA 1.2. – *The function* $\xi(\lambda)$ *is real-analytic for* $\lambda \in (0, 1)$.

Since $C_1 \oplus 0$ has no eigenvalues $\lambda \in (0, 1)$ and the resolvant $(C_0 - z)^{-1}$ allows an analytic continuation from the upper half-plane across the interval $(0, 1)$, the proof of lemma 1.2 goes like that of lemma 3.2 in [15]. We leave the details to the reader.

Next, we wish to apply the invariance principle (see [37], p. 30) with an admissible function

$$\psi(\lambda) = \begin{cases} (\lambda^{-\frac{1}{K}} - 1)^{1/2}, & \lambda \in (0,1), \\ 0 & , \lambda \notin (0,1). \end{cases}$$

To avoid the eigenvalue 0, we extend C_1, setting $C_1 f = f$ for $f \in (L^2(\Omega))^{\perp}$. The extended operator will be denoted by $C_1 \oplus I$. This modification on the interior of the obstacle does not affect the scattering operator, namely $S(C_1 \oplus I, C_0) = S(C_1 \oplus 0, C_0)$. We omit the easy proof of this fact.

Since $\psi(\lambda)$ has a finite limit at 1 and the point spectrum of C_0 and $C_1 \oplus I$ does not contain 0, we can apply the invariance principle, leading to the existence and completeness to the wave operators $W_{\pm}(\psi(C_1 \oplus I), \psi(C_0)) = W_{\mp}(B_1 \oplus 0, B_0)$. Consequently, we obtain

$$S(B_1 \oplus 0, B_0) = S^*(C_1 \oplus I, C_0). \qquad (1.1)$$

As above the modification of B_1 on $(L^2(\Omega))^{\perp}$ does not influence the scattering operator and we have

$$S(B_1 \oplus I, B_0) = S(B_1 \oplus 0, B_0). \qquad (1.2)$$

To obtain a link between $\xi(\lambda)$ and the scattering phase related to $B_1 \oplus I$ and B_0, we need to work in a suitable spectral representation for B_0 and C_0. Let $L^2(\mathbf{R}^+; N, \mu)$ be such representation for B_0, where $N = L^2(S^{n-1})$ and μ is a measure on $\mathbf{R}^+$. In this space C_0 acts as a multiplication by $(1+\lambda^2)^{-K}$. Therefore, changing the variable, we can find a spectral representation for C_0. Namely, consider the space $L^2((0,1); N, \tilde{\mu})$ with the measure

$$d\tilde{\mu}(\tau) = -\psi'(\tau)\, d\mu(\psi(\tau)).$$

In this space C_0 acts as a multiplication by τ. Using the unitary operator $\mathcal{U}: L^2(\mathbf{R}^+; N, \mu) \ni f(\tau) \longrightarrow f(\psi(\tau)) \in L^2((0,1); N, \tilde{\mu})$ and taking into account (1.1), (1.2), we conclude that

$$S(\lambda, B_1 \oplus I, B_0) = S^*((1+\lambda^2)^{-K}, C_1 \oplus I, C_0) \quad \text{for a.e. } \lambda > 0. \qquad (1.3)$$

According to lemma 1.2, we can choose a smooth scattering phase for B_0 and $B_1 \oplus I$, that is

$$\xi_1(\lambda, B_1 \oplus I, B_0) = -\xi((1+\lambda^2)^{-K}), \ \lambda > 0. \qquad (1.4)$$

In what follows, this scattering phase will be denoted by $\xi_1(\lambda)$.

2. Trace formula.

This section is devoted to the proof of theorem 2. Introduce the function

$$s_1(\lambda) = \begin{cases} \xi_1(\lambda), & \lambda > 0, \\ -\xi_1(-\lambda), & \lambda < 0. \end{cases} \tag{2.1}$$

First, we shall prove the following

PROPOSITION 2.1. — *For every* $\rho \in C_0^\infty(\mathbf{R})$ *we have*

$$2\operatorname{tr} \int_{-\infty}^{\infty} \rho(t)\,(\cos B_1 t \oplus 0 - \cos B_0 t)\,dt = \int_{-\infty}^{\infty} \frac{d\hat{\rho}}{d\lambda}(\lambda)\, s_1(\lambda)\, d\lambda. \tag{2.2}$$

Proof. — Given $\rho \in C_0^\infty(\mathbf{R})$, consider the cosine transform $\int_{-\infty}^{\infty} \rho(t) \cos\sqrt{\sigma} t\, dt$, $\sigma \geqslant 0$. Extending this function smoothly for $\sigma < 0$, we obtain a function $\phi_1(\sigma) \in \mathcal{S}(\mathbf{R})$. Let

$$\phi(\sigma) = \begin{cases} \varphi(\sigma)\,\phi_1(\sigma^{-\frac{1}{K}} - 1), & \sigma > 0, \\ 0 & , \sigma \leqslant 0 \end{cases}$$

where $\varphi \in C_0^\infty(\mathbf{R})$, $\varphi = 1$ in a neighborhood of the interval $[0,1]$. Obviously, $\phi \in C_0^\infty(\mathbf{R})$, hence applying the property (c), mentioned in section 1, we deduce that the operator

$$\phi(C_1 \oplus 0) - \phi(C_0) = \phi_1(B_1^2) \oplus 0 - \phi_1(B_0^2)$$

is trace class and

$$\operatorname{tr}(\phi_1(B_1^2) \oplus 0 - \phi_1(B_0^2)) = \int_0^1 \phi_1'(\sigma)\, \xi(\sigma)\, d\sigma.$$

According to (1.4), we get

$$\begin{aligned} \operatorname{tr} \int_{-\infty}^{\infty} \rho(t)\,&(\cos B_1 t \oplus 0 - \cos B_0 t)\, dt \\ &= \int_0^\infty \frac{d}{d\lambda}\left(\int_{-\infty}^{\infty} \rho(t) \cos\sqrt{\lambda} t\, dt\right) \xi_1(\sqrt{\lambda})\, d\lambda \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{d\hat{\rho}}{d\lambda}(\lambda)\, s_1(\lambda)\, d\lambda \end{aligned}$$

and this completes the proof of (2.2).

In order to prove theorem 2, it suffices to show, that $s_1(\lambda)$ coincides with the scattering phase related to the operators A_0 and A_D or A_N. To do this, we shall use essentially the approach in [37], section XI.10. In what follows, we restrict our attention to the Neumann problem. The modifications, needed to cover the Dirichlet problem, will be sketched in the end of this section.

For simplicity of the notations, we denote by B_N the operator $B_N \oplus I$. Introduce the unitary operator $\mathcal{J} : \mathcal{H}_0 \longrightarrow \mathcal{H}_N$, given by $\mathcal{J}(u, v) = (B_N^{-1} B_0 u, v)$. Next, we wish to reduce the existence and completeness of the wave operators $W_\pm(A_N, A_0; \mathcal{J})$ to that for the operators $W_\pm(B_N, B_0)$. Consider the unitary operators

$$T_k : \mathcal{H}_k \longrightarrow L^2(\mathbf{R}^n) \oplus L^2(\mathbf{R}^n), \quad k = 0, N,$$

given by

$$T_k = \frac{1}{\sqrt{2}} \begin{pmatrix} B_k & i \\ B_k & -i \end{pmatrix}.$$

A simple calculation shows that

$$W_\pm(A_N, A_0; \mathcal{J}) = T_N^{-1} \begin{pmatrix} W_\pm(B_N, B_0) & 0 \\ 0 & W_\pm(-B_N, -B_0) \end{pmatrix} T_0$$

which implies

$$T_0 S(A_N, A_0; \mathcal{J}) T_0^{-1} = \begin{pmatrix} S(B_N, B_0) & 0 \\ 0 & S(-B_N, -B_0) \end{pmatrix}. \tag{2.3}$$

On the other hand,

$$T_0 S(A_N, A_0; \mathcal{J}) T_0^{-1} = S(T_N A_N T_N^{-1}, T_0 A_0 T_0^{-1})$$

and

$$T_j A_j T_j^{-1} = \begin{pmatrix} B_j & 0 \\ 0 & -B_j \end{pmatrix}, \quad j = 0, N.$$

Let us introduce the space

$$L^2(\mathbf{R}; N, \mu^+ \oplus \mu^-) = L^2(\mathbf{R}^+; N, \mu^+) \oplus L^2(\mathbf{R}^-; N, \mu^-)$$

with $N = L^2(S^{n-1})$, $d\mu^\pm(\lambda) = (\pm\lambda)^{n-1}\, d\lambda$ on $\mathbf{R}^\pm$. Therefore, in the space $L^2(\mathbf{R}^\pm; N, \mu^\pm)$ the operator $\pm B_0$ acts as a multiplication by λ, while $T_0 A_0 T_0^{-1}$ acts as a multiplication by λ in $L^2(\mathbf{R}; N, \mu^+ \oplus \mu^-)$. On the other hand, in $L^2(\mathbf{R}; N, \mu^+ \oplus \mu^-)$

we have $T_N A_N T_N^{-1} = B_N \oplus (-B_N)$. Taking into account (2.3) and the equality $S(\lambda, -B_N, -B_0) = S^*(-\lambda, B_N, B_0)$, $\lambda < 0$, we conclude that

$$S(\lambda, A_N, A_0; \mathcal{J}) = \begin{cases} S(\lambda, B_N, B_0) & \text{on } L^2(\mathbf{R}^+; N, \mu^+), \\ S^*(-\lambda, B_N, B_0) & \text{on } L^2(\mathbf{R}^-; N, \mu^-). \end{cases}$$

This equality shows that $S(\lambda, A_N, A_0; \mathcal{J}) - I$ is trace class, hence $\det S(\lambda, A_N, A_0; \mathcal{J})$ and the related scattering phase $s(\lambda, A_N, A_0; \mathcal{J})$ exist. This observation shows that we can take

$$s(\lambda, A_N, A_0; \mathcal{J}) = \begin{cases} \xi(\lambda, B_N, B_0), & \lambda > 0, \\ -\xi(-\lambda, B_N, B_0), & \lambda < 0. \end{cases} \tag{2.4}$$

Finally, it remains to prove $S(A_N, A_0; \mathcal{J}) = S(A_N, A_0)$. To do this, we shall establish the asymptotic equivalence of $\mathcal{J}$ and the inclusion operator $I_0 : \mathcal{H}_0 \longrightarrow \mathcal{H}_N$, that is the relation

$$\lim_{t\to\pm\infty} (\mathcal{J} - I_0)\,\mathcal{U}_0(t)\,\varphi = 0, \quad \varphi \in \mathcal{H}_0, \tag{2.5}$$

where the limit is taken in $\mathcal{H}_N$. Then an application of the proposition 5c in [37] yields the above link between the scattering operators.

The relation (2.5) can be proved, applying the argument in [37]. For the sake of completeness, we briefly sketch the proof. Let $\mathcal{U}_0(t)\,\varphi = (u_1(t), u_2(t))$. Then

$$\|(\mathcal{J} - I_0)\,\mathcal{U}_0(t)\,\varphi\|^2_{\mathcal{H}_N} = \|(B_0 - B_N)\,u_1(t)\|_0^2,$$

where $\|\cdot\|_0$ denotes the norm in $L^2(\mathbf{R}^n)$. On the other hand,

$$u_1(t) = (\cos B_0 t)\,\varphi_1 + (\sin B_0 t)\,B_0^{-1}\varphi_2, \quad \varphi = (\varphi_1, \varphi_2).$$

Let $\mathcal{D} \subset \mathcal{S}(\mathbf{R}^n)$ be the space of functions whose Fourier transforms vanish in some neighborhood of 0. It suffices to show that

$$\|(B_0 - B_N)\,e^{-itB_0} w\|_0 \xrightarrow[t\to\pm\infty]{} 0, \quad w \in \mathcal{D}.$$

Choose $\psi \in C_0^\infty(\mathbf{R}^n)$ with $0 \leqslant \psi(x) \leqslant 1$, $\psi(x) = 1$ on a neighborhood of the obstacle $\mathcal{O}$. Since

$$(B_0 - B_N)\,(1 - \psi)\,e^{-itB_0} w = 0,$$

we need to study the term $(B_0 - B_N)\,\psi e^{-itB_0} w$. It is easy to see, that the operator $(B_0 - B_N)\,\psi\, B_0^{-2}$ is compact in $L^2(\mathbf{R}^n)$. Recall that B_0 has an absolutely continuous spectrum. Therefore, using a standard argument, we obtain $B_0^2 e^{-itB_0} w \xrightarrow[t \to \pm\infty]{\text{weakly}} 0$ and this completes the proof of (2.5).

Passing to the Dirichlet problem, notice that the trace formula (0.4) is proved by Bardos, Guillot and Ralston [2, 3] for n odd. For the sake of completeness, we shall sketch how the above argument can be applied to cover this case. First, extend B_D as $B_D f = f$ on $(L^2(\Omega))^{\perp}$ and set $\widetilde{\mathcal{H}}_D = \mathcal{H}_D \oplus (L^2(\mathcal{O}) \oplus L^2(\mathcal{O}))$. The extended operator will be denoted by B_D. As for the Neumann problem, we are going to the equality

$$s(\lambda, A_D, A_0; \mathcal{J}) = \begin{cases} \xi(\lambda, B_D, B_0), & \lambda > 0, \\ -\xi(-\lambda, B_D, B_0), & \lambda < 0. \end{cases}$$

Let $P : \mathcal{H}_0 \longrightarrow \mathcal{H}_D$ be the orthogonal projection. Setting $Pf = 0$ on $\mathcal{O}$, we can consider P as an operator from $\mathcal{H}_0$ into $\widetilde{\mathcal{H}}_D$. It remains to prove the asymptotic completeness

$$\lim_{t \to \pm\infty} (\mathcal{J} - P)\, \mathcal{U}_0(t)\, \varphi = 0, \quad \varphi \in \mathcal{H}_0,$$

where the limit is taken in $\widetilde{\mathcal{H}}_D$.

Let $\varphi \in C_0^\infty(\mathbf{R}^n)$, $\mathcal{U}_0(t)\,\varphi = (u_1(t), u_2(t))$. The projection P has the form $P(u_1(t), u_2(t)) = (u_1(t) - v(t), \chi_\Omega u_2(t))$, where χ_Ω is the characteristic function of Ω and $v(t)$ is defined as follows

$$\begin{cases} \Delta v(t) = 0 \quad \text{on } \Omega, \ v(t) \longrightarrow 0 \ \text{ as } |x| \longrightarrow \infty, \\ v(t) = u_1(t) \quad \text{on } \overline{\mathcal{O}}. \end{cases}$$

Using the local energy decay, it is easy to show, that as $|t| \longrightarrow \infty$, we have $\|(1 - \chi_\Omega) B_0 u_1(t)\|_0 \longrightarrow 0$, $\|B_0 \psi(x)\, u_1(t)\|_0 \longrightarrow 0$, $\|(1 - \chi_\Omega) u_2(t)\|_0 \longrightarrow 0$, where $\psi(x)$ is chosen as above. With these observations in mind, we can reduce the asymptotic completeness to the proof of

$$\|B_D(v(t) - \psi(x)\, u_1(t))\|_{L^2(\Omega)} \xrightarrow[t \to \pm\infty]{} 0. \tag{2.6}$$

Applying an integration by parts together with the local energy decay, we get (2.6).

3. Analysis of the behavior of $\sigma(t)$ for $|t| \longrightarrow \infty$.

The purpose of this section is to investigate the behavior of the Fourier transform $\sigma(t)$ of the scattering phase $s(\lambda)$. Our analysis does not depend on the boundary conditions and we shall write $s(\lambda)$, $\sigma(t)$ without especially saying what is the boundary condition.

First, let us recall some results concerning the kernel of the scattering operator S. Given $f = (f_1, f_2) \in \mathcal{H}_0$, set $\mathrm{R}f = \partial_s \widetilde{f_1} - \widetilde{f_2}$ where $\widetilde{f_i}(s, \omega) = \int_{\langle x, \omega \rangle = s} f_i(x)\, d\mathrm{S}$ is the Radon transform of f_i. The translation representation $\mathcal{R}_n$ of the unitary group $\mathcal{U}_0(t)$, found in [19, 20], depends on the parity of n. For n odd $\mathcal{R}_n$ becomes $\mathcal{R}_n f = d_n \partial_s^{(n-1)/2} \mathrm{R}f$, $d_n = 1/2^n \pi^{(1-n)/2}$. For n even there are two representations, related to the outgoing and incoming spaces $\mathrm{D}_\pm$ (see [20]). Namely, $\mathcal{R}_n^\pm = d_n p^\pm * \mathrm{R}f$ with

$$p^\pm(\xi) = \begin{cases} \xi^{(n-1)/2}, & \xi > 0, \\ -|\xi|^{(n-1)/2} e^{\mp i \left(\frac{n-3}{2}\right)\pi}, & \xi < 0. \end{cases}$$

The basic property of these representations is $\mathcal{R}_n^\pm \mathcal{U}_0(t) = \mathrm{T}_t \mathcal{R}_n^\pm$ where T_t denotes the operator of translation to t in the space $\mathrm{L}^2(\mathbf{R} \times \mathrm{S}^{n-1})$.

Associated to $\mathcal{R}_n^-$ is the scattering operator $\widetilde{\mathrm{S}} = \mathcal{R}_n^- \mathrm{S}(\mathcal{R}_n^-)^{-1}$ where for n odd we put $\mathcal{R}_n = \mathcal{R}_n^-$. Recall that $\widetilde{\mathrm{S}}$ is an unitary operator which commutes with T_t. Therefore, its kernel will be a distribution $\widetilde{s}(t - t', \theta, \omega) \in \mathcal{D}'(\mathbf{R} \times \mathrm{S}^{n-1} \times \mathbf{R} \times \mathrm{S}^{n-1})$ with $(t', \omega) \in \mathbf{R} \times \mathrm{S}^{n-1}$, $(t, \theta) \in \mathbf{R} \times \mathrm{S}^{n-1}$.

The representation of $\widetilde{s}(t, \theta, \omega)$, which we need, is closely related to the boundary values of the solution $w^s(t, x, \omega)$ to the mixed problem

$$\begin{cases} (\partial_t^2 - \Delta)\, w^s = 0, \\ \mathcal{B}(w^s + \delta(t - \langle x, \omega \rangle))|_{\mathbf{R} \times \partial\Omega} = 0, \\ w^s|_{t < -\mathrm{A}} = 0. \end{cases} \tag{3.1}$$

Here $\mathrm{A} = \sup\{|x|;\, x \in \mathcal{O}\}$ and $\mathcal{B}$ corresponds to Dirichlet or Neumann boundary conditions. The distribution $\widetilde{s}(t, \theta, \omega)$ has the form

$$\tilde{s}(t, \theta, \omega) = \delta(t)\,\delta(\theta - \omega) + d_n^2 \mathrm{M}_n \partial_t^{n-2}$$
$$\times \int_{\mathbf{R}} \int_{\partial\Omega} \delta(t + \tau - \langle x, \theta \rangle) \left[\frac{\partial w^s}{\partial \nu} - \langle \nu, \theta \rangle \frac{\partial w^s}{\partial \tau} \right] (\tau, x, \omega)\, d\tau d\mathrm{S}_x . \tag{3.2}$$

Here $d\mathrm{S}_x$ is the measure on $\partial\Omega$ and

$$\mathrm{M}_n = \begin{cases} \mathrm{Id}, & n \text{ odd}, \\ i^{n-1}\mathrm{H}, & n \text{ even}, \end{cases}$$

H being the Hilbert transform, i.e. $\widehat{\mathrm{H}f} = (\operatorname{sign} \lambda)\, \hat{f}(\lambda)$, where $\hat{f}(\lambda)$ is the Fourier transform of $f(t)$. The formula (3.2) has been established by Majda [22] for n odd and by Melrose [30] for arbitrary dimenmension $n \geqslant 2$. Note that a factor i in the formula (6.12) in [30] is omitted.

The treatment of (3.2) for the Neumann problem with $\gamma(x) \neq 0$ is a straigthforward repetition of that for $\gamma(x) = 0$. Moreover, the modification of $\mathrm{H_N}$ on $(\mathrm{L}^2(\Omega))^{\perp}$, which we have used in section 1, does not affect the action of $e^{it\mathrm{H_N}}$ on the space $\mathrm{D}_-^{\mathrm{A}} = \mathcal{U}_0(-\mathrm{A})\,\mathrm{D}_-$. It turns out that, this modification does not influence the formula (3.2). Indeed, the starting point for the proof of (3.2) is the expression

$$\tilde{\mathrm{S}}k = \lim_{t' \to \infty} \mathrm{T}_{-t'}\, \mathcal{R}_n^-\, e^{i2t'\mathrm{H_N}} (\mathcal{R}_n^-)^{-1}\, \mathrm{T}_{-t'} k .$$

Therefore, given $k \in \mathrm{C}_0^\infty(\mathbf{R} \times \mathrm{S}^{n-1})$, with $k = 0$ for $|s| > \mathrm{R}_0$, we have $\mathrm{T}_{-t'} k \in \mathcal{R}_n^- \mathrm{D}_{\mathrm{A}}^-$ for $t' > \mathrm{R}_0 + \mathrm{A}$.

Using a simple argument, we deduce from (3.2) the relation

$$\tilde{s}(t, \theta, \omega) = 0 \quad \text{for} \quad t > 2\mathrm{A}. \tag{3.3}$$

The reader should consult [22] for a stronger result. Similarly, for the distribution $\tilde{s}^*(t, \theta, \omega)$, related to the adjoint operator S^*, we obtain

$$\tilde{s}^*(t, \theta, \omega) = 0 \quad \text{for} \quad t < -2\mathrm{A}. \tag{3.4}$$

A more important information is contained in the following

PROPOSITION 3.1. – *Assume* $\mathcal{O}$ *non-trapping. Then there exists a number* $\mathrm{T}_0 > 0$ *such that* $\tilde{s}(t, \theta, \omega)$ *can be written as the sum* $\tilde{s}(t, \theta, \omega) = \delta(t)\,\delta(\theta - \omega) + \mathrm{M}_n(a(t, \theta, \omega) + b(t, \theta, \omega))$ *where* a *and* b *have the properties:*

i) $\operatorname{supp}_t a(t, \theta, \omega) \subset (-T_0, T_0)$,

ii) $\max \operatorname{supp}_t b(t, \theta, \omega) \leqslant -T_0 + 1$,

iii) $b(t, \theta, \omega)$ *is smooth and*

$$|\partial_\lambda^j \hat{b}(\lambda, \theta, \omega)| \leqslant C_N |\lambda|^{-N}, \quad j = 0, 1, \quad \forall N, \ |\lambda| \longrightarrow \infty. \tag{3.5}$$

Moreover, these properties are uniform with respect to

$$(\theta, \omega) \in S^{n-1} \times S^{n-1}.$$

Proof. – The non-trapping hypothesis and the results for propagation of singularities [26, 27, 29] imply the existence of a constant $T > 0$ such that $w^s(t, x, \omega) \in C^\infty$ for $t \geqslant T$, $|x| \leqslant 2A$. Given $\rho \in C_0^\infty(\mathbf{R})$ with $\operatorname{supp} \rho \subset (-\infty, -T - A)$, we obtain

$$(V(t, \theta, \omega), \rho(t))$$
$$= \left(\partial_t^{n-2} \int_{\mathbf{R}} \int_{\partial\Omega} \delta(t + \tau - \langle x, \theta \rangle) \left[\frac{\partial w^s}{\partial \nu} - \langle \nu, \theta \rangle \frac{\partial w^s}{\partial \tau}\right] d\tau dS_x, \rho(t)\right)$$
$$= (-1)^{n-2} \int_{\mathbf{R}} \int_{\partial\Omega} \left(\frac{\partial w^s}{\partial \nu} - \langle \nu, \theta \rangle \frac{\partial w^s}{\partial \tau}\right) \partial_t^{n-2} \rho(\langle x, \theta \rangle - \tau)\, d\tau dS_x.$$

Obviously, we have $t \geqslant T$ for $(\langle x, \theta \rangle - t) \in \operatorname{supp} \rho$. Consequently, the traces

$$\frac{\partial w^s}{\partial \nu}(\tau, x, \omega)\Big|_{\mathbf{R} \times \partial\Omega}, \quad \frac{\partial w^s}{\partial \tau}(\tau, x, \omega)\Big|_{\mathbf{R} \times \partial\Omega}$$

are smooth functions and we conclude that

$$(V(t), \rho(t)) = \int F(t)\, \rho(t)\, dt$$

with $F(t) \in C^\infty(\mathbf{R})$. Choose $T_0 > T + 2A + 1$ and a cut-off function $\varphi \in C_0^\infty(\mathbf{R})$ with $0 \leqslant \varphi(t) \leqslant 1$, $\varphi(t) = 1$ for $|t| \leqslant T_0 - 1$, $\varphi(t) = 0$ for $|t| \geqslant T_0$. Setting $a = \varphi V$, $b = (1 - \varphi) V$, we arrange i), ii) and the first part of iii). In order to obtain the estimate (3.5), some information about the rate of local energy decay is needed. Namely, assuming n odd, and sufficiently large τ, we have

$$\max_{|x| \leqslant 2A} \left| \left(\frac{\partial}{\partial \tau}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha w^s(\tau, x, \omega) \right| \leqslant C_N e^{-\delta\tau}, \ \delta > 0, \ \tau \geqslant T \tag{3.6}$$

for $j + |\alpha| \leqslant N$, where C_N depends on N and A. For the proof of this result we refer to [40, 23, 29, 36].

For n even a similar result with a weaker rate of decay holds. More precisely, we have

$$\max_{|x|\leqslant 2A} \left| \left(\frac{\partial}{\partial \tau}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha w^s(\tau , x , \omega) \right| \leqslant C_N \tau^{-\frac{n}{2}-j} , \quad \tau \geqslant T \quad (3.7)$$

for $j + |\alpha| \leqslant N$, where C_N depends on N and A. The proof of (3.7) is given by Melrose [29, 30] (see also [36] for a sharper result established for mixed problems with initial data in the energy space). The modifications, needed to treat the Neumann problem with $\gamma(x) \neq 0$, are obvious, since the Rellich's uniqueness theorem holds for the problem under consideration.

Now, it is clear, that the second part of iii) follows from (3.6) and (3.7). In particular, for n odd we can arrange $b(t , \theta , \omega) \in \mathcal{S}(\mathbf{R})$. Thus, the proposition 3.1 is proved.

A similar result is true for $\tilde{s}^*(t , \theta , \omega)$, which we state without proof.

PROPOSITION 3.2. – *Assume* $\mathcal{O}$ *non-trapping. Then there exists a number* $T_0 > 0$ *such that* $\tilde{s}^*(t , \theta , \omega)$ *can be written as the sum* $\tilde{s}^*(t , \theta , \omega) = \delta(t)\, \delta(\theta - \omega) + M_n\, (c(t , \theta , \omega) + d(t , \theta , \omega))$ *where* c *and* d *have the properties:*

i) $\operatorname{supp}_t c(t , \theta , \omega) \subset (- T_0 , T_0)$,

ii) $\min \operatorname{supp}_t d(t , \theta , \omega) \geqslant T_0 - 1$,

iii) $d(t , \theta , \omega)$ *is smooth and*

$$|\partial_\lambda^j \hat{d}(\lambda , \theta , \omega)| \leqslant C_N |\lambda|^{-N} , \quad j = 0 , 1 , \ \forall N , \ |\lambda| \longrightarrow \infty. \quad (3.8)$$

Moreover, these properties are uniform with respect to

$$(\theta , \omega) \in S^{n-1} \times S^{n-1} .$$

After this preparatory work, we turn to the analysis of $\sigma(t)$ as $|t| \longrightarrow \infty$. The basic tool will be the well-known formula (see for example [11])

$$\frac{d}{d\lambda} \operatorname{Log} \det S(\lambda) = - \operatorname{tr} \left(S(\lambda) \frac{d}{d\lambda} S^*(\lambda)\right), \quad \lambda \neq 0 . \quad (3.9)$$

First, consider the case n odd. According to the above propositions, we deduce the existence of the convolution

$$\tilde{s}(t, \theta, \omega) * it\, \tilde{s}^*(t, \theta, \omega) = \tilde{f}(t, \theta, \omega) + \tilde{r}(t, \theta, \omega)$$

where $\tilde{f} \in \mathcal{E}'(\mathbf{R})$ and $\tilde{r} \in \mathcal{S}(\mathbf{R})$. Let $k(\lambda, \theta, \omega)$ be the kernel of the operator $S(\lambda) - I$. Following the arguments of Majda [22], it follows easily that $k(\lambda, \theta, \omega)$ depends smoothly on θ and ω. Therefore, we get

$$\operatorname{tr}\left(S(\lambda) \frac{d}{d\lambda} S^*(\lambda)\right) = \int_{S^{n-1}} \frac{d}{d\lambda} \overline{k(\lambda, \omega, \omega)}\, d\omega$$
$$+ \iint_{S^{n-1} \times S^{n-1}} k(\lambda, \eta, \omega) \frac{d}{d\lambda} \overline{k(\lambda, \omega, \eta)}\, d\eta d\omega,$$

and taking into account (3.9), a representation formula for $t\sigma(t)$ can be found. Namely, there exists a distribution $f \in \mathcal{E}'(\mathbf{R})$ and a function $r \in \mathcal{S}(\mathbf{R})$ so that

$$t\sigma(t) = f(t) + r(t). \tag{3.11}$$

For n even some new terms appear in the form of $t\sigma(t)$. This phenomenon is essentially related to the singularity at $\lambda = 0$ of the scattering operator $S(\lambda)$ (see for more details [20]). On the other hand, our aim is to study the asymptotics of $\frac{ds}{d\lambda}(\lambda)$ as $\lambda \longrightarrow +\infty$, hence the terms which vanish for $\lambda > 0$ can be omitted. To make $\frac{ds}{d\lambda}$ in a suitable form, introduce a function $\varphi(\lambda) \in C_0^\infty(\mathbf{R})$, such that $\varphi(\lambda) = 1$ for $|t| \leqslant \delta$ and $\varphi(\lambda) = 0$ for $|t| \geqslant 2\delta > 0$. Making use of formula (3.9) for $\lambda \neq 0$ and performing some calculations, we can write $\frac{ds}{d\lambda}(\lambda)$ as follows:

$$\frac{ds}{d\lambda}(\lambda) = \varphi(\lambda) \frac{ds}{d\lambda}(\lambda) + d_n^2(1 - \varphi(\lambda)) \operatorname{sign}\lambda$$
$$\times \iint_{S^{n-1} \times S^{n-1}} (\hat{a} + \hat{b}) \frac{d}{d\lambda} (\operatorname{sign}\lambda(\hat{c} + \hat{d}))\, d\eta d\omega$$
$$- i^{n-1} d_n (1 - \varphi(\lambda)) \operatorname{sign}\lambda \int_{S^{n-1}} \frac{d}{d\lambda} (\hat{c} + \hat{d})\, d\omega = \varphi(\lambda) \frac{ds}{d\lambda}(\lambda)$$
$$+ (1 - \varphi(\lambda)) (\hat{\tilde{f}} + \hat{\tilde{r}}) - i^{n-1} d_n (1 - \varphi(\lambda)) (\operatorname{sign}\lambda - 1)$$
$$\int_{S^{n-1}} \frac{d}{d\lambda} (\hat{c} + \hat{d})\, d\omega.$$

Taking the Fourier transform, we obtain

$$t\sigma(t) = f(t) + r(t) + g(t) \tag{3.12}$$

where $f(t) \in \mathcal{E}'(\mathbf{R})$, $r(t) \in C^\infty(\mathbf{R})$ and $\hat{r}(\lambda)$ is rapidly decreasing and $\operatorname{supp} \hat{g}(\lambda) \subset (-\infty, -\delta)$.

In the next section we show that $\sigma(t) \in C^\infty(\mathbf{R}\setminus 0)$. Therefore, the asymptotics of $\frac{ds}{d\lambda}(\lambda)$ as $\lambda \longrightarrow +\infty$ can be computed, evaluating $\rho(t)\,\widehat{t\sigma(t)}$. Here $\rho(t) \in C_0^\infty(\mathbf{R})$ with $\operatorname{supp} \rho$ suffiently close to 0. This assertion follows immediately from (3.12) and the observation that

$$\int_{-\infty}^{-\delta} \hat{\rho}(\mu - \lambda)\, \hat{g}(\mu)\, d\mu = O(\lambda^{-N}), \ \forall N, \ \lambda \longrightarrow +\infty .$$

Obviously, the same result holds for n odd.

4. Singularities of $\sigma(t)$.

In this section we prove that the non-trapping condition implies $\sigma(t) \in C^\infty$ for $t \neq 0$. We preserve the notations of the previous sections and denote by B_1 the operator B_D or B_N, extended on $L^2(\mathbf{R}^n)$. Let $\Pi : L^2(\mathbf{R}^n) \longrightarrow L^2(\Omega)$ be the orthogonal projection. Then we obtain

$$\begin{aligned}
\operatorname{tr} \int \rho(t)\, e^{-i\lambda t} (\cos B_1 t - \cos B_0 t)\, dt
&= \operatorname{tr} \int \rho(t)\, e^{-i\lambda t}\Pi(\cos B_1 t - \cos B_0 t)\, \Pi\, dt \\
&- \operatorname{tr} \int \rho(t)\, e^{-i\lambda t}(I - \Pi) \cos B_0 t\, (I - \Pi)\, dt \\
&+ \operatorname{tr} \int \rho(t)\, e^{-i\lambda t}(I - \Pi)\, (\cos B_1 t - \cos B_0 t)\, \Pi\, dt \\
&- \operatorname{tr} \int \rho(t)\, e^{-i\lambda t}\Pi \cos B_0 t (I - \Pi)\, dt .
\end{aligned} \tag{4.1}$$

It is not hard to see, that the last two terms in (4.1) vanish. The second term involves a distribution which is smooth for $t \neq 0$. To study the first one, introduce the kernels $u_0(t, x, y)$ and $u_1(t, x, y)$ of the operators $\cos B_0 t$ and $\cos B_1 t$. Let $\rho \in C_0^\infty(\mathbf{R}^n)$ with $\operatorname{supp} \rho \subset (-T, T)$. Hereafter, we fix T and choose $\varphi(x) \in C_0^\infty(\mathbf{R}^n)$ such that $0 \leqslant \varphi(x) \leqslant 1$, $\varphi(x) = 1$ for $|x| < T + A$, $\varphi(x) = 0$ for $|x| > T + 2A$. Then a domain dependence argument implies

$$\int \rho(t)\, e^{-i\lambda t}\Pi(\cos B_1 t - \cos B_0 t)\,\Pi\, dt$$
$$= \iint \rho(t)\, e^{-i\lambda t}(u_1(t, x, x) - u_0(t, x, x)\, \chi_\Omega(x)\, \varphi(x)\, dt dx,$$

χ_Ω being the characteristic function of Ω. Since $u_0(t, x, x) \in C^\infty$ for $t \neq 0$, it remains to study the distribution

$$C_0^\infty(\mathbf{R}) \ni \psi \longrightarrow \iint \psi(t)\, u_1(t, x, x)\, \chi_\Omega(x)\, \varphi(x)\, dt dx$$

for $|t| < T$. On the other hand, for $|t| < T$, $|x| > 2T + 2A$ we have $u_1(t, x, y)\, \chi_\Omega(y)\, \varphi(y) = 0$, hence we can restrict our attention to the distribution $v(t, x, y)$, determined as solution to the mixed problem

$$\begin{cases} (\partial_t^2 - \Delta)\, v(t, x, y) = 0, \\ \mathcal{B}v\,|_{\mathbf{R}\times\partial\Omega} = 0,\ v\,|_{|x|=2T+2A} = 0, \\ v\,|_{t=0} = \delta(x - y)\, \varphi(y),\ D_t v\,|_{t=0} = 0. \end{cases} \tag{4.2}$$

Setting $M = \Omega \cap \{x\, ;\, |x| \leqslant 2T + 2A\}$, we are going to study the singularities of the distribution $\int_M v(t, x, x)\, dx$ for $|t| < T$. A similar problem has been studied by Melrose and Andersson [1] under the assumption that $\partial\Omega$ is strictly convexe or concave. The analysis, carried out in [1], can be applied in our case, where some modifications concerning the more complicated propagation of singularities are needed.

First, note that the reflection of singularities on $|x| = 2T + 2A$ does not produce closed generalized geodesics with length less than T. Therefore, the analogue of proposition (8.15) in [1] holds, since its proof does not involve an analysis of the singularities near $\partial\Omega$.

Next, a precise examination of the proof of proposition (8.20) in [1] shows that the geometry of $\partial\Omega$ is essential only for the application of the results concerning the regularity up to the boundary of solution $w(x, z) = E'g(x, z)$ to the following mixed problem

$$\begin{cases} (\partial_t^2 - \Delta)\, w = 0 \\ \mathcal{B}w\,|_{\mathbf{R}\times\partial\Omega} = g(x, z), \\ w\,|_{t<<0} = 0. \end{cases} \tag{4.3}$$

Here $z, 0 \leqslant z \leqslant \mu$, is a parameter, $WF(g) \subset \Gamma_0$ uniformly for

$0 \leqslant z \leqslant \mu$ and Γ_0 is a conic neighborhood of $\rho_0 = (\hat{x}, 0, \hat{\xi}, \hat{\tau})$. $((\xi, \tau)$ denote the variables dual to $(x, t))$. The regularity up to the boundary is described by the generalized wave front $\mathrm{WF}_b(w)$, introduced in [26]. We need the following.

PROPOSITION 4.1. – *Let* $\rho = (\hat{x}, \hat{t}, \hat{\xi}, \hat{\tau})$, $\hat{x} \in \partial\Omega$, $0 < \hat{t} < \mathrm{T}$. *If* Γ_0 *is sufficiently small there exists a conic neighborhood* Γ *of* ρ *such that*

$$\mathrm{WF}_b(w) \cap \Gamma = \emptyset \tag{4.4}$$

uniformly for $0 \leqslant z \leqslant \mu$.

Proof. – Let Γ be a small conic neighborhood of ρ. Consider all generalized bicharacteristics issued from Γ. We claim that for sufficiently small Γ_0 and Γ these bicharacteristics do not intersect Γ_0. In order to prove the claim, we need to define the relation C_t for $t < 0$, which is completely analogous to that introduced for $t > 0$ in [27, 28]. The obvious modifications are left to the reader. We list three properties of C_t :

i) $\zeta \in \mathrm{C}_t \eta$ if and only if there is a generalized bicharacteristics γ connecting η and ζ, i.e. $\gamma : [t, 0] \longrightarrow \mathrm{T}^*(\mathrm{M})$ with $\gamma(0) = \eta$, $\gamma(t) = \zeta$,

ii) $\mathrm{C}_{t_1} \circ \mathrm{C}_{t_2} = \mathrm{C}_{t_1+t_2}$, $t_1 < 0$, $t_2 < 0$,

iii) C_t is a closed relation.

Remark 4.2. – In the case $\sum_b^\infty = \emptyset$, the property iii) can be strengthed, since $\mathrm{C}_t \eta$ coincedes with the generalized Hamiltonian flow $\mathrm{F}(t, \eta)$ which is continuous with respect to t and η (see [26]).

Suppose there are sequences $\rho_n \longrightarrow \rho_0$, $\eta_n \longrightarrow \rho_0$, t_n so that

$$\eta_n \in \mathrm{C}_{t_n} \rho_n, \quad t_n < 0. \tag{4.5}$$

According to the outgoing condition, we can assume $-\mathrm{T}_0 \leqslant t_n < 0$. Passing to subsequences, let $t_n \longrightarrow t_0$ where $t_n \geqslant t_0$ or $t_n \leqslant t_0$ for every n. In the first case, consider a sequence $z_n \in \mathrm{C}_{t_0 - t_n} \eta_n$. Using the behavior of the generalized bicharacteristics, discussed in [27, 28], we deduce from $\eta_n \longrightarrow \rho_0$ that $z_n \longrightarrow \rho_0$. Therefore, the properties i)-iii), mentioned above, imply

$$z_n \in C_{t_0 - t_n} \circ C_{t_n} \rho_n = C_{t_0} \rho_n .$$

Hence $\rho_0 \in C_{t_0} \rho$, which contradicts the non-trapping hypothesis.

In the case $t_n \leqslant t_0$ we have $C_{t_n - t_0} \circ C_{t_0} = C_{t_n}$. This leads to the existence of a sequence z_n such that

$$z_n \in C_{t_0} \rho_n , \quad \eta_n \in C_{t_n - t_0} z_n .$$

Now the remark (3.16) in [27] shows that $z_n \longrightarrow \rho_0$ and we obtain again $\rho_0 \in C_{t_0} \rho$, leading to a contradiction. Thus, the claim is proved, and the neighborhoods Γ_0, Γ do not depend on z.

Now we are in position to apply the microlocal version of the results of Melrose and Sjöstrand [26, 27] uniformly with respect to z. Assuming $x_0 \in \mathrm{WF}_b(w) \cap \Gamma$ and following the procedure exposed in [28], we can construct a generalized bicharacteristics γ issued from x_0 and determined for $t \leqslant 0$. As we have proved, γ hits the boundary at points $y \notin \Gamma_0$. The wave front $\mathrm{WF}_b(w)$ propagates along γ, hence w is singular for t large negative, which contradicts the outgoing condition. This completes the proof of proposition 4.1.

Repeating the arguments in [1] and using the proposition 4.1, we deduce that the singularities of $\int_M v(t, x, x)\, dx$ are related to the lengths of the periodic generalized geodesics. The non-trapping hypothesis excludes the existence of such geodesics and we conclude that $\sigma(t) \in C^\infty$ for $t \neq 0$.

5. Asymptotic expansion of the scattering phase.

In this section we show that $s(\lambda)$ has an asymptotic expansion with respect to λ and we compute the first term in this expansion. Moreover, using special coordinates near the boundary, we prepare the calculation of the second and third term given in the next section.

Our analysis is based on the trace formula, proved in section 2. Hereafter, we write B_1 for B_D or B_N and denote by $s(\lambda)$ the scattering phase related to $S(\lambda, A_D, A_0, P)$ or $S(\lambda, A_N, A_0)$. We have

$$2\operatorname{tr} \int \rho(t)\, e^{-i\lambda t} (\cos B_1 t \oplus 0 - \cos B_0 t)\, dt = -\Big(\hat{\rho} * \frac{ds}{d\lambda}\Big)(\lambda) + r(\lambda)$$

where $r(\lambda)$ is rapidly decreasing.

To handle the second term in (4.1), we shall find the Fourier transform of $u_0(t, x, x)$, working with the Fourier-Laplace transform

$$\hat{u}_0(\tau, \xi, y) = (2\pi)^{-n-1} \int_0^\infty \int_{\mathbf{R}^n} e^{-i(t\tau + x\xi)} u_0(t, x, y)\, dt dx\,, \ \operatorname{Im}\tau < 0\,,$$

$$\hat{u}_0(\tau, \xi, y) = (2\pi)^{-n-1} \int_{-\infty}^0 \int_{\mathbf{R}^n} e^{-i(t\tau + x\xi)} u_0(t, x, y)\, dt dx\,, \ \operatorname{Im}\tau > 0\,.$$

The same tool will be used in the next section for some more complicated calculations. It is a simple exercise to show that

$$\hat{u}_0(\tau, \xi, y) = \pm\, i\tau(2\pi)^{-n-1}(\tau^2 - |\xi|^2)\, e^{-iy\xi}\,, \quad \pm\, \operatorname{Im}\tau > 0\,.$$

Hence the Fourier transform $\hat{u}_0(\lambda, \xi, y)$ becomes

$$\hat{u}_0(\lambda, \xi, y) = -\, i\lambda(2\pi)^{-n-1}\,[((\lambda - i0)^2 - |\xi|^2)^{-1} - ((\lambda + i0)^2 - |\xi|^2)^{-1}]\, e^{-iy\xi} = \frac{(2\pi)^{-n}}{2}\, e^{-iy\xi}\, \delta(|\lambda| - |\xi|)\,.$$

Therefore,

$$\hat{u}_0(\lambda, x, x) = \frac{(2\pi)^{-n}}{2}\, |\lambda|^{n-1} \operatorname{vol} S^{n-1} = \frac{n}{2}\, \frac{(4\pi)^{-n/2}}{\Gamma(n/2+1)}\, |\lambda|^{n-1}\,.$$

On the other hand, $u_0(t, x, x)$ has a singularity only for $t = 0$, which implies

$$\widehat{\rho u_0}(\lambda, x, x) = \frac{n}{2}\, \frac{(4\pi)^{-n/2}}{\Gamma(n/2 + 1)}\, |\lambda|^{n-1} + O(|\lambda|^{-N})\,, \ \forall N\,.$$

Now we turn to the asymptotics of

$$\mathcal{J}(\lambda) = 2\operatorname{tr}\int \rho(t)\, e^{-i\lambda t}\, \Pi(\cos B_0 t - \cos B_1 t)\, \Pi\, dt\,.$$

Using the finite speed of the propagations and choosing the support of $\rho(t)$ sufficiently small, we see that the kernel of the operator $\rho(t)\, \Pi(\cos B_0 t - \cos B_1 t)\, \Pi$ vanishes for x and y outside a sufficiently small neighborhood of $\partial\Omega$. Applying a finite partition of unity $\{\varphi_j(y)\}_{j=1}^{M_0}$, we can reduce the calculation to the case where $\partial\Omega$ has a simple form. Consider a function $\varphi(y) \in C_0^\infty(\mathbf{R}^n)$ and assume that in a neighborhood $\mathcal{U} \subset \partial\Omega$ of $\partial\Omega \cap \operatorname{supp}\varphi$ the boundary $\partial\Omega$ is given by $y_1 = g(y')$, $y' = (y_2, \ldots, y_n)$. It is convenient to work with the coordinates (r, x') connected with (y_1, y') as follows

$$(y_1, y') = \phi(r, x') = \psi(x') + r\mathrm{N}(\psi(x')).$$

Here $\psi(x') = (g(x'), x')$ and $\mathrm{N} : \mathcal{U} \longrightarrow \mathrm{S}^{n-1}$ is the Gauss map. The next lemma follows from a simple geometrical argument and we omit the proof.

LEMMA 5.1. – *In the coordinates* (r, x') *the operator* $\mathrm{D}_t^2 + \Delta$ *has the form*

$$\mathrm{P} = \mathrm{D}_t^2 - a(r, x')\,\mathrm{D}_r^2 - \mathrm{R}(r, x', \mathrm{D}_{x'}) - \mathrm{R}_1(r, x', \mathrm{D}_r, \mathrm{D}_{x'}) \tag{5.2}$$

where R *is an elliptic operator with symbol* $\mathrm{R}(r, x', \xi')$, *homogeneous of order 2 in* ξ', *while* R_1 *is a first order differential operator.*

For the calculations, we need a sharper result concerning the Taylor expansions of R and R_1. For this purpose we shall prove the following

LEMMA 5.2. – *We have*

$$({}^t\mathrm{D}\phi(r, x'))^{-1} = \begin{pmatrix} 1 & g_{x_2} & \dots & g_{x_n} \\ -g_{x_2} & 1 + rg_{x_2x_2} & \dots & rg_{x_2x_n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ g_{x_n} & rg_{x_nx_n} & \dots & 1 + rg_{x_nx_n} \end{pmatrix} + \mathrm{O}(r^2 + |\nabla g|^2).$$

Proof. – We start with the computation of $\det \mathrm{D}\phi(r, x')$. It follows easily that $\mathrm{D}\phi(r, x') = (\mathrm{I} + r\mathrm{DN})\,(\mathrm{N}, \mathrm{D}_{x'}\psi)$ hence $\det \mathrm{D}\phi(r, x') = \det(\mathrm{I} + r\mathrm{DN})\,\det(\mathrm{N}, \mathrm{D}_{x'}\psi)$. On the other hand, $\det(\mathrm{I} + r\mathrm{DN}) = \prod_{j=1}^{n} (1 + r\lambda_j) = \sum_{j=0}^{n} c_j r^j$ where λ_j are the eigenvalues of the Gauss map. In particular, $c_0 = 1$, $c_1 = -\mathrm{H}(x')$ where $\mathrm{H}(x')$ is the mean curvature at $(g(x'), x')$. Remark that we have $\mathrm{H}(x') = \Delta g(x')$ if $\nabla g(x') = 0$. Furthermore, a simple calculation yields $\det(\mathrm{N}, \mathrm{D}_{x'}\psi) = (1 + |\nabla g(x')|^2)^{1/2}$, and we conclude that

$$\det \mathrm{D}\phi(r, x') = (1 + |\nabla g(x')|^2)^{1/2}\,(1 - r\mathrm{H}(g(x'), x') + \dots + c_n r^n). \tag{5.3}$$

Next we have

$$
{}^t\mathrm{D}\phi(r\,,x') = \begin{pmatrix} 1 & -g_{x_2} & \dots & -g_{x_n} \\ g_{x_2} & 1 - rg_{x_2x_2} & \dots & -rg_{x_2x_n} \\ \cdot & & & \\ \vdots & & & \\ g_{x_n} & -rg_{x_2x_n} & \dots & 1 - rg_{x_nx_n} \end{pmatrix} + \mathrm{O}(r^2 + |\nabla g|^2)\,.
$$

Applying (5.3), we are going to

$$
({}^t\mathrm{D}\phi(r\,,x'))^{-1} = (1+r\mathrm{H}) \begin{pmatrix} 1-r\mathrm{H} & g_{x_2} & \dots & g_{x_n} \\ -g_{x_2} & 1-r(\mathrm{H}-g_{x_2x_2}) & \dots & rg_{x_2x_n} \\ \cdot & & & \\ \vdots & & & \\ -g_{x_n} & rg_{x_nx_2} & \dots & 1-r(\mathrm{H}-g_{x_nx_n}) \end{pmatrix}
$$
$$
+\mathrm{O}(r^2+|\nabla g|^2
$$

which proves the assertion.

Now lemma 5.2 enables us to write the operator P in the form

$$
\mathrm{P} = \mathrm{D}_t^2 - \mathrm{D}_r^2 - \mathrm{D}_{x'}^2 - 2r\sum_{j,k} g_{x_jx_k}(x')\,\mathrm{D}_{x_jx_k}
$$
$$
- i\mathrm{H}(g(x')\,,x')\,\mathrm{D}_r + \sum_{|\alpha|=2} a_\alpha \mathrm{D}^\alpha + \sum_{|\alpha|=1} b_\alpha \mathrm{D}^\alpha
$$

with $a_\alpha = \mathrm{O}(r^2 + |\nabla g|^2)$, $b_\alpha = \mathrm{O}(|r| + |\nabla g|)$. Here it is important, that the coefficients of P, which are essential for the calculations, coincide with some geometrical invariants.

Following the techniques, developed by Ivrii [13], we shall study the asymptotics of the expression $\mathcal{J}(\lambda)$. For brevity of the notations, we shall use below the coordinates $x = (x_1\,,x')$, $y = (y_1\,,y')$, where x_1 stands for r. Consider the distributions $u_0(t\,,x\,,y)$, $u(t\,,x\,,y)$, determined as solutions to the problems

$$
\begin{cases} \mathrm{P}u_0 = 0\,, \\ u_0|_{t=0} = \delta(x-y), \quad \mathrm{D}_t u_0|_{t=0} = 0\,, \end{cases} \tag{5.5}
$$

$$
\begin{cases} \mathrm{P}u = 0\,, & \mathcal{B}u_{|\mathbf{R}\times\partial\Omega} = 0\,, \\ u_{|t=0} = \delta(x-y), & \mathrm{D}_t u_{|t=0} = 0\,. \end{cases} \tag{5.6}
$$

Setting $u_1 = u_0 - u$, we shall investigate

$$\mathrm{I}(\lambda) = 2\int_0^\infty \int_{\mathbf{R}^{n-1}} \widehat{\rho u_1}(\lambda, x, x)\,\mathrm{D}\left(\frac{y}{x}\right)(x)\,\varphi(x)\,dx_1\,dx'.$$

For every $y' \in \mathcal{U}$ consider the differential operator

$$\overline{\mathrm{P}}(y', \mathrm{D}_t, \mathrm{D}_x) = \mathrm{P}^0(0, y', \mathrm{D}_t, \mathrm{D}_x)$$

where P^0 is the principal symbol of P. Taking the Taylor expansion of the coefficients of the operator $\mathrm{K} = \overline{\mathrm{P}} - \mathrm{P}$, we obtain

$$\mathrm{K} = \sum_{\substack{j+|\alpha|\leqslant m \\ |\beta|\leqslant 2}} a_{\alpha,\beta,j}(x'-y')^\alpha x_1^j\,\mathrm{D}^\beta + \sum_\beta a'_\beta \mathrm{D}^\beta \tag{5.7}$$

with $a'_\beta = \mathrm{O}((|x'-y'| + x_1)^{m+1})$. In a similar way, we handle the function $\gamma(x)$ and get

$$\gamma(g(x'), x') = \sum_{|\alpha|<m} \gamma_\alpha (x'-y')^\alpha + \mathrm{O}(|x'-y'|^{m+1}). \tag{5.8}$$

Following the approach of Ivrii [13], we shall build an approximation to u_0 and u_1, using the parametrices related to the operator $\overline{\mathrm{P}}$. Denote by $\overline{u}_0$ the solution to the Cauchy problem

$$\begin{cases} \overline{\mathrm{P}}\,\overline{u}_0 = 0, \\ \overline{u}_0|_{t=0} = \delta(x-y), \quad \mathrm{D}_t\overline{u}_0|_{t=0} = 0. \end{cases} \tag{5.9}$$

Let E_0, E', E be the parametrices to the following problems

$$\begin{cases} \overline{\mathrm{P}}\mathrm{E}_0 = \mathrm{I}, \\ \mathrm{E}_0|_{t=0} = \mathrm{D}_t\mathrm{E}_0|_{t=0} = 0, \end{cases}$$

$$\begin{cases} \overline{\mathrm{P}}\mathrm{E}' = 0, \quad \mathcal{B}_\mathrm{D}\mathrm{E}' = \mathrm{I} \ \text{ or } \ \mathcal{B}_\mathrm{N}\mathrm{E}' = \mathrm{I}, \\ \mathrm{E}'|_{t=0} = \mathrm{D}_t\mathrm{E}'|_{t=0} = 0, \end{cases}$$

$$\begin{cases} \overline{\mathrm{P}}\mathrm{E} = \mathrm{I}, \quad \mathcal{B}_\mathrm{D}\mathrm{E} = 0 \ \text{ or } \ \mathcal{B}_\mathrm{N}\mathrm{E} = 0, \\ \mathrm{E}|_{t=0} = \mathrm{D}_t\mathrm{E}|_{t=0} = 0. \end{cases}$$

Here $\mathcal{B}_\mathrm{D}u = u(0, x')$, while $\mathcal{B}_\mathrm{N}u = \dfrac{\partial u}{\partial x_1}(0, x')$. These parametrices depend on y' but we omit this in our notations.

It is easy to show that

$$u_0 = \overline{u}_0 + E_0 K u_0 = \sum_{k=0}^{m} (E_0 K)^k \overline{u}_0 + (E_0 K)^{m+1} u_0 , \tag{5.10}$$

$$u_1 = E' \mathcal{B}_D u_0 + EK u_1 = \sum_{k=0}^{m} (EK)^k E' \mathcal{B}_D u_0 + (EK)^{m+1} u_1 , \qquad (5.11)_D$$

where E′ and E are determined with Dirichlet boundary conditions,

$$\begin{aligned} u_1 &= E'(\mathcal{B}_N + \gamma \mathcal{B}_D) u_0 + (EK - E'\gamma \mathcal{B}_D) u_1 \\ &= \sum_{k=0}^{m} (EK - E'\gamma\mathcal{B}_D)^k E'(\mathcal{B}_N + \gamma\mathcal{B}_D) u_0 + (EK - E'\gamma\mathcal{B}_D)^{m+1} u_1 , \end{aligned} \qquad (5.11)_N$$

where E′ and E are determined with Neumann boundary conditions. We associate to every term $a_{\alpha,\beta,j}(x' - y')^{\alpha}_{x_1}{}^{j} D^{\beta}$ in the development of K a weight $\omega = |\alpha| + j - |\beta| + 2$. Similarly, we associate weight $|\alpha| + 2$ to every term $\gamma_\alpha (x' - y')^\alpha$ in the development (5.8). The weight of a product of terms of this kind by definition will be the sum of the weights of all terms. Thus, we obtain the representations

$$u_0 = u_0^{(1)} + \ldots + u_0^{(m)} + u_0^{(m+1)} ,$$
$$u_1 = u_1^{(1)} + \ldots + u_1^{(m)} + u_1^{(m+1)} ,$$

where $u_k^{(j)}$, $k = 0, 1$, has weight j, $0 \leqslant j \leqslant m$, while $u_k^{(m+1)}$ is a sum of terms with weights $\omega \geqslant m + 1$ and the remainders in the corresponding developments.

Performing a rather lengthly computation, it can be shown that

$$\int \widehat{\rho u_1^{(j)}}(\lambda, x, x)\, \varphi(x)\, D\left(\frac{y}{x}\right) dx = O(|\lambda|^{n-2-j}), \quad 0 \leqslant j \leqslant m .$$

The main steps of this calculation are sketched in [13]. The analysis of the remainder term $u_1^{(m+1)}$ is the hardest part in the approach, proposed by Ivrii. This analysis is based on the so called normal singularity at $t = 0$ of the distribution $F = \int_M \rho(t)\, u_1(t, x, x)\, dx$ where the integral is taken over a bounded set M as was discussed in section 4. More precisely, there exist two numbers $\epsilon_0 > 0$, $s_0 \in \mathbf{R}$ such that, provided $\operatorname{supp} \rho \subset (-\epsilon_0, \epsilon_0)$, we have $(tD_t)^k F \in H^{s_0}(-\epsilon_0, \epsilon_0)$ for every integer $k \geqslant 0$. The argument, carried out in [13], can be applied without any change to the problem under consideration and we obtain

$$\int \widehat{\rho u}_1^{(m+1)}(\lambda, x, x)\ \varphi(x)\ \mathrm{D}\left(\frac{y}{x}\right) dx = \mathrm{O}(|\lambda|^{s_1 - m/2})$$

where s_1 does not depend on m.

Finally, these observations show that

$$\mathrm{I}(\lambda) = 2 \sum_{j=0}^{m} \int_{\mathbf{R}^{n-1}} (1 + |\nabla g(x')|^2)^{1/2}\ \mathrm{L}_j(\lambda, x')\ dx' + \mathrm{O}(|\lambda|^{s_1 - m/2})$$

where

$$\mathrm{L}_j(\lambda, x') = \int_0^\infty \widehat{\rho u}_1^{(j)}(\lambda, x, x)\ \varphi(x)\ (1 - x_1 \mathrm{H} + \ldots + c_n x_1^n)\ dx_1 .$$

6. Computation of the second and third terms.

In this section we study the asymptotics of $\mathrm{L}_j(\lambda, x')$, $j = 0, 1$. The calculations depend on the form of the operator $\overline{\mathrm{P}}(y', \mathrm{D}_t, \mathrm{D}_x)$ which can be essentially simplified, provided $\nabla g(y') = 0$. The general case will be covered in the end of the section by an approximation argument adapted to the developments $(5.11)_{\mathrm{D}}$ and $(5.11)_{\mathrm{N}}$.

Now, let us consider the terms with weight 0 and 1. We have

$$\overline{\mathrm{P}} = \mathrm{D}_t^2 - \mathrm{D}_x^2 ,$$

$$\mathrm{K}_1(x, \mathrm{D}) = 2x_1 \sum_{j,k} g_{x_j x_k} \mathrm{D}_{x_j x_k} + i\Delta g(y')\ \mathrm{D}_{x_1} .$$

Denote by $p(\tau, \xi) = \tau^2 - |\xi|^2$ the symbol of $\overline{\mathrm{P}}$ and let $\lambda_\pm(\tau, \xi')$ be the roots of the equation $p(\tau, \xi) = 0$, $\mathrm{Im}\tau < 0$, with respect to ξ_1. By convention, we put $\pm\ \mathrm{Im}\lambda_\pm(\tau, \xi') > 0$.

Let $\hat{v}(\tau, \xi)$ denotes the Fourier-Laplace transform of $v(t, x)$ for $\mathrm{Im}\tau < 0$. From the previous section we know that

$$\hat{\bar{u}}_0(\tau, \xi) = -i\tau(2\pi)^{-n-1} e^{-iy\xi}(p(\tau, \xi))^{-1} .$$

The parametrix E_0 has the form $\widehat{\mathrm{E}_0 f} = (p(\tau, \xi))^{-1} \hat{f}(\tau, \xi)$. On the other hand, the parametrix E' with Dirichlet boundary condition becomes

$$(\mathrm{E}'f)\,(t, x) = \int_{\mathrm{L}_-} \int_{\mathbf{R}^{n-1}} e^{i(t\tau + x_1\lambda_+(\tau, \xi') + x'\xi')}\ \hat{f}(\tau, \xi')\ d\tau d\xi'$$

where $\mathrm{L}_- = \{\tau ; \mathrm{Im}\tau = \tau_0 < 0\}$. Similarly, the parametrix, associated to the Neumann problem, is given by

$(\mathrm{E}'f)(t, x)$

$$= \int_{\mathrm{L}_-} \int_{\mathbf{R}^{n-1}} e^{i(t\tau + x_1\lambda_+(\tau,\xi') + x'\xi')} (i\lambda_+(\tau, \xi'))^{-1} \hat{f}(\tau, \xi')\, d\tau d\xi'.$$

Finally, we obtain $\mathrm{E} = (\mathrm{I} - \mathrm{E}'\mathcal{B}_j)\,\mathrm{E}_0$, $j = \mathrm{D}, \mathrm{N}$.

Next, in our exposition we restrict our attention to the Neumann problem, since the analysis of the Dirichlet problem is similar and simpler.

In the development $(5.11)_{\mathrm{N}}$ the terms with weights 0, 1 have the form

$$u_1^{(0)}(t, x, y) = \mathrm{E}'\mathcal{B}_{\mathrm{N}}\overline{u}_0, \tag{6.1}$$

$$\begin{aligned} u_1^{(1)}(t, x, y) = {} & \mathrm{E}'\mathcal{B}_{\mathrm{N}}\mathrm{E}_0\mathrm{K}_1\overline{u}_0 + \mathrm{E}\mathrm{K}_1\mathrm{E}'\mathcal{B}_{\mathrm{N}}\overline{u}_0 \\ & + \gamma(g(x'), x')\,(\mathrm{E}'\mathcal{B}_{\mathrm{D}}\overline{u}_0 - \mathrm{E}'\mathcal{B}_{\mathrm{D}}\mathrm{E}'\mathcal{B}_{\mathrm{N}}\overline{u}_0). \end{aligned} \tag{6.2}$$

First, we deal with (6.1). Taking into account the expressions for $\hat{\overline{u}}_0(\tau, \xi, y)$ and E_0, we obtain

$\widehat{\mathcal{B}_{\mathrm{N}}\overline{u}_0}(\tau, \xi', y)$

$$= -\tau(2\pi)^{-n-1} e^{-iy'\xi'} \lim_{\mathrm{R}\to\infty} \int_{\Gamma_{\mathrm{R}}^-} \frac{\xi_1(\xi_1 - \lambda_+(\tau, \xi'))^{-1}}{\xi_1 - \lambda_-(\tau, \xi')}\, e^{-iy_1\xi_1} d\xi_1,$$

where $\Gamma_{\mathrm{R}}^- = [-\mathrm{R}, \mathrm{R}] \cup \{z\,;\, |z| = \mathrm{R},\ \mathrm{Im}\, z < 0\}$. Computing the integral, we are going to

$$\widehat{\mathcal{B}_{\mathrm{N}}\overline{u}_0}(\tau, \xi', y) = \frac{i}{2}\,\tau(2\pi)^{-n} e^{-iy'\xi'} e^{-iy_1\lambda_-}.$$

Therefore,

$$\hat{u}_1^{(0)}(\tau, x_1, \xi', y) = -(2\pi)^{-n} e^{-iy'\xi'} e^{-i(x_1+y_1)\lambda_-}(2\lambda_-)^{-1}. \tag{6.3}$$

Now we turn to the analysis of (6.2). It is convenient to set

$$\mathrm{K}_{11} = 2x_1 \sum_{j,k} g_{x_jx_k}(y')\,\mathrm{D}_{x_jx_k}, \quad \mathrm{K}_{12} = i\mathrm{H}(g(y'), y')\,\mathrm{D}_{x_1}.$$

As the form of K_{11} shows, we have

$$\mathrm{E}_0\mathrm{K}_{11}\overline{u}_0 = -i\tau(2\pi)^{-n-1} \sum_{j,k} g_{x_jx_k}\,\xi_j\,\xi_k(2y_1 p^{-2} + 4i\xi_1 \rho^{-3}).$$

Furthermore,

$$\mathcal{B}_{\mathrm{N}}\widehat{\mathrm{E}_0\mathrm{K}_{11}}\overline{u}_0(\tau, \xi', y) = -i\tau(2\pi)^{-n-1} e^{-iy'\xi'} \sum_{j,k} g_{x_jx_k}\,\xi_j\,\xi_k$$

$$\times\, 2 \lim_{\mathrm{R}\to\infty} \int_{\Gamma_{\mathrm{R}}^-} \left(y_1 \frac{i\xi_1(\xi_1 - \lambda_+)^{-2}}{(\xi_1 - \lambda_-)^2} + \frac{4\xi_1(\xi_1 - \lambda_+)^{-3}}{(\xi_1 - \lambda_-)^3}\right) e^{-iy_1\xi_1} d\xi_1.$$

Performing some calculations, we get

$$E'\mathcal{B}_N \widehat{E_0 K_{11}}\, \overline{u}_0 = \frac{\tau}{4}(2\pi)^{-n} e^{-iy'\xi'} e^{-i(x_1+y_1)\lambda_-} \times \sum_{j,k} g_{x_j x_k} \xi_j \xi_k (i - y_1\lambda_- - iy_1^2\lambda_-^2)\lambda_-^{-4}.$$

Repeating the same procedure, we obtain

$$E'\mathcal{B}_N \widehat{E_0 K_{12}}\, \overline{u}_0 = \frac{\tau}{4}(2\pi)^{-n} e^{-iy'\xi'} e^{-i(x_1+y_1)\lambda_-} H(i + y_1\lambda_-)\lambda_-^{-2}$$

and this completes the calculation of the first term in (6.2).

To handle the second one, we make use of the equality

$$EK_{11} E'\mathcal{B}_N \overline{u}_0 = (I - E'\mathcal{B}_N) E_0 K_{11} E'\mathcal{B}_N \overline{u}_0 .$$

First, we get

$$EK_{11} \widehat{E'\mathcal{B}_N}\, \overline{u}_0 = -\frac{\tau}{\lambda_-}(2\pi)^{-n-1} e^{-iy'\xi'} e^{-iy_1\lambda_-} \sum_{j,k} g_{x_j x_k} \xi_j \xi_k \times \lim_{R\to\infty} \int_{\Gamma_R^+} \left(e^{ix_1\xi_1} + \frac{i\xi_1 e^{-ix_1\lambda_-}}{\lambda_-} \right) \frac{(\xi_1 - \lambda_-)^{-1}}{(\xi_1 + \lambda_-)^3}\, d\xi_1$$

with $\Gamma_R^+ = [-R, R] \cup \{z ; |z| = R,\ \mathrm{Im} > 0\}$. Taking the integral, we have

$$EK_{11} \widehat{E'\mathcal{B}_N}\, \overline{u}_0 = \frac{\tau}{4}(2\pi)^{-n} e^{-iy'\xi'} e^{-i(x_1+y_1)\lambda_-} \times \sum_{j,k} g_{x_j x_k} \xi_j \xi_k (i - x_1\lambda_- - ix_1^2\lambda_-^2)\lambda_-^{-4}.$$

In a similar way, we find

$$EK_{12} \widehat{E'\mathcal{B}_N}\, \overline{u}_0 = \frac{\tau}{4}(2\pi)^{-n} e^{-iy'\xi'} e^{-i(x_1+y_1)\lambda_-} H(i - x_1\lambda_-)\lambda_-^{-2}.$$

Finally, treating the last two terms in (6.2), we obtain

$$E'\mathcal{B}_D \overline{u}_0 - E'\mathcal{B}_D E'\mathcal{B}_N \overline{u}_0 = 0 .$$

Summarizing the above calculations, we are going to

$$u_1^{(1)}(\tau, y_1, \xi', y) = \frac{i\tau}{2}(2\pi)^{-n} e^{-iy'\xi'} e^{-2iy_1\lambda_-} \lambda_-^{-2} \times \Big[\sum_{j,k} g_{x_j x_k}(y')\, \xi_j \xi_k (\lambda_-^{-2} + iy_1\lambda_-^{-1} - y_1^2) + H(g(y'), y') + 2\gamma(g(y'), y') \Big] . \tag{6.4}$$

In order to prepare the calculation of $\hat{L}_j(\tau, \xi')$, $j = 0, 1$, we shall take the integration over y_1. It is clear, that without loss of the generality, we may assume that $\varphi(x) = \varphi_1(x_1)\,\varphi_2(x')$ where $\varphi_1 \in C_0^\infty(\mathbf{R})$, $\varphi_2 \in C^\infty(\mathbf{R}^{n-1})$ and $\varphi_1 = 1$ in some neighborhood to $x_1 = 0$. Therefore,

$$\hat{L}_0(\tau, \xi', y') = \frac{i\tau}{2}\,(2\pi)^{-n} e^{-iy'\xi'}\,\varphi_2(y')\,\lambda_-^{-2} \times\Big(1 + \frac{i}{4}\,H(g(y'), y')\,\lambda_-^{-1} + \text{lower order terms}\Big), \tag{6.5}$$

$$\hat{L}_1(\tau, \xi', y') = \frac{\tau}{2}\,(2\pi)^{-n} e^{-iy'\xi'}\,\varphi_2(y')\,\lambda_-^{-3}\Big[\lambda_-^{-2}\sum_{j,k} g_{x_j x_k}(y')\,\xi_j\xi_k + \frac{1}{2}\,H(g(y'), y') + \gamma(g(y'), y') + \text{lower order terms}\Big]. \tag{6.6}$$

After this preparatory work, we obtain the Fourier transform $\hat{L}_j(\lambda, \xi', y')$, making use of the equality

$$\hat{L}_j(\lambda, \xi', y') = \hat{L}_j(\lambda - i0, \xi', y') - \hat{L}_j(\lambda + i0, \xi', y').$$

Taking the inverse Fourier transform, it is important to note, that the distributions $\hat{L}_j(\lambda, \xi', y')$, $j = 0, 1$ have compact support. We omit the easy proof of this fact.

Next, our aim is to find the inverse Fourier transform of

$$\hat{v}_{j,k}(\tau, \xi', y') = e^{-iy'\xi'}\,\xi_j\xi_k\,[((\tau - i0)^2 - |\xi'|^2)^{-5/2} - ((\tau + i0)^2 - |\xi'|^2)^{-5/2}],$$

where the branch $0 \leqslant \operatorname{Arg} z < 2\pi$ for $\operatorname{Arg} z$ is taken. Before we proceed with the analysis of $\hat{v}_{j,k}$, it is necessary to justify the choice of this branch. For this purpose we go back and recall the form of the parametrix E'. It is well-known [26, 13], that $WF_b(E'u)$ does not contain points which belong to the elliptic region, given by the inequality $|\tau| < |\xi'|$. Therefore, we need to arrange the equality $((\tau - i0)^2 - |\xi'|^2)^{1/2} = ((\tau + i0)^2 - |\xi'|^2)^{1/2}$ and for this reason we take $0 \leqslant \operatorname{Arg} z < 2\pi$.

It is convenient to introduce the distributions

$$T_z = [((\tau - i0)^2 - \rho^2)^z - ((\tau + i0)^2 - \rho^2)^z]\,H(\rho) = \begin{cases} |\tau^2 - \rho^2|\,\operatorname{sign}\tau(e^{2\pi iz} - 1)\,H(\rho), & |\tau| > \rho, \\ 0, & |\tau| < \rho, \end{cases}$$

where $H(\rho)$ is the Heaviside function. Moreover, T_z is an analytic function of z for $\operatorname{Re} z > -1$ and meromorfic for $z \in \mathbf{C}$ with poles at $z = -1, -2, \ldots$. To simplify the notations, let us introduce also the distributions $Q_z = ((\tau - i0)^2 - \rho^2)^z - ((\tau + i0)^2 - \rho^2)^z$. Since for $|\tau| > 0$, $\hat{v}_{j,k}(\tau, \xi', y')$ has no singularity at $\xi' = 0$, we may work with polar coordinates $\xi' = \rho\omega$, $x' - y' = r\theta$, $\omega \in S^{n-1}$, $\theta \in S^{n-1}$. The inverse Fourier transform of $\hat{v}_{j,k}(\tau, \xi', y')$ is a smooth function and the trace $x' = y'$ can be evaluated, putting $r = 0$. Consequently, we obtain

$$v_{j,k}(\tau, y', y') = \langle \rho^2 \omega_j \omega_k T_{-5/2}, \rho^{n-2} \rangle$$

$$= \delta_{jk} \frac{\operatorname{vol} S^{n-2}}{n-1} \langle T_{-5/2}, \rho^n \rangle = -\delta_{jk} \frac{\operatorname{vol} S^{n-2}}{3} \langle T_{-3/2}, \rho^{n-2} \rangle .$$

First, we treat the case n odd and set $n = 2k+1$, $k \geqslant 2$. A simple calculation yields

$$T_{-3/2} = ((-1)^{k+1}(2k-3)!!)^{-1} \left(\frac{1}{\rho} \frac{\partial}{\partial \rho}\right)^k T_{(2k-3)/2},$$

which implies

$$\langle T_{-3/2}, \rho^{2k-1} \rangle = \frac{(2k-2)!!}{(2k-3)!!} \langle H(\rho) \frac{\partial}{\partial \rho} Q_{(2k-3)/2}, 1 \rangle$$

$$= \frac{2^k (n-2)(k-1)! \operatorname{sign} \tau \, \tau^{n-4}}{(2k-1)!!} .$$

Therefore, taking the product with $\operatorname{vol} S^{n-2} = \dfrac{2\pi^{(n-1)/2}}{(k-1)!}$, we get

$$\langle T_{-3/2}, \rho^{2k-1} \rangle \operatorname{vol} S^{n-2} = (\operatorname{sign} \tau) \tau^{n-4} \frac{2(n-2)(4\pi)^{-n/2}}{\Gamma(n/2)} . \tag{6.7}$$

Now, let $n \geqslant 4$ be even. We have

$$\langle T_{-3/2}, \rho^{n-4} \rangle = -(n-3) \langle T_{-1/2}, \rho^{n-4} \rangle$$

$$= 2(\operatorname{sign} \tau) \tau^{n-4} (n-3) \int_0^1 t^{n-4} (1-t^2)^{-1/2} \, dt .$$

Computing the last integral, we are going to

$$\langle T_{-3/2}, \rho^{n-2} \rangle = 2(\operatorname{sign} \tau) \tau^{n-4} \pi/2 \, (n-3)!! \, ((n-4)!!)^{-1},$$

which implies (6.7).

From (6.5) and (6.6) we bring together the terms, involving $(\lambda_-)^{-3}$, and obtain the expression

$$\tau(2\pi)^{-n} e^{-iy'\xi'} \varphi_2(y') \langle T_{-3/2}, \rho^{n-2} \rangle \operatorname{vol} S^{n-2} \times \frac{1}{8} (H(g(y'), y') + 4\gamma(g(y'), y')).$$

The first term in (6.5) can be treated by an argument, similar to that, used for $\hat{u}_0(\lambda, \xi, y)$ in section 5. Then summarizing all calculations and applying (6.7), we have for $\tau > 0$ the following asymptotics

$$L_0(\tau, y') + L_1(\tau, y') = -\frac{(4\pi)^{-(n-1)/2}(n-1)}{8\Gamma\left(\frac{n-1}{2}+1\right)} \varphi_2(y')\tau^{n-2}$$
$$-\frac{(4\pi)^{-n/2}(n-2)}{\Gamma(n/2)}\left(\frac{H(g(y'), y')}{12} - \gamma(g(y'), y')\right)\varphi_2(y')\tau^{n-3} + O(\tau^{n-4}).$$

The remainder of the section is devoted to the analysis of the case $\nabla g(y') \neq 0$. Let z be new coordinates, given by $z = Cy$, where C is an orthogonal matrix. Following the argument in section 5, introduce the functions

$$(y_1, y') = \phi(r, x') = \psi(x') + rN(\psi(x')),$$
$$(z_1, z') = \widetilde{\phi}(r, z') = \widetilde{\psi}(z') + rN(\widetilde{\psi}(z'))$$

with $\psi(x') = (g(x'), x')$, $\widetilde{\psi}(z') = (\widetilde{g}(z'), z')$. The wave operator in these coordinates has the form

$$P = D_t^2 - a(r, x') D_r^2 - \langle R(r, x') D_{x'}, D_{x'} \rangle + b(r, x') D_r + P_1(r, x', D_{x'}),$$
$$\widetilde{P} = D_t^2 - \widetilde{a}(r, z') D_r^2 - \langle \widetilde{R}(r, z') D_{z'}, D_{z'} \rangle + \widetilde{b}(r, z') D_r + \widetilde{P}_1(r, z', D_{z'}).$$

Here $R, \widetilde{R}$ are $(n-1)\times(n-1)$ matrices, $P_1, \widetilde{P}_1$ are differential operators of first order with homogeneous symbols and $a, \widetilde{a}$, $b, \widetilde{b}$ are smooth functions. Consider the Taylor expansion for P and $\widetilde{P}$ respectively near $(0, y')$ and $(0, \widetilde{y}')$. We write down the terms with weight 0 or 1:

$$\overline{P}(y', D_t, D_x) = D_t^2 - a(0, y') D_r^2 - \langle R(0, y') D_{x'}, D_{x'} \rangle,$$
$$\overline{\widetilde{P}}(\widetilde{y}', D_t, D_z) = D_t^2 - \widetilde{a}(0, \widetilde{y}') D_r^2 - \langle \widetilde{R}(0, \widetilde{y}') D_{z'}, D_{z'} \rangle,$$

$$\mathrm{K}_1 = -r\left\langle \frac{\partial \mathrm{R}}{\partial r}(0, y')\mathrm{D}_{x'}, \mathrm{D}_{x'}\right\rangle - r\frac{\partial a}{\partial r}(0, y')\mathrm{D}_r^2 + b(0, y')\mathrm{D}_r + \sum_{j=2}^{n}(x_j - y_j)\mathrm{R}_j(\mathrm{D}_x) + \mathrm{P}_1'(\mathrm{D}_{x'}),$$

$$\widetilde{\mathrm{K}}_1 = -r\left\langle \frac{\partial \widetilde{\mathrm{R}}}{\partial r}(0, \widetilde{y}')\mathrm{D}_{z'}, \mathrm{D}_{z'}\right\rangle - r\frac{\partial \widetilde{a}}{\partial r}(0, \widetilde{y}')\mathrm{D}_r^2 + \widetilde{b}(0, \widetilde{y}')\mathrm{D}_r + \sum_{j=2}^{n}(z_j - \widetilde{y}_j)\widetilde{\mathrm{R}}_j(\mathrm{D}_z) + \widetilde{\mathrm{P}}_1'(\mathrm{D}_{z'}).$$

Here R_j, $\widetilde{\mathrm{R}}_j$ are differential operators of second order with homogeneous symbols, while P_1', $\widetilde{\mathrm{P}}_1'$ are first order differential operators with homogeneous symbols. Let $\widetilde{y}' = h(y')$ where $h(y')$ is given by the equality $(r, h(x')) = \widetilde{\phi}^{-1}\phi(r, x')$. Our purpose is to compare the operators $\overline{\mathrm{P}}$, $\overline{\widetilde{\mathrm{P}}}$ as well as K_1 and $\widetilde{\mathrm{K}}_1$. Writing $\widetilde{\mathrm{P}}$ in coordinates (r, x') and applying the Taylor formula for $({}^t\mathrm{D}h)^{-1}$, we obtain

$$\widetilde{a}(0, \widetilde{y}') = a(0, y'),\ \frac{\partial \widetilde{a}}{\partial r}(0, \widetilde{y}') = \frac{\partial a}{\partial r}(0, y'),\ \widetilde{b}(0, \widetilde{y}') = b(0, y'),$$

$$(\mathrm{D}h)^{-1}(\widetilde{y}')\,\widetilde{\mathrm{R}}(0, \widetilde{y}')\,({}^t\mathrm{D}h)^{-1}(\widetilde{y}') = \mathrm{R}(0, y'),$$

$$(\mathrm{D}h)^{-1}(\widetilde{y}')\,\frac{\partial \widetilde{\mathrm{R}}}{\partial r}(0, \widetilde{y}')\,({}^t\mathrm{D}h)^{-1}(\widetilde{y}') = \frac{\partial \mathrm{R}}{\partial r}(0, y').$$

These relations imply

$$\overline{\mathrm{P}}(y', \mathrm{D}_t, \mathrm{D}_r, \mathrm{D}_{x'}) = \overline{\widetilde{\mathrm{P}}}(h(y'), \mathrm{D}_t, \mathrm{D}_r, ({}^t\mathrm{D}h)^{-1}(y')\mathrm{D}_{x'}) \tag{6.9}$$

and consequently

$$u_1^{(0)}(t, r, y', r, y') = \widetilde{u}_1^{(0)}(t, r, h(y'), r, h(y')) \tag{6.10}$$

where $\widetilde{u}_1^{(k)}(t, z, y)$, $k = 0, 1$, are the distributions related to $\widetilde{\mathrm{P}}$.

In order to deduce a similar connection between $u_1^{(1)}$ and $\widetilde{u}_1^{(1)}$, we need to study the terms given by (6.2). Introducing the operator

$$\widetilde{\mathrm{K}}_1^0(r, \mathrm{D}_r, \mathrm{D}_{z'}) = -r\left\langle \frac{\partial \widetilde{\mathrm{R}}}{\partial r}(0, \widetilde{y}')\mathrm{D}_{z'}, \mathrm{D}_{z'}\right\rangle - r\frac{\partial \widetilde{a}}{\partial r}(0, \widetilde{y}')\mathrm{D}_r^2 + b(0, \widetilde{y}')\mathrm{D}_r,$$

we obtain the representations

$$\mathrm{K}_1(y', r, x', \mathrm{D}_t, \mathrm{D}_r, \mathrm{D}_{x'}) = \widetilde{\mathrm{K}}_1^0(r, \mathrm{D}_t, \mathrm{D}_r, ({}^t\mathrm{D}h)^{-1}(y'), \mathrm{D}_{x'}) + \sum_{j=2}^{n} (x_j - y_j)\,\mathrm{M}_j + \mathrm{L}_1(\mathrm{D}_{x'}),$$

$$\widetilde{\mathrm{K}}_1(h(y'), r, z', \mathrm{D}_t, \mathrm{D}_r, \mathrm{D}_{z'}) = \widetilde{\mathrm{K}}_1^0(r, \mathrm{D}_t, \mathrm{D}_r, \mathrm{D}_{z'}) + \sum_{j=2}^{n} (x_j - y_j)\,\widetilde{\mathrm{M}}_j + \widetilde{\mathrm{L}}_1(\mathrm{D}_{z'})$$

where M_j, $\widetilde{\mathrm{M}}_j$, L_1, $\widetilde{\mathrm{L}}_1$ are differential operators with homogeneous symbols and $\operatorname{ord} \mathrm{M}_j = \operatorname{ord} \widetilde{\mathrm{M}}_j = 2$, $\operatorname{ord} \mathrm{L}_1 = \operatorname{ord} \widetilde{\mathrm{L}}_1 = 1$. Using the solutions $\widetilde{u}_0$, $\widetilde{\widetilde{u}}_0$, related to $\overline{\mathrm{P}}$, $\overline{\overline{\mathrm{P}}}$, the associated parametrices and comparing the terms corresponding to the operator $\widetilde{\mathrm{K}}_1^0$, we conclude that

$$u_1^{(1)}(t, r, y', r, y') = \widetilde{u}_1^{(1)}(t, r, h(y'), r, h(y')). \tag{6.11}$$

Here the coefficients of $\widetilde{\mathrm{K}}_1^0$ do not depend on z', which enables us to find a simple relation between the form of $\widetilde{\mathrm{K}}_1^0$ in the coordinate systems under consideration. The other terms in K_1 and $\widetilde{\mathrm{K}}_1$ involve only the operators $(x_j - y_j)\,\mathrm{D}_{x_k x_1}$, $(x_j - y_j)\,\mathrm{D}_r^2$, D_{x_j}, which do not contribute to the asymptotics of $\widetilde{u}_1^{(1)}(\lambda, y, y)$.

Now consider

$$\widetilde{\mathrm{L}}_j(\lambda, h(y')) = \int_0^\infty \widetilde{u}_1^{(j)}(\lambda, r, h(y'), r, h(y'))\,\varphi_1(r) \times \mathrm{D}\left(\frac{\widetilde{y}_1, \widetilde{y}'}{r, z'}\right)(r, h(y'))\,dr, \quad j = 0, 1.$$

Making use of (6.10), (6.11), it is easy to establish the equality

$$(1 + |\nabla g(y')|^2)^{1/2}\,\mathrm{L}_j(\lambda, y') = (1 + |\nabla \widetilde{g}(h(y'))|^2)^{1/2}\,\widetilde{\mathrm{L}}_j(\lambda, h(y')).$$

Therefore, the case $\nabla g(y') \neq 0$ can be reduced to that with $\nabla g(y') = 0$, and this completes the computation of the second and third term. The result of theorem 1 follows immediately from the asymptotics of $\frac{ds}{d\lambda}(\lambda)$.

The above reduction can be avoided if a stronger version of the so called normal singularity at $t = 0$ is proved. Namely, it is necessary to show, that the distribution

$$F(t, x') = \int_0^\infty \rho(t)\, u(t, x_1, x', x_1, x')\, \varphi_1(x_1)\, \varphi_2(x')\, dx_1$$

satisfies the property (5.12), uniformly for $x' \in \operatorname{supp} \varphi_2$. This information can be deduced from the arguments, exposed in [13], but some complementary work is needed.

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Manuscrit reçu le 15 juillet 1981
révisé le 1er février 1982.

V. PETKOV & G. POPOV,
Institute of Mathematics of Bulgaria
Academy of Sciences
P.O. Box 373
1090 Sofia (Bulgaria).