Regular holomorphic images of balls

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REGULAR HOLOMORPHIC IMAGES OF BALLS

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Dedicated to Walter Rudin on the occasion
of his sixtieth birthday, May 2, 1981.

A few years ago we proved [2] that every paracompact connected N-
dimensional complex manifold is the image of an N-dimensional polydisc
under a finite regular holomorphic map. When we wrote [2] it was not
clear to us that an analogous result could be obtained for the ball. In the
present paper we settle this question as follows:

THEOREM. — For each \( N = 1, 2, \ldots \), there is a positive integer \( \lambda_N \) such
that if \( \mathcal{M} \) is a connected, paracompact N-dimensional complex manifold, then
there is a regular holomorphic map \( \Phi \) from the unit ball \( B_N \) in \( \mathbb{C}^N \) onto \( \mathcal{M} \)
such that for each \( p \in \mathcal{M} \), the fiber \( \Phi^{-1}(p) \) contains not more than \( \lambda_N \)
points.

As we shall see, for \( \lambda_N \) we can take the value

\[
(1 + 4^{12N}(12N)^{2N}((2N + 1)4^N + 2).
\]

This is not the best possible value for \( \lambda_N \); at the cost of considerable
complication in the proof below, we could improve this bound somewhat,
but the gain would be more apparent than real. We have no inkling of the
best possible value of \( \lambda_N \) nor of any way of finding it. (We point out
however that D. H. Bushnell [1] has improved our estimate for the polydisc
case from \( (2N + 1)4^N + 2 \) to \( (2N + 1)^2 + 1 \). This has the immediate effect
of reducing our \( \lambda_N \) to \( (1 + 4^{12N}(12N)^{2N}((2N + 1)^2 + 1) \) as we shall see in the
course of the proof.)

We treat the ball case by reducing it to the case of the polydisc. Two
new ingredients in this work are these. First, we invoke a result from [3] to

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recognize that certain monotone unions of balls are again balls. Second, based on some estimates in the plane, we have a lemma to the effect that certain dumbbell-shaped sets in \( \mathbb{C}^N \) admit rough approximations by domains biholomorphically equivalent to balls.

The paper is organized as follows: In Section 1 we state two lemmas. In Section 2 we give the proof of the theorem assuming the lemmas. Sections 3 and 4 are devoted to the proofs of the lemmas, and Section 5 contains some concluding remarks.

The case \( N = 1 \) of the theorem was proved in [2]. Hence we will assume in the rest of this paper that \( N \geq 2 \).

1. Two lemmas.

We shall use the notation that for \( x \in \mathbb{C} \), \( x \) denotes the point \((x,0,\ldots,0) \in \mathbb{C}^N\). Also, for \( r \in (0,\infty) \), \( K_r \) denotes the compact set in \( \mathbb{C}^N \) consisting of the union of \( B_N \), the interval \([0,4]\) and the ball \( 4 + rB_N \) of radius \( r \), center 4.

**Lemma 1.** — If \( U \) is a neighborhood of \( K_2 \), then there is an open set \( W \) biholomorphically equivalent to \( B_N \) with \( K_{1/16} \subseteq W \subseteq U \).

It would be of interest to know whether the dumbbell \( K_1 \) has a fundamental neighborhood basis consisting of balls, i.e., of domains biholomorphically equivalent to balls.

Denote by \( E \) the union of the ray \( \{x : x \in \mathbb{R}, x \geq 0\} \) and the balls \( 4j + B_N, j = 0, 1, 2 \ldots \).

**Lemma 2.** — If \( D \) is a domain in \( \mathbb{C}^N \), then there is a regular holomorphic map \( \Phi \) from a neighborhood \( V \) of \( E \) onto \( D \) such that for each \( p \in D \), the fiber \( \Phi^{-1}(p) \) consists of no more than

\[
c_N = 1 + 4^{12N}(12N)^{2N}
\]

points and such that

\[
\bigcup_{j=0}^{\infty} \Phi(4j + \frac{1}{64}B_N) \supseteq D.
\]

2. Proof of the theorem.

Let \( \mathcal{M} \) be a connected paracompact complex manifold of dimension \( N \). The Main Theorem of [2] provides a regular holomorphic map from
the unit polydisc in \( \mathbb{C}^N \) onto \( \mathcal{M} \) the multiplicity of which does not exceed \((2N+1)4^N + 2\). Thus, to prove the theorem, it would suffice to construct a regular holomorphic map from \( B_N \) onto a polydisc with multiplicity no more than \( 1 + 4^{12N}(12N)^2 \). We shall do somewhat more: We shall treat not only the case of the polydisc but rather the case of an arbitrary domain in \( \mathbb{C}^N \). Our reason for doing this is twofold. The case of a general domain in \( \mathbb{C}^N \) is not especially harder than the case of the polydisc, and by treating the case of general domains in \( \mathbb{C}^N \) directly, we obtain a relatively simple proof of our theorem for these special domains. Thus, we shall prove the following result.

**Lemma 3.** — If \( D \) is a domain in \( \mathbb{C}^N \), then there is a regular holomorphic map \( \Psi \) from \( B_N \) onto \( D \) such that for each \( p \in D \), the fiber \( \Psi^{-1}(p) \) consist of no more than \( c_N \) points.

The \( c_N \) of this lemma is the same as the \( c_N \) of lemma 2.

**Proof.** — Let \( E, V \) and \( \Phi : V \to D \) be as in lemma 2.

We shall construct a domain \( \Omega \) biholomorphically equivalent to \( B_N \) with

\[
\bigcup_{j=0}^{\infty} \left( 4j + \frac{1}{64} B_N \right) \subset \Omega \subset V.
\]

We may assume without loss of generality that

\[ V \subset \{ z \in \mathbb{C}^N : z = (z_1, \ldots, z_N) \text{ with } |\text{Im } z_j| < 2 \text{ for } j = 1, 2, \ldots, N \}. \]

The domain on the right is biholomorphically equivalent to a polydisc and so is \textit{taut} in the sense of [4]. By lemma 1 of [3], a monotone union of balls in \( V \) is (biholomorphically equivalent to) a ball. Accordingly, we shall construct a sequence \( \{ \Omega_j \}_{j=0}^{\infty} \) of domains with the following properties:

1. \( \Omega_j \subset V \) for all \( j \).
2. \( 4j + \frac{1}{16} B_N \subset \Omega_j \) for all \( j \).
3. \( \Omega_0 \subset \Omega_1 \subset \Omega_2 \subset \ldots \).
4. \( \Omega_j \cap \bigcup_{k>j} (4k + \bar{B}_n) = \emptyset \) for all \( j \).
(5) \( \Omega_j \) is biholomorphically equivalent to \( B_N \) under a biholomorphic map that extends to carry a neighborhood of \( B_N \) biholomorphically onto a domain in \( V \).

Conditions (1), (2) and (3) are the ones of interest; (4) and (5) are used to smooth the induction. Granted such a sequence, \( \Omega = \bigcup_{j=0}^{\infty} \Omega_j \) is a domain in \( V \) biholomorphically equivalent to a ball, and it is carried onto \( D \) by the map \( \Phi \) with the desired bound on the multiplicity.

To construct the sequence \( \{ \Omega_j \}_{j=0}^{\infty} \), let \( \Omega_0 = B_N \). Assume \( \Omega_0, \Omega_1, \ldots, \Omega_k \) have been constructed so as to satisfy conditions (1)-(5). Let \( \psi_k : B_{4} \to \Omega_k \) be a biholomorphic map that extends, as in (5), to carry a neighborhood of \( B_{4} \) biholomorphically onto a neighborhood of \( \Omega_k \). Denote by \( \chi \) a smooth map from a neighborhood of \( K_2 \) into \( V \) that agrees with \( \psi_k \) near \( B_{4} \) and that agrees on a neighborhood of \( 4 + 2B_{4} \) with the affine map \( z \to 4(k+1) + \frac{1}{2}(z-4) \) that carries the ball \( 4 + 2B_{4} \) onto the ball \( 4(k+1) + B_{4} \). We require, in addition, that \( \chi \) carry the interval \( (0,4) \) diffeomorphically into \( V \setminus (\bigcup_{m \geq k+1} (4m + B_{N})) \). We may approximate \( \chi \) uniformly on a neighborhood of \( K_2 \) by a holomorphic map \( \psi_{k+1} \) from \( C^N \) to \( C^N \)-note that \( K_2 \) is polynomially convex and we may insist, moreover, that the approximation be close in the \( \mathcal{C}^1 \) sense on \( [0,4] \). By making the approximation close enough, we shall have that \( \psi_{k+1} \) is one-to-one on \( K_2 \). By replacing \( \psi_{k+1} \) by \( \psi_{k+1} \) defined by \( \psi_{k+1}(z) = \psi_{k+1}(z + z_0) \) for a suitably chosen, small \( z_0 \) if necessary, we can be sure that \( \psi_{k+1} \) is regular on a neighborhood of \( K_2 \). As \( \Omega_{k-1} \subset \subset \Omega_k \), there is a domain \( \Omega_{k+1} \) biholomorphically equivalent to a ball with \( \Omega_k \subset \subset \Omega_{k+1} \) and such that \( \Omega_{k+1} \) satisfies (1)-(5). Granted that our approximation of \( \chi \) by \( \psi_{k+1} \) is close enough, we shall have \( \psi_{k+1}(B_{N}) \supset \supset \Omega_{k} \) and

\[
\psi_{k+1}(4 + \frac{1}{16} B_{N}) \supset \supset 4(k+1) + \frac{1}{64} B_{N}.
\]

As \( \psi_{k+1} \) is one-to-one and regular on \( K_2 \), there is a neighborhood \( U \) of \( K_2 \) carried biholomorphically into \( V \setminus \bigcup_{m \geq k+1} (4m + B_{N}) \) by \( \psi_{k+1} \). Lemma 1 provides a domain \( W \) biholomorphically equivalent to \( B_{N} \) with \( K_{1/16} \subset \subset \subset \subset W \subset U \). For \( \Omega_{k+1} \) we take \( \psi_{k+1}(W) \). The sequence
\( \Omega_0, \ldots, \Omega_{k-1}, \Omega_k, \Omega_{k+1} \) enjoys the properties (1)-(5). Thus, our sequence \( \{\Omega_j\}_{j=0}^{\infty} \) can be constructed inductively; at each step we alter the last \( \Omega_k \) constructed, but having altered it once, it remains unaltered through the rest of the construction.

This proves lemma 3 and shows that to prove the theorem, it suffices to prove lemmas 1 and 2.


At one point in the argument below we will need the following elementary geometric fact.

**Lemma 1.1.** — For \( d \in (0, \infty) \), let \( S_d \) be the union of all the discs in the open unit disc with centers on the real diameter and hyperbolic radius \( d \). If \( \alpha(d) = 2 \tanh d/(1-\tanh d)^2 \), then \( S_d \subset \{ z = x + iy : |y| < \alpha(d)(1-x) \} \).

**Proof.** — Denote by \( \rho \) the hyperbolic distance function on the disc so that if \( [z,w] = \frac{z-w}{1-wz} \), then \( \rho(z,w) = \frac{1}{2} \log \frac{1 + [z,w]}{1 - [z,w]} \). If \( x \in (-1,1) \) satisfies \( \rho(0,x) = d \), then \( x = \pm \tanh d \). Under the non-Euclidean translation \( z \rightarrow \frac{z + t}{1 + tz} = \varphi_t(z) \), the circle \( C_d(0) = \{ z : \rho(z,0) = d \} \) goes onto the circle \( C_d(t) = \{ z : \rho(z,t) = d \} \). If we denote by \( \tau \) the number \( \tanh d \), then the Euclidean radius \( \delta \) of \( C_d(t) \) is determined by \( 2\delta = \varphi_t(\tau) - \varphi_t(-\tau) \) so that

\[
\delta = \frac{\tau(1-t^2)}{1-t^2\tau^2}.
\]

This means that for \( z = x + iy \) inside \( C_d(t) \), we have

\[
|y|/(1-x) < \tau(1-t^2)(1-\delta^2)^{-1}\left(1 - \frac{\tau + x}{1+t\tau}\right) = \tau(1+t)/(1-t\tau)(1-\tau) \leq 2\tau/(1-\tau)^2
\]

which is the desired inequality.

The proof of lemma 1 depends on some estimates of conformal
mappings in the plane. Fix small $\delta_1$ and $\delta_2$, and let $\Delta_n$ be the domain shown in figure 1. Thus, explicitly, $\Delta_n$ is the union of the open disc of radius $1 + \delta_1$ centered at 0, the disc of radius $\frac{1}{8}$ centered at 4, the rectangle

$\{x + iy : 0 < x < 3 - \frac{3}{4}, -\delta_2 < y < \delta_2\}$

and the rectangle

$\{x + iy : 0 < x < 4, -\frac{1}{n} < y < \frac{1}{n}\}$.

Denote by $\varphi_n$ the conformal map from the unit disc $\Delta$ to $\Delta_n$, $\varphi_n$ normalized so that $\varphi_n(0) = 0$ and $\varphi_n'(0) > 0$. Let $\psi_n : \Delta_n \to \Delta$ be the inverse of $\varphi_n$. By symmetry, $\varphi_n$ and $\psi_n$ both map the real line to itself, and we have that $\psi_n(4, \frac{1}{8}) = 1$ for all $n$.

Lemma 1.2. — The sequence $\{\psi_n(4)\}_{n=1}^{\infty}$ converges to $1 \in b\Delta$.

Proof. — By symmetry, we know that the points $\psi_n(4)$ are real.
A theorem of Carathéodory [5, p. 33] implies that the sequence \( \{\varphi_n\}_{n=1}^{\infty} \) converges, uniformly on compacta in \( \Delta \), to the conformal map, \( \varphi \), from \( \Delta \) onto the kernel of the sequence \( \{\Delta_n\}_{n=1}^{\infty} \) which is defined to be the largest domain \( D, \ 0 \in D \), with the property that if \( S \) is a closed subset of \( D \) with \( S \subset \Delta_n \) for all large \( n \), then \( S \subset D \). The kernel of the sequence \( \{\Delta_n\}_{n=1}^{\infty} \) is the union of the disc of radius \( 1 + \delta_1 \) centered at 0 and the rectangle

\[
\{x+iy : 0 < x < \frac{3}{4}, \quad -\delta_2 < y < \delta_2 \}.
\]

The map \( \varphi \) is normalized to satisfy \( \varphi(0) = 0, \ \varphi'(0) > 0 \), of course. Carathéodory's theorem asserts, moreover, that the sequence \( \{\psi_n\}_{n=1}^{\infty} \) converges uniformly on compacta in \( D \) to \( \varphi^{-1} \). This implies that if \( K \subset \Delta \) is compact, then \( K \subset \psi_n(D) \) for sufficiently large \( n \). As \( \psi_n \) is one-to-one on \( \Delta_n \), it follows that \( \psi_n(4) \) moves to \( b\Delta \) as \( n \to \infty \), and this implies that \( \psi_n(4) \to 1 \), as we wished to show.

A consequence of this lemma, by way of Schwarz's lemma, is that if \( V \) is the disc \( \left\{ z : |z-4| < \frac{1}{16} \right\} \), then the sets \( \psi_n(V) \) move to 1 as \( n \to \infty \):

If \( \varepsilon > 0 \), there is \( n(\varepsilon) \) so that for \( w \in V, \ |\psi_n(w)| - 1 < \varepsilon \) provided \( n > n(\varepsilon) \).

**Lemma 1.3.** — For large \( n \), the set \( \psi_n(V) \) is contained in the Stolz angle

\[
\left\{ z : |z| < 1, \quad |\text{Im} \ z| < 4(1 - \text{Re} \ z) \right\}.
\]

**Proof.** — Put \( t_n = \psi_n(4) \) so that the sequence \( \{t_n\}_{n=1}^{\infty} \) in (0,1) converges to 1. The points on the circle \( bV \) are all at the same distance, \( d = \frac{1}{2} \log 3 \), from the point 4, distance computed in the Carathéodory metric on the disc \( V' = \left\{ z : |z-4| < \frac{1}{8} \right\} \). The holomorphic map \( \psi_n : V' \to \Delta \) decreases the Carathéodory distance, so if \( \rho \) is again the Carathéodory (or hyperbolic) distance function on \( \Delta \), then

\[
\psi_n(V) \subset \left\{ z : \rho(t_n,z) < \frac{1}{2} \log 3 \right\}.
\]

The result now follows from lemma 1.1, for \( \alpha \left( \frac{1}{2} \log 3 \right) = 4 \).
For $\delta \in (0, \pi)$, set $J_\delta = \{ e^{i\theta} : |\theta| < \delta \}$, and define $k_\delta$ to be the harmonic extension of $\chi_{J_\delta} \log \frac{1}{\sqrt{1 - z_1}}$ to the unit disc, $\Delta$, where $\chi_{J_\delta}$ denotes the characteristic function of $J_\delta$.

**Lemma 1.4.** — There is $\delta_0 > 0$ small enough that for all $\delta \in (0, \delta_0)$,

$$(1 - |z_1|^2) e^{2k_\delta(z_1)} \leq 2$$

for $z_1 \in \Delta$.

*Proof.* — The function $\log \frac{1}{\sqrt{1 - z_1}}$ is harmonic on the unit disc and is positive at $z_1 \in \Delta$ if and only if $|1 - z_1| < 1$. For $z_1 = e^{it}$, this happens if and only if $t \in \left( -\frac{\pi}{3}, \frac{\pi}{3} \right)$ (mod $2\pi$). Fix $\varepsilon > 0$ small enough that for $z_1 \in \Delta$ with $\text{Re} z_1 \geq 1 - \varepsilon$, $\log \frac{1}{\sqrt{1 - z_1}} > 0$. Since $k_\delta \to 0$ uniformly on $\{z_1 \in \Delta : \text{Re} z_1 \leq 1 - \varepsilon\}$ when $\delta \to 0$, it suffices to prove the inequality for $z_1 \in \Delta$ with $\text{Re} z_1 \geq 1 - \varepsilon$.

Fix $\delta_0 > 0$ so small that the harmonic extension $h_\delta$ of

$$(1 - \chi_{J_\delta}) \log \frac{1}{\sqrt{1 - z_1}}$$

is positive on $\Delta \cap \{\text{Re} z_1 = 1 - \varepsilon\}$ for all $\delta \in (0, \delta_0)$. The function $h_\delta$ is nonnegative on $\Delta \cap \{z_1 : \text{Re} z_1 \geq 1 - \varepsilon\}$ because it has nonnegative values on the boundary of this set. For $\delta \in (0, \delta_0)$ and $\text{Re} z_1 \geq 1 - \varepsilon$, $z_1 \in \Delta$, we have

$$(1 - z_1 \bar{z}_1) e^{2k_\delta(z_1)} \leq (1 - z_1 \bar{z}_1) e^{2(h_\delta(z_1) + h_\delta(z_1))}$$

$$= (1 - z_1 \bar{z}_1) \left| \frac{1}{1 - z_1} \right|$$

$$= 2.$$  

The lemma is proved.

We can now prove lemma 1. Fix a neighborhood $U$ of the set $K_2$. Choose $\delta_0 > 0$ so small that $|z - w| > \delta_0$ if $z \in K_2, w \notin U$. Fix a small $\delta > 0$, and let $J_\delta$ be the arc of the unit circle as above. Define $k_\delta$ to be
the function holomorphic on the unit disc determined by \( \text{Im } k_\delta(0) = 0 \) and \( \text{Re } k_\delta = k_\delta \) as in the preceding lemma. Thus,

\[
(1) \quad k_\delta(z) = \frac{-1}{2\pi} \int_{-\delta}^{\delta} \frac{e^{it} + z}{e^{it} - z} \log |1 - e^{it}| \, dt.
\]

Define \( \tilde{\phi}_n : B_N \to \mathbb{C}^N \) by

\[
\tilde{\phi}_n(z_1, \ldots, z_N) = (\phi_n(z_1), (1 + \delta_2)z_2e^{k_\delta(z_1)}, \ldots, (1 + \delta_2)z_Ne^{k_\delta(z_1)})
\]

with \( \phi_n \) the conformal map considered above and \( \delta_2 \) as in the definition of \( A_n \). The map \( \tilde{\phi}_n \) is regular and one-to-one from \( B_N \) onto a domain \( W \) in \( \mathbb{C}^N \). We shall show that for a suitable choice of the parameters on which \( \phi_n \) depends and a suitable choice of the interval \( J_\delta \), the domain \( W \) has the properties we seek.

We shall need the fact that as \( \delta_2 \to 0 \), \( \phi_n(z_1) \to (1 + \delta_1)z_1 \) uniformly on compacta in \( \Delta \) as follows from [5, p. 33]. It is clear that the convergence is, in fact, uniform on compacta in \( \Delta \setminus \{1\} \).

Fix \( \delta_2 < \delta_1 < \delta_0 \) and \( n > \frac{1}{\delta_2} \).

We define \( \tilde{\phi}_n^* : B_N \to \mathbb{C}^N \) by

\[
\tilde{\phi}_n^*(z) = (\phi_n(z_1), (1 + \delta_1)z_1, \ldots, (1 + \delta_1)z_N).
\]

Provided \( \delta_1 \) and \( \delta_2 \) are small enough, \( \tilde{\phi}_n^* \) carries \( B_N \) into \( U \). Since \( |z_1|^2 + \cdots + |z_N|^2 < 1 \) when \( z \in B_N \), it follows that for \( z \in B_N \), if \( z_1 \) is near 1, then \( |z_2|^2 + \cdots + |z_N|^2 \) is small. Also, when \( z_1 \) is away from 1, \( \phi_n(z_1) \) is near \( (1 + \delta_1)z_1 \), provided \( n \) is large enough. It follows that, as soon as \( \delta_1 \) is small enough, we have \( B_N \subset \subset \tilde{\phi}_n^*(B_N) \subset U \).

From (1) we see that \( k_\delta(z) \to 0 \) uniformly on compacta in \( \Delta \setminus \{1\} \) as \( \delta \to 0^+ \). Therefore, for small \( \delta_1 \), \( \tilde{\phi}_n \) is a small perturbation of \( \tilde{\phi}_n^* \) on a given compact subset of \( B_N \setminus \{1\} \), and so, provided \( \delta \) is small enough, \( \tilde{\phi}_n \) carries the part of \( B_N \) on which \( \text{Re } \phi_n(z_1) < 3 \frac{1}{2} \) onto a domain in \( U \) that contains \( B_N \).

For \( z \in B_N \), the distance, \( d(z) \), from \( \tilde{\phi}_n(z) \) to 4 is given by

\[
d^2(z) = |\phi_n(z_1) - 4|^2 + (1 + \delta_1)^2(|z_2|^2 + \cdots + |z_N|^2)e^{2\text{Re } k_\delta(z_1)}.
\]
When $\text{Re } \varphi_n(z_1) \geq 3/2$, $d(z)$ is less than 2, for $|z_1 - 4|$ is bounded by $3/4$
and
$$(1 + \delta_1)^2(|z_2|^2 + \cdots + |z_N|^2)e^{2\text{Re} \varphi(z_1)}$$
is no more than $2(1 + \delta_1)^2$ as follows from lemma 1.4. Thus, for $z \in B_N$ with $\text{Re } \varphi_n(z_1) \geq 3/2$, $\varphi_n(z) \in U$. The conclusion is that $\varphi_n(B_N) \subset U$ as soon as $\delta_1$, $\delta_2$ and $\delta$ are sufficiently small.

Finally, by taking $n$ large enough, we find that $\varphi_n(B_N)$ contains a ball of radius $1/16$ centered at 4. To see this, denote by $V_n$ the set

$$\left\{ z \in B_N : |\varphi_n(z_1) - 4| < \frac{1}{16} \right\},$$

and write $bV_n = \Sigma_1 \cup \Sigma_2$ where $\Sigma_1 = bV_n \cap bB_N$ and $\Sigma_2 = bV_n \cap \{ z \in B_N : |\varphi_n(z_1) - 4| = \frac{1}{16} \}$.

For $z \in \Sigma_2$, $d(z) \geq 1/16$, and for $z \in \Sigma_1$, we have

$$d^2(z) = |\varphi_n(z_1) - 4|^2 + (1 + \delta_1)^2(1 - |z_1|^2)e^{2\text{Re} \varphi(z_1)}.$$

The function $\log \left| \frac{1}{\sqrt{1 - z_1}} \right|$ is harmonic on the unit disc so

$$2\text{Re} \, \tilde{k}_\delta(z_1) = \log \left| \frac{1}{\sqrt{1 - z_1}} \right| - \frac{1}{\pi} \int_{e^{i\theta} \notin \delta} \frac{1 - |z_1|^2}{|e^{i\theta} - z_1|^2} \log |1 - e^{it}| \, dt$$

$$= \log \left| \frac{1}{\sqrt{1 - z_1}} \right| + \nu(z_1)$$

where $\nu(z_1) \to 0$ as $z \to e^{it} \in \text{int } J_\delta$ and uniformly when $e^{it}$ is constrained to lie in a compact subset of $\text{int } J_\delta$. Thus, provided only that $n$ is large, we have that for $z \in \Sigma_1$,

$$2\text{Re} \, \tilde{k}_\delta(z_1) > \log \left| \frac{1}{\sqrt{1 - z_1}} \right| - 1$$
which implies
\[ d^2(z) > (1 + \delta_1)^2 \frac{1 - |z_1|^2}{|1 - z_1|} e^{-1}. \]

By lemma 1.3, \( z_1 \) lies in the Stolz angle \( |\text{Im } z_1| < 4(1 - \text{Re } z_1) \), so we have the estimate that for large \( n \),
\[ d^2(z) > (1 + \delta_1)^2 \frac{1}{3e} \]
whence \( d(z) > \frac{1}{\sqrt{3e}} > \frac{1}{16} \).

This completes the proof of lemma 1.


We begin with a covering lemma.

**Lemma 2.1.** Given a domain \( D \subset C^N \), there is a family \( \{V_j\}_{j=0}^{\infty} \) of open balls in \( C^N \) with these properties:
1. \( V_j \subset \subset D \).
2. Every point in \( D \) has a neighborhood intersecting at most \( c(N) = 4^{12N}(12N)^{2N} \) of the \( V_j \).
3. If \( V_j \) denotes the ball concentric with \( V_j'' \) and of radius \( \frac{1}{128} \) times that of \( V_j'' \), then \( \bigcup_{j=0}^{\infty} V_j = D \).

**Proof.** According to [6, pp. 167 ff.], there is a locally finite sequence \( Q_0, Q_1, \ldots \) of closed cubes with mutually disjoint interiors such that
4. \( Q_j \subset \subset D \).
5. \( \bigcup_{j=0}^{\infty} Q_j = D \).
6. The sides of the \( Q_j \) are parallel to the coordinate axes.
7. If \( Q_j^* \) is the cube centered at the center of \( Q_j \) and of edge \( \frac{9}{8} \) the edge of \( Q_j \), then \( Q_j^* \subset \subset D \), and no point of \( D \) lies in more than \( (12)^{2N} \) of the \( Q_j^* \).
Fix attention on $Q_0$. By a change of coordinates, we may suppose it to be the unit cube in the underlying $\mathbb{R}^{2N}$ so that

$$Q_0^* = \left\{ x \in \mathbb{R}^{2N} : -\frac{1}{16} \leq x_1, \ldots, x_{2N} \leq 1 + \frac{1}{16} \right\}.$$

Put $v = 4^6 N$ and let $K_1, \ldots, K_{2^N}$ be the subcubes of $Q_0$ of edge $\frac{1}{v}$ with vertices $\left( \frac{\mu_1}{v}, \ldots, \frac{\mu_{2N}}{v} \right)$ for integers $\mu_1, \ldots, \mu_{2N}$. Let $V_{0,r}$ be the open ball of radius $\frac{1}{v} \sqrt{2N}$ centered at the center $p_r$ of $K_r$, and let $V_{0,r}''$ be the open ball centered at $p_r$ of radius $\frac{128}{v} \sqrt{2N}$. The balls $V_{0,r}$ cover $Q_0$ and the balls $V_{0,r}''$ are contained in $Q_0^*$. To see the latter point, notice that the distance from $p_r$ to $bQ_0^*$ is at least $\frac{1}{2v} + \frac{1}{16}$, so for $x \in V_{0,r}''$ to lie outside $Q_0^*$, we must have $\frac{1}{2v} + \frac{1}{16} < \frac{128}{v} \sqrt{2N}$. As $v = 4^6 N, N \geq 2$, this inequality is not satisfied.

We perform the analogous construction, appropriately scaled, on each of the other cubes $Q_1, Q_2, \ldots$. In this way, we obtain two families of concentric open balls

$$\{V_{j,r}''\}_{j=0,1,2,\ldots}^{2^N} \quad \text{and} \quad \{V_{j,r}''\}_{j=1,2,\ldots}^{2^N}$$

such that $\bigcup_{j,r} V_{j,r}'' \subset \subset D$ for all $j, r$, and the diameter of $V_{j,r}''$ is 128 times the diameter of $V_{j,r}''$.

As no point of $D$ is in more than $(12)^{2N}$ of the $Q_j^*$, a given point can belong to $V_{j,r}''$ for at most $(12)^{2N}$ distinct indices $j$. For a given $j$, there are $v^{2N} = 4^{12N} N^{2N}$ of the $V_{j,r}''$, so we see that no point of $D$ lies in more than $c(N)$ of the $V_{j,r}''$. The lemma is proved.

Finally, we turn to the proof of lemma 2.

Let the sequence $\{V_{j}''\}_{j=0}^\infty$ and $\{V_j''\}_{j=0}^\infty$ be as in lemma 2.1. For each $j$, let $V_j \subset \subset V_j''$ be a concentric ball whose radius is very nearly the same as that of $V_j''$. 

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Specify a pair of diametrically opposite points on $bV_j$, say $s_j$ and $t_j$. We may choose these points so that all of the $s_j$'s are distinct, all of the $t_j$'s are distinct, and so that no $s_j$ is a $t_j$. For $j = 0, 1, 2, \ldots$, let $\alpha_j$ be an invertible affine map of $\mathbb{C}^N$ to itself that carries $4j + B_N$ to $V_j$ with $4j - 1$ and $4j + 1$ going to $s_j$ and $t_j$ respectively. Let $\varphi : \mathbb{R} \to D$ be a smooth, regular one-to-one map such that for all $x$ in a neighborhood (in $\mathbb{R}$) of the interval $[4j - 1, 4j + 1]$, $\varphi(x) = \alpha_j(x)$ and such that the curves

$$L_j = ([4j - 1, 4j + 1]), \ j = 0, 1, 2, \ldots$$

have mutually disjoint neighborhoods $U_j \subset D$. (NB. We do not require that the family $\{L_j\}_{j=0}^\infty$ be locally finite in $D$; this additional condition cannot be met unless $bD$ is connected.)

If $\Phi : \mathbb{C}^N \to \mathbb{C}^N$ is an entire map that approximates $\varphi$ sufficiently well on $\mathbb{R}$ and $\alpha_j$ sufficiently well on a neighborhood of $4j + B_N$, then $\Phi$ will be regular and one-to-one on a neighborhood of $4j + B_N$ and will be one-to-one from a neighborhood of $[4j - 1, 4j + 1]$ into $U_j$. Moreover, we may take $\Phi$ to be regular on a neighborhood in $\mathbb{C}^N$ of the ray $\{x : x > 0\}$. (For the details of this kind of approximation, see [2, lemma 2.3] where the context is only formally different from the present one.) Notice finally that if the approximation is close enough on $4j + B_N$, we shall have that

$$\Phi(4j + \frac{1}{64} B_N) \supset V'_j \quad \text{and} \quad \Phi(4j + B_N) \subset V'_j.$$

It follows that if $V$ is a suitably small neighborhood of the set $E$, then $\Phi$ is regular on $V$, that $\Phi(V) = D$, and that no point is the image of more than $1 + 4^{12N(12N)^2N}$ points in $V$.

This completes the proof of lemma 2 and hence the proof of the theorem.

5. Remarks.

As we noted in the introduction, our estimate for the bound $\lambda_N$ is not sharp, but we do not know what the sharp bound is.

In the other direction, it seems likely that on purely topological grounds, there should be an interesting a priori lower bound for $\lambda_N$. The fact that e.g., the ball and the polydisc are holomorphically distinct but topologically
equivalent shows that topological considerations are probably insufficient to determine the true lower bound for $\lambda_N$.

Granted the result of this paper and that of [2], it seems not unreasonable to ask for a pair of domains $\Omega', \Omega''$ in $\mathbb{C}^N$, $\Omega'$ a bounded domain of holomorphy, such that $\Omega'$ admits no regular holomorphic map onto $\Omega''$. Of course, if such $\Omega'$ and $\Omega''$ exist, then for $\Omega''$ we may take either a polydisc or a ball.

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