

ANNALES DE L'INSTITUT FOURIER

T. R. RAMADAS

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Annales de l'institut Fourier, tome 32, n° 1 (1982), p. 263-276

http://www.numdam.org/item?id=AIF_1982__32_1_263_0

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ON THE SPACE OF MAPS INDUCING ISOMORPHIC CONNECTIONS

by T.R. RAMADAS

1. Introduction .

In this paper we prove the following

THEOREM. — *Let M be a smooth compact manifold, P a principal bundle on M with the unitary group $U(k)$ as structure group, A a smooth connection on P , and $\text{Aut } A$ the group of gauge transformations [i.e., automorphisms of P which act trivially on M] which leave A invariant. Let B be the Grassmanian of k -planes in a separable Hilbert space \mathcal{H} , E the Stiefel bundle of orthonormal k frames in \mathcal{H} , and ω the canonical universal connection on E . Denote by $\Sigma(A)$ the space of maps $p : M \rightarrow B$ such that the pull-back bundle $p^*(E)$, with the connection $p^*\omega$, is isomorphic to (P, A) .*

Then the space $\Sigma(A)$, with the C^∞ topology, has the homotopy type of $B_{(\text{Aut } A)}$ where $B_{(\text{Aut } A)}$ is the base-space of a universal bundle for $\text{Aut } A$.

The connectedness of $\Sigma(A)$ is shown in [6]. We use some ideas from this paper.

To motivate this result, consider the case when P is a principal G -bundle with G a compact Lie group. Let $\text{Aut } P$ denote the group of gauge transformations of P . Denote by \mathcal{C} the space of C^∞ connections on P . The group $\text{Aut } P$ acts on \mathcal{C} , though not freely in general. Denote by \mathcal{C} the quotient.

By [4] there exists a finite dimensional principal G -bundle $E(G, M) \rightarrow B(G, M)$ with connection such that the following diagram commutes, and the map φ is onto :

$$\begin{array}{ccc}
 \text{Mor}_G(P, E(G, M)) & \xrightarrow{\varphi} & \mathcal{C} \\
 \text{Aut } P \downarrow & & \downarrow \\
 \text{Mor}_P(M, B(G, M)) & \xrightarrow{\varrho} & \underline{\mathcal{C}}
 \end{array}$$

Here $\text{Mor}_G(P, E(G, M))$ is the space of G -morphisms of P into E and $\text{Mor}_P(M, B(G, M))$ is the component of $C^\infty(M, B(G, M))$ which induces pull-back bundles isomorphic to P . ϱ is the map given by pulling back the universal connection on $E(G, M)$.

We wish to investigate the fibres of the map ϱ . It is possible to do so when we consider instead of $E(G, M)$ a universal bundle E_G with connection such that E_G is contractible. Suppose then, that in the above diagram we replace $E(G, M)$ by E_G and $B(G, M)$ by B_G . Let $A \in \mathcal{C}$ and \underline{A} its class in $\underline{\mathcal{C}}$. We argue heuristically :

The spaces \mathcal{C} and $\text{Mor}_G(P, E_G)$ are both contractible. This would imply that $\varphi^{-1}(A)$ is contractible (all the mappings being assumed to be good fibrations). The group $\text{Aut } A$ acts on $\varphi^{-1}(A)$ to give $\varphi^{-1}(\underline{A})$. If all goes well this implies

- a) $\varphi^{-1}(A) \rightarrow \varphi^{-1}(\underline{A})$ is a universal $\text{Aut } A$ bundle. The fibre over A of the map ϱ has the same homotopy type as $B_{(\text{Aut } A)}$.
- b) If G has trivial centre and all connections are generic (i.e. $\text{Aut } P$ acts freely on \mathcal{C}) ϱ has a section.

The quotient space $\underline{\mathcal{C}}$ is relevant in studies of Yang-Mills theories, at present very popular in Physics. It has been pointed out [1] that the Universal Connection Theorem could possibly provide connections between Yang-Mills theories and so-called σ -models which concern themselves with the space $\text{Mor}(M, B)$. Also in the cases when ϱ has a section, it could give an alternative to "gauge-fixing" which has been shown to be impossible in general [3, 7, 5].

The paper is organized as follows. In § 2 we imbed E and B as closed submanifolds of Hilbert spaces. In § 3 we describe a one parameter family of isometries $A_t: \mathcal{H} \rightarrow \mathcal{H}$, and also give the

C^∞ topology to be used on the function spaces $\text{Mor}_{U(k)}(P, E)$ and $\text{Mor}_P(M, B)$. In § 4 we prove that $\varphi^{-1}(A)$ is contractible [Proposition 4.1] using the isometries A_t . Then we prove [Proposition 4.3] that $\varphi^{-1}(A) \rightarrow \varphi^{-1}(\underline{A})$ is a locally trivial principal fibre space with $\text{Aut } A$ as structure group. This involves, among other things, proving that the above projection is closed [Lemma 4.4], which is done by studying a certain differential equation. The completeness of the C^∞ topology is crucial, and the imbeddings obtained in § 2 simplify proofs throughout.

I would like to thank M.S. Narasimhan for several suggestions and much encouragement. I also thank M.S. Raghunathan, S. Ramanan and V. Sunder for their help.

2. The bundle of orthonormal k -frames in a Hilbert space.

Fix an integer $k > 0$. Let \mathcal{H} be an infinite dimensional separable Hilbert space over the complex numbers. Denote by E the space of orthonormal k -frames in \mathcal{H} . The group $U(k)$ acts on E on the right and the quotient is the Grassmannian B of k -dimensional subspaces of \mathcal{H} . In fact E is a universal principal bundle for $U(k)$. It also carries a natural connection, which is a universal connection for $U(k)$.

It will be useful, in the following, to have characterizations of E and B as closed submanifolds of Hilbert spaces.

We shall identify a point p in B with the orthogonal projector onto the corresponding subspace, denoted by $H(p)$. Thus $H(p) = \{x \in \mathcal{H} \mid px = x\}$. For $p_0 \in B$, define

$$\mathcal{P}_0 = \{p \in B \mid H(p_0) \cap \ker p = \{0\}\}.$$

Then we have a bijection $L_0: \mathcal{P}_0 \rightarrow \mathcal{P}(H(p_0), \ker p_0)$ such that for $p \in \mathcal{P}_0$ its image $L \equiv L_0(p)$ has $H(p)$ as graph.

LEMMA 2.1 [2]. — *The charts $\{(\mathcal{P}_0, L_0)\}$ give B the structure of a C^∞ Hilbert manifold.*

Let \mathcal{H}_2 denote the Hilbert space of Hilbert-Schmidt operators on \mathcal{H} .

PROPOSITION 2.2. — Let ψ denote the injection $B \rightarrow \mathcal{J}_2$ given by associating to each k -dimensional subspace its orthogonal projector. Then ψ is a C^∞ immersion, and a homeomorphism onto its image.

Proof. — Follows from Lemmas 2.3 and 2.4.

Remark. — This shows that B , with the manifold structure given in Lemma 2.1 is a submanifold of \mathcal{J}_2 .

LEMMA 2.3. — On a chart (\mathcal{R}_0, L_0) ψ is given by (1 - 3). It is a C^∞ immersion.

Proof. — Let $L \in \mathcal{L}(H(p_0), \ker p_0)$ and let $p = \psi L_0^{-1}(L)$. Write

$$p = A + LA \quad (1)$$

where $A : \mathcal{H} \rightarrow H(p_0)$. Then we claim that A satisfies

$$A = p_0 + L^+(1 - p_0) - L^+LA \quad (2)$$

which can be solved to give

$$A = \frac{1}{1 + L^+L} (p_0 + L^+(1 - p_0)). \quad (3)$$

To see that p given by (2.1)-(2.3) is indeed equal to $\psi L_0^{-1}(L)$, we verify:

a) Image of $p = \{x + Lx \mid x \in H(p_0)\}$. The map is clearly into this set. In fact it is onto since A is invertible on $H(p_0)$.

b) $p^2 = p$. This follows since $Ap = p$, which in turn is clear because Ap satisfies the same equation as p .

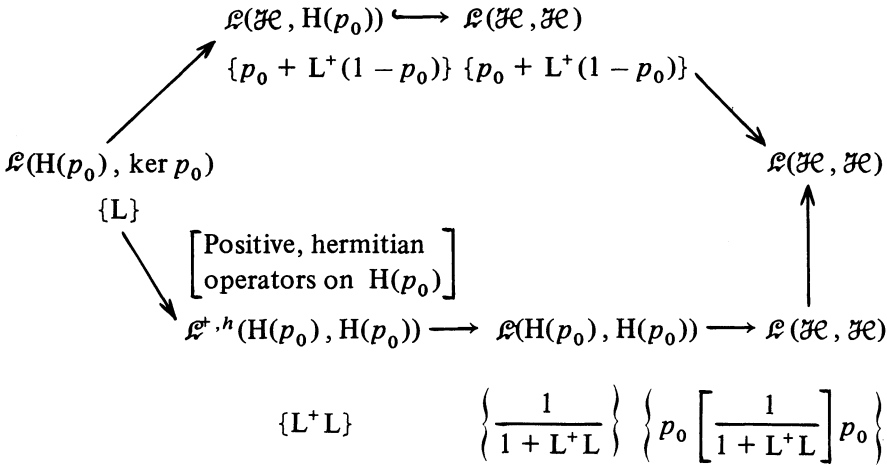
$$\begin{aligned} Ap &= p_0 p + L^+(1 - p_0) p - L^+LAp = A + L^+LA - L^+LAp \\ &= p_0 + L^+(1 - p_0) - L^+LAp. \end{aligned}$$

c) p is an orthogonal projector, for

$$\ker p = \{y - L^+y \mid y \in \ker p_0\}$$

which is the orthogonal subspace to $\text{Im } p$.

(i) ψ is C^∞ : To see this split ψ into the steps:



ψ is in fact real-analytic.

(ii) It is enough to check the differential at $L = 0$. Here $\delta p = \delta L^+(1 - p_0) + p_0 \delta L$ which is clearly injective. Also the image, being defined by $p_0 \delta p p_0 = (1 - p_0) \delta p (1 - p_0) = 0$ and $\delta p^+ = \delta p$, is closed, and hence admits a supplement.

LEMMA 2.4. — *The inverse ψ^{-1} is given by (4) and is continuous.*

Proof. — Consider a chart (\mathcal{R}_0, L_0) . Let $p \in \mathcal{R}_0$ and let $Q = (p_0|_{H(p)})^{-1}$. Then for $x \in H(p)$, $Qx = x + (1 - p_0)pQx$. This gives, for $L = (1 - p_0)Q$, $L = (1 - p_0)p(1 + L)$.

This can be solved to give $p \xrightarrow[\psi^{-1}]{} L$ such that

$$Lx = (1 - p_0) \frac{1}{1 - (1 - p_0)p} x, x \in H(p_0). \tag{4}$$

The continuity of ψ^{-1} follows easily.

We turn now to E . This can be identified with a closed subset of $\mathcal{L}(\mathbf{C}^k, \mathcal{H})$: $E = \{U : \mathbf{C}^k \longrightarrow \mathcal{H} \mid U^+U = 1\}$. Standard arguments show:

LEMMA 2.5. — *E is a closed submanifold of $\mathcal{L}(\mathbf{C}^k, \mathcal{H})$. It is a principal $U(k)$ bundle on B . The $u(k)$ -valued one-form $U^+ dU$ is a connection on E .*

LEMMA 2.6. — E is contractible and hence a universal $U(k)$ bundle. The connection is a universal $U(k)$ connection.

Proof. — Both statements are well-known. The first follows also from the remarks after Lemma 4.2. The second is a consequence of the Universal Connection Theorem.

3. Some preliminary remarks and definitions.

(i) A one-parameter-family of isometries on \mathcal{H} .

Following [6], we introduce, on \mathcal{H} , a one-parameter family of isometries which we will use later. Define, for $t \in [0, 1]$ an isometry $A_t : \mathcal{H} \rightarrow \mathcal{H}$ as follows. Fix an orthonormal basis, so that $\mathcal{H} \approx \{\text{square-summable sequences in } \mathbf{C}\}$. Then let $A_0 = \text{Identity}$

$$A_t(a_0, a_1, a_2, \dots) = (a_0, a_1 \dots a_{n-2}, a_{n-1} \cos \theta_n(t), a_{n-1} \sin \theta_n(t) \\ a_n \cos \theta_n(t), a_n \sin \theta_n(t) a_{n+1} \cos \theta_n(t), a_{n+1} \sin \theta_n(t) \dots)$$

for $\frac{1}{n+1} \leq t \leq \frac{1}{n}$ where $\theta_n(t) = \frac{\pi}{2} n[(n+1)t - 1]$.

The A_t are continuous in t w.r. to the strong operator topology. Note that

$$A\left(\frac{1}{2}\right)(a_0, a_1, \dots) = (a_0, 0, a_1 0, \dots) \in \mathcal{H}_{\text{even}}$$

$$A(1)(a_0, a_1, \dots) = (0, a_0, 0, a_1 \dots) \in \mathcal{H}_{\text{odd}}$$

where $\mathcal{H}_{\text{even}}$ and \mathcal{H}_{odd} denote obvious subspaces of \mathcal{H} .

(ii) The topology of the function spaces $\text{Mor}_{U(k)}(P, E)$ $\text{Mor}(M, B)$.

We topologize $\text{Mor}_{U(k)}(P, E)$ as a (closed) subset of

$$C^\infty(P, \mathcal{L}(\mathbf{C}^k, \mathcal{H})),$$

and $\text{Mor}(M, B)$ as a (closed) subset of $C^\infty(M, \mathcal{Y}_2)$. The C^∞ topology is described below :

Let X be a compact manifold and \mathcal{H} a Hilbert space. Let X_1, \dots, X_q be a set of vector fields on X which together span the tangent space at each point of X . For a multi index $\alpha = (\alpha_1, \dots, \alpha_2)$

set $D^\alpha = X_1^{\alpha_1}, \dots, X_q^{\alpha_q}$. We make $C^\infty(X, \mathcal{F})$ a Frechet space w.r. to the seminorms $\|f\|_\alpha = \sup_x \|D^\alpha f\|$ where the heavy bars $\| \|$ denote the Hilbert space norm. The topology is clearly independent of the choice of X_1, \dots, X_q . If $N \subset \mathcal{F}$ is a closed submanifold then $C^\infty(X, N)$ is a closed subset of $C^\infty(X, \mathcal{F})$ and we give it the relative topology, which makes it a complete metric space.

We choose now, once and for all, a set of vector fields X_1, \dots, X_p spanning the tangent space of M at each point. Let $\hat{X}_1, \dots, \hat{X}_p$ be their lifts to P w.r. to some connection, and let $\hat{Y}_1, \dots, \hat{Y}_{k^2}$ be vertical vector fields on P , the images of a fixed basis Y_1, \dots, Y_{k^2} in $u(k)$ by the group action. We will use these to determine the seminorms. Note that $[\hat{X}_i, \hat{Y}_\ell] = 0 \ \forall X_i$ and Y_ℓ . We will let $\alpha_L = (\alpha_1, \dots, \alpha_{k^2})$ and $\alpha = (\alpha_1, \dots, \alpha_p)$, and write the seminorms as $\|f\|_{\alpha_L, \alpha} = \sup_{x \in P} \|D^{\alpha_L} D^{\alpha f}\|$.

When there is no need to distinguish between the vertical and horizontal vectors we simply denote (α_L, α) by γ .

LEMMA 3.1. — *Mor_{U(k)}(P, E) and Mor(M, B) are closed subsets of $C^\infty(P, \mathcal{L}(C^k, \mathcal{E}))$ and $C^\infty(M, \mathcal{Y}_2)$ respectively. The map $\text{Mor}_{U(k)}(P, E) \rightarrow \text{Mor}(M, B)$ is continuous.*

Proof. — For $g \in U(k)$ the map $C^\infty(P, E) \rightarrow C^\infty(P, E)$ given by $f \mapsto f^g, f^g(x) \equiv f(xg)g^{-1} (x \in P)$, is continuous. This follows since

$$\begin{aligned} \|f_1^g - f_2^g\|_{\alpha_L, \alpha} &= \sup_{x \in P} \|D_x^{\alpha_L} D_x^\alpha (f_1(xg)g^{-1} - f_2(xg)g^{-1})\| \\ &= \sup_{x \in P} \|D_x^{\alpha_L} D_x^\alpha (f_1(xg) - f_2(xg))\| \\ &= \sup_{xg \in P} \|D_{xg}^{[\alpha_L, g]} D_{xg}^\alpha (f_1(xg) - f_2(xg))\| \\ &= \|f_1 - f_2\|_{[\alpha_L, g], \alpha} \end{aligned}$$

where $D^{[\alpha_L, g]}$ denotes the differential operator

$$D^{[\alpha_L, g]} = (g^{-1} \hat{Y}_1 g)^{\alpha_1} \dots (g^{-1} \hat{Y}_{k^2} g)^{\alpha_{k^2}}.$$

Here $g^{-1} \hat{Y}_i g$ is the image of the Lie algebra element $g^{-1} \hat{Y}_i g$. This proves the first statement. To prove the second statement, let $f_n \rightarrow f$ in $\text{Mor}_{U(k)}(P, E)$ and let $p_n = f_n f_n^+$. Then

$$\begin{aligned} \|p_n - p\|_\alpha &= \sup_{x \in B} \|D^\alpha(p_n - p)\| \quad (\text{where } D^\alpha = X_1^{\alpha_1} \dots X_n^{\alpha_n}) \\ &= \sup_{x \in P} \|D^\alpha(p_n - p)\| \quad (\text{where } D^\alpha = \hat{X}_1^{\alpha_1} \dots \hat{X}_n^{\alpha_n}) \\ &= \sup_{x \in P} \left\| \sum_{\beta < \alpha} \binom{\alpha}{\beta} (D^{\alpha-\beta} f_n D^\beta f_n^+ - D^{\alpha-\beta} f D^\beta f^+) \right\| \\ &\leq \alpha \sum_{\beta < \alpha} \|f_n\|_\beta \|f_n - f\|_{\alpha-\beta} + \|f\|_{\alpha-\beta} \|f_n - f\|_\beta. \end{aligned}$$

This proves $p_n \rightarrow p$ in $\text{Mor}(M, B)$.

4. The topology of the fibres.

We will be interested in the fibres of the map φ . Consider first a fibre of φ .

PROPOSITION 4.1. — *Let $A \in \mathcal{C}$. Then $\varphi^{-1}(A)$ is contractible. In other words the space of morphisms $P \rightarrow E$ which induce a fixed connection on P is contractible.*

Proof. — The proof proceeds in two steps.

(i) Define a map

$$\xi : \varphi^{-1}(A) \times [0, 1/2] \rightarrow \varphi^{-1}(A)$$

by

$$\xi_t(f)(x) = A_t \circ f(x) \begin{cases} f \in \varphi^{-1}(A) \\ x \in P \\ t \in [0, 1/2]. \end{cases}$$

The map is into $\varphi^{-1}(A)$ since,

$$\begin{aligned} \text{a) } \xi_t(f)(xg) &= A_t \circ f(xg) \quad (g \in U(k)) = A_t \circ f(x) \circ g \\ &= \xi_t(f)(x) \circ gU \end{aligned}$$

$$\text{b) } \xi_t(f)^+ d\xi_t(f) = f^+ df = A.$$

By lemma 4.2 below ξ is continuous.

(ii) There exists a $f_0 \in \varphi^{-1}(A)$ s.t. $\forall x \in P, f_0(x)$ maps \mathbf{C}^k into \mathcal{H}_{odd} [Apply A_1 to any $f \in \varphi^{-1}(A)$ to get such an f_0]. Define for $t \in [1/2, 1]$ a map $\eta : \varphi^{-1}(A) \times [1/2, 1] \rightarrow \varphi^{-1}(A)$ by

$$\eta_t(f)(x)v = (\sin t\pi) A_{1/2} f(x)v - \cos t\pi f_0(x)v.$$

Again the map is into $\varphi^{-1}(A)$. Note that $A_{1/2}f$ maps into $\mathcal{H}_{\text{even}}$. This means that $\forall (x, t), \eta_t f(x)$ defines an isometry of \mathbf{C}^k into \mathcal{H} , for, given $v, v' \in \mathbf{C}^k$,

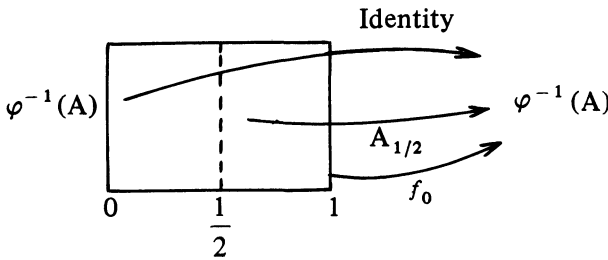
$$\begin{aligned}
 (\eta_t f(x) v, \eta_t f(x) v') &= \sin^2 t\pi (A_{1/2} f(x) v, A_{1/2} f(x) v') \\
 &\quad + (\cos^2 t\pi) (f_0(x) v, f_0(x) v') = (v, v')
 \end{aligned}$$

where $(,)$ denotes the inner product.

The points a), b) above can be checked easily. Lemma 4.2 gives continuity.

(iii) Compose ξ and η to get the contraction

$$\psi : \varphi^{-1}(A) \times [0, 1] \longrightarrow \varphi^{-1}(A). \text{ (See diagram)}$$



LEMMA 4.2. — *The maps ξ, η constructed in the proof of Proposition 4.1 are continuous (in the product topology).*

Proof. — Consider the map ξ . Let (f_n, t_n) be a sequence in $\varphi^{-1}(A) \times [0, 1/2]$. Then

$$\begin{aligned}
 \|\xi_{t_n}(f_n) - \xi_t(f)\|_\gamma &= \sup_{x \in \mathbb{P}} \|A_{t_n} \circ D^\gamma f_n - A_t \circ D^\gamma f\| \\
 &= \sup_{x \in \mathbb{P}} \|A_{t_n} \circ D^\gamma (f_n - f) + (A_{t_n} - A_t) \circ D^\gamma f\| \\
 &\leq \|f_n - f\|_\gamma + \|(A_{t_n} - A_t)f\|_\gamma.
 \end{aligned}$$

This shows continuity of ξ . The continuity of η follows similarly.

Remark. — The proof of Proposition 4.1 can be extended to prove contractivity of $\text{Mor}_{U(k)}(\mathbb{P}, E)$. In particular, taking $\mathbb{P} = U(k)$, we see that E itself is contractible.

We turn now to the fibres of the map ϱ . Note that if $A \in \mathcal{C}$ and $\underline{A} \in \underline{\mathcal{C}}$ is its class, then $\varphi^{-1}(A)$ projects onto $\varrho^{-1}(\underline{A})$. Also if $\text{Aut } A$ is the subgroup of Aut that leaves A fixed $\text{Aut}(A)$ acts freely on $\varphi^{-1}(A)$, the quotient being in bijection with $\varrho^{-1}(\underline{A})$.

$\text{Aut } A$ is the space of maps $\hat{g} : P \longrightarrow U(k)$ such that

$$(i) \hat{g}(xh) = h^{-1}g(x)h \quad x \in P, h \in U(k)$$

$$(ii) A = \hat{g}^{-1}A\hat{g} + \hat{g}^{-1}d\hat{g}.$$

Since $\hat{g} \in \text{Aut } A$ is determined by its value at a fixed point in P , we shall, fixing $y_0 \in P$ (projecting onto $x_0 \in M$) identify $\text{Aut } A \ni \hat{g} \sim \hat{g}(y_0) \in U(k)$.

Thus $\text{Aut } A$ is a closed subgroup of $U(k)$ [This is seen either from the equation (ii) above, or noting the fact that under the above identification $\text{Aut } A$ is the centralizer of the holonomy group at y_0] and hence a Lie subgroup.

From now on we assume that the vector fields $\hat{X}_1 \dots \hat{X}_p$ have been lifted to P w.r. to A . Note that then $\hat{X}_i(\hat{g}) = 0$ for $\hat{g} \in \text{Aut } A$.

PROPOSITION 4.3. — $\varphi^{-1}(A) \longrightarrow \varrho^{-1}(\underline{A})$ is a locally trivial principal fibre space with $\text{Aut } A$ as structure group.

Proof. — The proof proceeds in four steps.

a) $\text{Aut}(A)$ acts continuously on $\varphi^{-1}(A)$. For suppose $(f_n, \hat{g}_n) \in \varphi^{-1}(A) \times \text{Aut } A$ and $(f_n, \hat{g}_n) \longrightarrow (f, \hat{g})$. Then for any α_L, α

$$\begin{aligned} \|f_n \circ \hat{g}_n - f \circ \hat{g}\|_{\alpha_L, \alpha} &\leq \| (f_n - f) \circ \hat{g}_n \|_{\alpha_L, \alpha} + \| f \circ (\hat{g}_n - \hat{g}) \|_{\alpha_L, \alpha} \\ &= \sup_x \| D^{\alpha_L}([D^\alpha(f_n - f)] \hat{g}_n) \| + \sup_x \| D^{\alpha_L}([D^\alpha f](\hat{g}_n - \hat{g})) \| \\ &\hspace{15em} (\text{since } D^\alpha \hat{g} = 0) \\ &= \sup_x \left\| \sum_{\beta_L < \alpha_L} \binom{\alpha_L}{\beta_L} D^{\alpha_L - \beta_L} D^\alpha (f_n - f) D^{\beta_L} \hat{g}_n \right\| \\ &\quad + \sup_x \left\| \sum_{\beta_L < \alpha_L} \binom{\alpha_L}{\beta_L} D^{\alpha_L - \beta_L} D^\alpha f D^{\beta_L} (\hat{g}_n - \hat{g}) \right\| \\ &\leq \alpha_L! \left\| \sum_{\beta_L < \alpha_L} \|f_n - f\|_{\alpha_L - \beta_L, \alpha} \|\hat{g}_n\|_{\beta_L} + \|f\|_{\alpha_L - \beta_L, \alpha} \|\hat{g}_n - \hat{g}\|_{\beta_L} \right\|. \end{aligned}$$

Now, for any $\hat{Y}_i, \hat{g} \in \text{Aut } A$

$$\hat{Y}_i(\hat{g}) = \lim_{t \rightarrow 0} \frac{\hat{g}(x \exp t Y_i) - \hat{g}(x)}{t} = [\hat{g}(x), Y_i].$$

Also, if \hat{g}_1, \hat{g}_2 are in $\text{Aut } A$, $d(\text{Tr}(\hat{g}_1 - \hat{g}_2)^+ (\hat{g}_1 - \hat{g}_2)) = 0$, so that $\|\hat{g}_1(x) - \hat{g}_2(x)\| = \|\hat{g}_1(y_0) - \hat{g}_2(y_0)\|$.

So, we have

$$\begin{aligned} \|f_n \circ \hat{g}_n - f \circ \hat{g}\|_{\alpha_L, \alpha} &\leq \alpha_L! \sum_{\beta_L \leq \alpha_L} \|f_n - f\|_{\alpha_L - \beta_L, \alpha} \|\hat{g}_n\|_{\beta_L} \\ &\quad + \|f\|_{\alpha_L - \beta_L, \alpha} C_{\beta_L} \|\hat{g}_n(p_0) - \hat{g}(p_0)\| \end{aligned}$$

where C_{β_L} is a constant depending on the multiindex β_L .

b) Denote by \mathbf{G} the graph of the equivalence relation defined by $\text{Aut } A$ on $\varphi^{-1}(A)$. Then the map $\mathbf{G} \rightarrow \text{Aut } A$ is continuous. This follows since the map is given by $(f_1, f_2) \mapsto f_1^+(y_0) f_2(y_0)$ which is clearly continuous.

c) The projection $\varphi^{-1}(A) \rightarrow \varrho^{-1}(\underline{A})$ is continuous and closed. Continuity follows from lemma 3.1 and lemma 4.4 shows that it is closed. Thus $\varrho^{-1}(\underline{A})$ has the quotient topology w.r. to the projection.

d) Thus we have shown that $\varphi^{-1}(A) \rightarrow \varrho^{-1}(\underline{A})$ is a principal fibre space. Now note that there is a $\text{Aut } A$ -morphism

$$\begin{array}{ccc} \varphi^{-1}(A) & \longrightarrow & E \\ \downarrow & & \downarrow \\ \varphi^{-1}(\underline{A}) & \longrightarrow & E/\text{Aut } A \end{array}$$

given by $f \mapsto f(y_0)$. Since $E \rightarrow E/\text{Aut } A$ is locally trivial, the proof is complete.

LEMMA 4.4. — *The map $\varphi^{-1}(A) \rightarrow \varrho^{-1}(\underline{A})$ is closed.*

Proof. — Let $f_n \in \varphi^{-1}(A)$ s.t. $p_n = f_n f_n^+ \rightarrow p$ in $\varrho^{-1}(\underline{A})$.

It is enough to prove that $\{f_n\}$ contains a convergent subsequence. Since $p_n(x_0) \rightarrow p(x_0)$ and E has compact fibres one

can assume $f_n(y_0) \rightarrow g_0 \in E$ without loss of generality. Note that the f_n satisfy

$$df_n = f_n A + dp_n f_n. \tag{5}$$

We now prove that the f_n are Cauchy in the C^0 norm so that \exists a C^0 function f such that $f_n \rightarrow f$. Put $D = f_n - f_m$. Then from (5) we have

$$d(DD^+) = DD^+ dp_n + dp_n DD^+ + d(p_n - p_m) f_m D^+ + Df_m^+ d(p_n - p_m).$$

Evaluating on a vector field X_t , taking the trace and then absolute value of both sides we get

$$\begin{aligned} |X_t \text{Tr}(DD^+)| &\leq |\text{Tr}(DD^+ X_t p_n)| + |\text{Tr}(X_t(p_n) DD^+)| \\ &\quad + |\text{Tr}(X_t(p_n - p_m) f_m D^+)| + |\text{Tr}(Df_m^+ X_t(p_n - p_m))| \\ &\leq 2 \{ \|D\|^2 \|X_t p_n\| + \|D\| \|X_t(p_n - p_m)\| \} \end{aligned}$$

or,

$$|X_t \|D\|^2| \leq 2 \{ \|D\|^2 \|X_t p_n\| + \|X_t(p_n - p_m)\| \}. \tag{6}$$

Consider now the set $\{X_i, Y_q\}$ which we collectively denote by $\{Z_j\}$. They give a map from $P \times \mathbb{R}^N$ (where $N = k^2 + p$) to the tangent bundle TP which is onto:

$$(x, (t_1 \dots t_N)) \mapsto (x, \sum_i t_i Z_i(x)).$$

Take the obvious metric on the vector bundle $P \times \mathbb{R}^n$. This induces a splitting of the above map as well as a Riemannian metric on P . Then we have the following obvious result: if X is a vector field on P of norm ≤ 1 and we express $X = \sum a_i Z_i$ with respect to the above splitting then $|a_i| \leq 1 \forall i$.

Now let $y \in P$ and let $\Gamma(y)$ be a minimal geodesic joining y_0 to y [such a geodesic exists for P compact] parametrized with respect to arc-length. Then the length of $\Gamma(y) < T$ for some constant T independent of y . Now let X_t be the tangent vector field to Γ (which is necessarily of norm one). This gives

$$\begin{aligned} \|X_t(p_n - p_m)\| &= \sum_i \|p_n - p_m\|_i \text{ where } \|p\|_i = \sup_x \|Z_i p\| \\ &= \sum_{|\alpha|=1} \|p_n - p_m\|_\alpha. \end{aligned}$$

Thus we have, from (6)

$$|X_t \|D\|^2| = 2 \{ a \|D\|^2 + b \|D\| \}$$

with
$$a = \sum_{|\alpha|=1} \|p\|_\alpha + c, \quad c > 0$$

and
$$b = \sum_\alpha \|p_n - p_m\|_\alpha.$$

Consider the ordinary differential equation

$$\frac{du^2}{dt^2} = 2(au^2 + bu)$$

$$u(0) = D(y_0).$$

The solution is clearly:

$$u(t) = D(y_0) e^{at} + \frac{(e^{at} - 1)}{a} b.$$

Consider the set $K = \{t \geq 0 \mid \|D(t)\| > u(t)\}$. K is open, and hence a union of disjoint open intervals. Let t_0 be its least boundary point. Clearly $D(t_0) = u(t_0)$. From the polygonal approximations to $\|D(t_0)\|^2$ and $u^2(t)$ it is clear that in an interval $(t_0, t_0 + \epsilon)$ we have $\|D(t)\| \leq u(t)$. Thus $K = \emptyset$. We have finally,

$$\|D(y)\| \leq D(y_0) e^{aT} + \frac{(e^{aT} - 1)}{a} b$$

which clearly shows that $\{f_n\}$ are Cauchy in the C^0 norm.

Let f be the C^0 limit. We now turn back to (5) and ‘bootstrap’ the above result to show that f is C^∞ and $f_n \rightarrow f$ in the C^∞ topology. Assume, therefore, that f is C^k and $f_n \rightarrow f$ in C^k . For any multi-index γ ($|\gamma| \geq 1$) define γ' and $X^{(\gamma)}$ [here $X^{(\gamma)}$ is one of the vector fields Z_i] by $D^\gamma = D^{\gamma'} X^{(\gamma)}$ so that $D^{\gamma'}$ is of order $|\gamma| - 1$. Let $|\gamma| = k + 1$. Then

$$\begin{aligned} D^\gamma f_n &= D^{\gamma'} X^{(\gamma)}(f_n) = D^{\gamma'}(f_n A(X^{(\gamma)}) + X^{(\gamma)}(p_n) f_n) \\ &= \sum_{\delta < \gamma'} \binom{\gamma'}{\delta} [D^{\gamma'-\delta} f_n D^\delta A(X^{(\gamma)}) + D^{\gamma'-\delta} X^{(\gamma)}(p_n) D^\delta f_n]. \end{aligned}$$

Then

$$\begin{aligned} \|Df_n - \sum_{\delta < \gamma'} \binom{\gamma'}{\delta} [D^{\gamma'-\delta} f D^\delta A(X^{(\gamma)}) + D^{\gamma'-\delta} X^{(\gamma)}(p) D^\delta f]\| \\ \leq \gamma! \sum_{\delta < \gamma'} \|f_n - f\|_{\gamma-\delta} \|A(X^{(\gamma)})\|_\delta + \|p_n\|_{\gamma'-\delta, X^{(\gamma)}} \|f_n - f\|_\delta \\ + \|p_n - p\|_{\gamma'-\delta, X^{(\gamma)}} \|f\|_\delta \end{aligned}$$

where $\|f\|_{\gamma'-\delta, X^{(\gamma)}} \equiv \sup_x \|D^{\gamma'-\delta} X^{(\gamma)} f\|.$

This shows $D^\gamma f_n$ tends uniformly to a C^0 function, and hence f is C^{k+1} . By induction f is C^∞ and $f_n \rightarrow f$ in $C^\infty(P, E)$. The proof also shows $df = fA + pf$.

Since $\text{Mor}_{U^{(k)}}(P, E)$ is closed, $f \in \text{Mor}_{U^{(k)}}(P, E)$ and $p = ff^+$ by continuity of the projection $\text{Mor}_{U^{(k)}}(P, E) \rightarrow \text{Mor}_p(M, B)$. (One can now easily show that $f^+df = A$, thus showing that the fibre $\varphi^{-1}(A)$ is closed. This is because we have nowhere in the proof used the fact that $p \in \varphi^{-1}(A)$).

The Theorem stated in the Introduction now follows.

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Manuscrit reçu le 17 août 1981.

T.R. RAMADAS,
School of Mathematics
Tata Institute of Fundamental
Research
Homi Bhabha Road
Bombay 400005 (India).