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Annales de l'institut Fourier, tome 32, n° 1 (1982), p. 211-234
<http://www.numdam.org/item?id=AIF_1982__32_1_211_0>

Ann. Inst. Fourier, Grenoble
32, 1 (1982), 211-234

THE LAGRANGE RIGID BODY MOTION

by T. RATIU and P. van MOERBEKE

In this paper, we discuss the three-dimensional rigid body motion about a fixed point under the influence of gravity; the main emphasis will be put on its symplectic structure, its constants of the motion and the origin of these from a group theoretical point of view. It can be expressed as a Hamiltonian vector field on the 6-dimensional space $SO(3) \times so(3)$. The invariance of the problem under rotation about the direction of gravity and about the axis of symmetry, leads to conservation of angular momentum with regard to the gravity axis. This invariant together with a trivial extra-invariant leads to a reduction of this problem to a smooth manifold symplectically diffeomorphic to a four-dimensional submanifold of $so(3) \times so(3)$. The Hamiltonian vector field in this reduced manifold leads precisely to the customary Euler-Poisson equations $\dot{M} = M \times \Omega + \Gamma \times \chi$ and $\dot{\Gamma} = \Gamma \times \Omega$, where M, Ω, Γ and χ denote respectively the angular momentum, angular velocity, the coordinates of the unit vector in the direction of gravity and the coordinates of the center of mass, all expressed in body-coordinates.

Another description of the situation comes from considering generic coadjoint orbits of the Lie algebra of the semi-direct product group $SO(3) \times so(3)$; this is to say the four-dimensional manifold considered above appears among these orbits; according to the Kostant-Kirillov-Souriau method, these orbits have a natural symplectic structure, which turns out to coincide with the one before; the invariants characterizing these orbits are exactly the ones discussed previously. This will be the object of section 1.

The symmetry, about the axis through the center of gravity and the fixed point (Lagrange top), leads to conservation of angular momentum with regard to the axis of symmetry. Moreover it commutes with all the previous invariants. Carrying out another reduction procedure given by this new invariant, one obtains a Hamiltonian system on a 2-dimensional manifold diffeomorphic to $\mathbf{R}^2$ which linearizes on a cylinder $S^1 \times \mathbf{R}$, the S^1-part of the flow being given by Lagrange's classical solution in the form of an elliptic integral.

As will be discussed in section 2, the Euler-Poisson equations can for the Lagrange top be written as a single polynomial equation in an indeterminate h:

$$(\Gamma + Mh + Ch^2)^{\cdot} = (\Gamma + Mh + Ch^2) \times (\Omega + \chi h)$$

for some multiple of the center of mass χ. This equation ties up with a Kac-Moody extension of $so(3)$. The same Kostant-Kirillov-Souriau method of orbits, leads to the construction of a symplectic structure on a specific orbit; a theorem of Adler, Kostant and Symes yields Hamiltonians in involution with regard to this symplectic structure on the orbit; the Lagrange top flow in the form above appears among one of these Hamiltonian vector fields, although the symplectic structures are different. The complete integrability follows at once from these considerations. This somewhat roundabout approach has the virtue that the linearization can be carried out at once: the expression $\Gamma + Mh + Ch^2$ defines naturally an elliptic curve, whose Jacobian, i.e. the curve itself, linearizes the Lagrange flow. This will be explained in section 3.

1. The Euler-Poisson equations.

This section derives the Euler-Poisson equations of the rigid body motion about a fixed point under the influence of gravity as a reduction of the physical equations on the phase space $SO(3) \times so(3)$. It is also shown that these equations are Hamiltonian on coadjoint orbits of a semi-direct product.

1.1. Consider a rigid body that is moving about a fixed point, the origin of an orthonormal coordinate system $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ of $\mathbf{R}^3$, with ε_3 collinear to the vector gravitational acceleration. Normalize units such that the gravitational force exerted on the body is one. Assume that the mass distribution of the body is given by a positive measure μ on $\mathbf{R}^3$ and let $\widetilde{\chi} = (\chi_1, \chi_2, \chi_3)$ be its center of mass. Denote by $f(t, \mathbf{x})$ the position at time t of a particle which at time zero was at $\mathbf{x}$. "Rigidity" means that $f(t, \mathbf{x}) = \mathrm{A}(t)\mathbf{x}$ where $\mathrm{A}(t)$ is an orthogonal matrix. Assuming the motion to be smooth, $\mathrm{A}(0) = \mathrm{Id}$ yields $\mathrm{A}(t) \in \mathrm{SO}(3)$. Denoting $\Omega(t) = \mathrm{A}(t)^{-1}\dot{\mathrm{A}}(t) \in so(3)$ (the left translate of the tangent vector $\dot{\mathrm{A}}(t)$ to $\mathrm{SO}(3)$ at $\mathrm{A}(t)$), the kinetic energy is

$$\mathrm{K}(t) = \frac{1}{2}\int_{\mathbf{R}^3} \|\dot{f}(t, \mathbf{x})\|^2 \, d\mu(\mathbf{x}) = \frac{1}{2}\langle \Omega(t), \Omega(t)\rangle$$

where

$$\langle \xi, \eta \rangle = \int_{\mathbf{R}^3} \xi\mathbf{x} \cdot \eta\mathbf{x} \, d\mu(\mathbf{x}), \qquad \xi, \eta \in so(3).$$

The map

$$\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{R}^3 \longmapsto \hat{\mathbf{x}} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \in so(3)$$

is a Lie algebra isomorphism with inverse $\xi \in so(3) \longmapsto \widetilde{\xi} \in \mathbf{R}^3$ satisfying for $\mathbf{x}, \mathbf{y} \in \mathbf{R}^3$, the relations

$$(\mathbf{x} \times \mathbf{y})\hat{} = [\hat{\mathbf{x}}, \hat{\mathbf{y}}], \ \mathbf{x} \cdot \mathbf{y} = -\frac{1}{2}\operatorname{Tr}(\hat{\mathbf{x}}\hat{\mathbf{y}}) = \kappa(\hat{\mathbf{x}}, \hat{\mathbf{y}}), \ (\mathrm{Ad}_{\mathrm{A}}\xi)\tilde{} = \mathrm{A}\widetilde{\xi},$$

where $\mathrm{Ad}_{\mathrm{A}}\xi$ is the adjoint action for $\mathrm{A} \in \mathrm{SO}(3)$ on $\xi \in so(3)$.

Since κ is non-degenerate there exists a unique κ-symmetric positive-definite isomorphism I of $so(3)$ such that $\kappa(\mathrm{I}\xi, \eta) = \langle \xi, \eta \rangle$. Let $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$ be a basis of eigenvectors of I with corresponding eigenvalues I_1, I_2, I_3. If $\mathrm{J} = \operatorname{diag}(\mathrm{J}_1, \mathrm{J}_2, \mathrm{J}_3)$, $\mathrm{I}_1 = \mathrm{J}_2 + \mathrm{J}_3$, $\mathrm{I}_2 = \mathrm{J}_1 + \mathrm{J}_3$, $\mathrm{I}_3 = \mathrm{J}_1 + \mathrm{J}_2$, then $\mathrm{I}(\xi) = \xi\mathrm{J} + \mathrm{J}\xi$ and the kinetic energy becomes $\mathrm{K}(t) = \frac{1}{2}\kappa(\mathrm{M}, \Omega)$, $\mathrm{M} = \mathrm{I}(\Omega)$.

The potential energy $\mathrm{V}(t)$ is given by the height of the center of mass $\mathrm{A}(t)\widetilde{\chi} = (\mathrm{Ad}_{\mathrm{A}(t)}\chi)\tilde{}$ above the $(\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2)$-plane, i.e. $\mathrm{V}(t) = \mathrm{A}(t)\widetilde{\chi} \cdot \boldsymbol{\varepsilon}_3 = \kappa(\mathrm{Ad}_{\mathrm{A}(t)}\chi, \hat{\boldsymbol{\varepsilon}}_3)$. Thus the total energy $\mathrm{E}:$ $\mathrm{SO}(3) \times so(3) \longrightarrow \mathbf{R}$ is $\mathrm{E}(\mathrm{A}, \Omega) = \frac{1}{2}\kappa(\dot{\mathrm{M}}, \Omega) + \kappa(\mathrm{Ad}_{\mathrm{A}}\chi, \hat{\boldsymbol{\varepsilon}}_3)$.

Bearing in mind that $\mathrm{SO}(3) \times so(3)$ is identified with the tangent bundle T(SO(3)) via left translations which in turn is isomorphic to the cotangent bundle $T^*(SO(3))$ by the left-invariant metric $\langle \cdot , \cdot \rangle$ on SO(3), the canonical one-form θ has the expression $\theta(A,\eta)(v_A, \zeta) = \langle T_A L_{A^{-1}}(v_A), \eta \rangle$, $\eta, \zeta \in so(3)$, $A, B \in SO(3)$, $v_A \in T_A(SO(3))$, $L_A(B) = AB$, and $\omega = -d\theta$ is the symplectic form on $SO(3) \times so(3)$. E defines thus a Hamiltonian vector field on $SO(3) \times so(3)$ whose trajectories describe the rigid body motion.

The physical interpretation of the above data is :

a) $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are the principal axes of inertia of the body; it represents an orthonormal frame fixed with regard to the body;

b) I_1, I_2, I_3 are the principal moments of inertia of the body corresponding to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ respectively;

c) $\widetilde{\chi} \in \mathbf{R}^3$ is the center of mass of the body and it will be expressed from now on only in the body frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$.

1.2. To obtain from the above physical energy function the classical Euler-Poisson equations of motion, a reduction of the above Hamiltonian system by a momentum map will be performed; see Abraham-Marsden [1], § 4.2, § 4.3, or Marsden [8].

Let $H \cong S^1$ denote the isotropy subgroup at $\hat{\varepsilon}_3$ of the adjoint action of SO(3) on $so(3)$ and define the H-action

$$\Phi(A, (B, \xi)) = (AB, \xi) \quad \text{on} \quad SO(3) \times so(3).$$

The Lie algebra $\mathcal{H}$ of H coincides with the centralizer of $\hat{\varepsilon}_3$ and is isomorphic to $\mathbf{R}\varepsilon_3$. If $\xi \in \mathcal{H}$, the infinitesimal generator is $\left.\frac{d}{dt}\right|_{t=0} \Phi(\exp t\xi, (A, \eta)) = (TR_A(\xi), 0)$ (where R_A is right translation by A in SO(3)). It is easy to see that the canonical one-form θ and the Hamiltonian E are H-invariant. Thus by Noether's theorem the H-action has a momentum map

$$\mathcal{J} : SO(3) \times so(3) \longrightarrow \mathcal{H}^* \quad \text{given by}$$
$$\mathcal{J}(A, \eta) \cdot \xi = \theta(A, \eta)(TR_A(\xi), 0) = \kappa(\mathrm{Ad}_A I(\eta), \xi),$$

$(A, \eta) \in SO(3) \times so(3)$, $\xi \in \mathcal{H}$, which is a constant of the motion of the Hamiltonian system defined by E (see [1], theorems 4.2.2, 4.2.10). Since H acts freely and properly on $SO(3) \times so(3)$ and $\mathcal{J}$ has no critical points, the Marsden-Weinstein reduction theorems ([9], [1] theorems 4.3.1, 4.3.5) give a symplectic form $\overline{\omega}$ on the quotient manifold $\mathcal{J}^{-1}(\kappa(a\hat{\varepsilon}_3, .))/\mathrm{H}$, with $\kappa(a\hat{\varepsilon}, .) \in \mathcal{H}^*$, $a \in \mathbf{R}$, and a Hamiltonian system on it naturally induced by E.

THEOREM 1.1. – *The reduced symplectic manifold*

$$(\mathcal{J}^{-1}(\kappa(a\hat{\varepsilon}_3, .))/\mathrm{H}, \bar{\omega}), \quad a \in \mathbf{R},$$

is symplectically diffeomorphic to the following 4-dimensional submanifold of $so(3) \times so(3)$

$$\mathfrak{M}_a = \{(\Gamma, \mathrm{M}) \in so(3) \times so(3) \mid \kappa(\mathrm{M}, \Gamma) = a, \ \kappa(\Gamma, \Gamma) = 1\}$$

whose tangent space at (Γ, M) *equals*

$$\mathrm{T}_{(\Gamma,\mathrm{M})}\mathfrak{M}_a = \{(-[\alpha, \Gamma], \zeta) \in so(3) \times so(3) \mid \kappa(\zeta, \Gamma) = \kappa(\mathrm{M}, [\alpha, \Gamma])\}.$$

The symplectic form ρ *on* $\mathfrak{M}_a$ *is*

$$\rho(\Gamma, \mathrm{M})((-[\alpha, \Gamma], \xi), (-[\beta, \Gamma], \zeta)) = -\kappa(\xi, \beta) + \kappa(\zeta, \alpha) + \kappa(\mathrm{M}, [\alpha, \beta]).$$

The reduced Hamiltonian F *defined by* E *is*

$$\mathrm{F}(\Gamma, \mathrm{M}) = \frac{1}{2}\kappa(\mathrm{M}, \Omega) + \kappa(\chi, \Gamma), \ \mathrm{I}(\Omega) = \mathrm{M} = \Omega\mathrm{J} + \mathrm{J}\Omega$$

and the corresponding Hamiltonian vector field equals

$$\mathrm{X}_\mathrm{F}(\Gamma, \mathrm{M}) = ([\Gamma, \Omega], \ [\mathrm{M}, \Omega] + [\Gamma, \chi]),$$

i.e. F *defines the* Euler-Poisson equations

$$\dot{\Gamma} = [\Gamma, \Omega], \quad \dot{\mathrm{M}} = [\mathrm{M}, \Omega] + [\Gamma, \chi], \quad \mathrm{M} = \Omega\mathrm{J} + \mathrm{J}\Omega.$$

Moreover, if $\mathfrak{N} = \mathrm{SO}(3)/\mathrm{H} = \mathrm{Ad}_{\mathrm{SO}(3)}\hat{\varepsilon}_3 \cong \mathrm{S}^2$, $(\mathfrak{M}_a, \rho)$ *is symplectically diffeomorphic to* $(\mathrm{T}^*\mathfrak{N}, \omega_0)$, ω_0 *the canonical symplectic structure of* $\mathrm{T}^*\mathfrak{N}$. $\check{\Xi}_a$ *the pull-back to* $\mathrm{T}^*\mathfrak{N}$ *of the projection to* $\mathfrak{N}$ *of the* H*-invariant one-form*

$$\Xi_a(\mathrm{A}) = a\langle \mathrm{TR}_\mathrm{A}\hat{\varepsilon}_3, . \rangle / \langle \mathrm{Ad}_\mathrm{A}^{-1}\hat{\varepsilon}_3, \mathrm{Ad}_\mathrm{A}^{-1}\hat{\varepsilon}_3\rangle$$

has zero differential. Under this isomorphism the Hamiltonian F *goes over into a Hamiltonian on* $\mathrm{T}^*\mathfrak{N}$, *the sum of the kinetic energy of the induced metric from* $\mathrm{T}^*\mathrm{SO}(3)$ *and the potential*

$$\check{\mathrm{V}}_a(\Gamma) = \kappa(\Gamma, \chi) + a^2/2\langle \Gamma, \Gamma\rangle, \ \Gamma \in \mathrm{Ad}_{\mathrm{SO}(3)}\hat{\varepsilon}_3,$$

which is the projection of the effective potential

$$\mathrm{V}_a(\mathrm{A}) = \mathrm{E}(\Xi_a(\mathrm{A})) = \kappa(\mathrm{Ad}_\mathrm{A}\chi, \hat{\varepsilon}_3) + (a^2/2)\langle \mathrm{Ad}_\mathrm{A}^{-1}\hat{\varepsilon}_3, \mathrm{Ad}_\mathrm{A}^{-1}\hat{\varepsilon}_3\rangle$$

on SO(3).

Proof. – The map $\psi : SO(3) \times so(3) \longrightarrow \mathrm{Ad}_{SO(3)}\hat{e}_3 \times so(3)$ defined by $\psi(A, \xi) = (\mathrm{Ad}_{A^{-1}}\hat{e}_3, I(\xi))$ is H-invariant; therefore it induces a smooth map

$$\bar{\psi} : (SO(3) \times so(3))/H = (SO(3)/H) \times so(3) \longrightarrow \mathrm{Ad}_{SO(3)}\hat{e}_3 \times so(3)$$

which is easily seen to be a diffeomorphism satisfying

$$\bar{\psi} \circ \pi = \psi, \quad \text{for } \pi : SO(3) \times so(3) \longrightarrow (SO(3)/H) \times so(3)$$

the canonical projection. Using the transitivity of the SO(3)-action on the 2-sphere in $\mathbf{R}^3$, it is straightforward that

$$\psi(\mathcal{J}^{-1}(\kappa(a\hat{e}_3, .))) = \bar{\psi}(\mathcal{J}^{-1}(\kappa(a\hat{e}_3, .))/H) = \mathfrak{M}_a .$$

Since $\mathcal{J}^{-1}(\kappa(a\hat{e}_3, .))/H$ is a submanifold of $(SO(3)/H) \times so(3)$, this proves that $\mathfrak{M}_a$ is a submanifold of $so(3) \times so(3)$.

Since

$$T_{(\Gamma,M)}\mathfrak{M}_a \subset T_{(\Gamma,M)}(\mathrm{Ad}_{SO(3)}\hat{e}_3 \times so(3)) = \{(-[\alpha, \Gamma], \zeta) \in so(3) \times so(3) \mid \alpha, \zeta \in so(3)\},$$

the formula for $T_{(\Gamma,M)}\mathfrak{M}_a$ follows by differentiating the defining relations for $\mathfrak{M}_a$.

Since

$$\omega = -d\theta, \quad \omega(A, \xi)((v, \zeta), (w, \eta)) = -\langle \zeta, T_A L_{A^{-1}}(w)\rangle + \langle \eta, T_A L_{A^{-1}}(v)\rangle + \langle \xi, [T_A L_{A^{-1}}(v), T_A L_{A^{-1}}(w)]\rangle,$$

for $A \in SO(3)$, $\xi, \eta, \zeta \in so(3)$, $v, w \in T_A(SO(3))$,

$\bar{\omega}$ is characterized by the relation $\pi^*\bar{\omega} = i^*\omega$, where i: $\mathcal{J}^{-1}(\kappa(a\hat{e}_3, .)) \hookrightarrow SO(3) \times so(3)$ is the canonical inclusion.

Defining $\rho = \bar{\psi}_*\bar{\omega}$, it follows that ρ is uniquely characterized by $\psi^*\rho = i^*\omega$. Let $(\Gamma, M) = \psi(A, \Omega) = (\mathrm{Ad}_{A^{-1}}\hat{e}_3, I(\Omega))$ and put $v = TL_A(\alpha)$, $w = TL_A(\beta) \in T_A(SO(3))$, α, $\beta \in so(3)$. An easy computation shows that

$$T_{(A,\Omega)}\psi(v, \xi) = (-[T_A L_{A^{-1}}(v), \mathrm{Ad}_{A^{-1}}\hat{e}_3], I(\xi)).$$

Thus using the above formula for ω we have :

$$\begin{aligned}
\rho(\Gamma, M)((-[\alpha, \Gamma], \xi), (-[\beta, \Gamma], \zeta))
&= (\psi^*\rho)(A, \Omega)((v, I^{-1}(\xi)), (w, I^{-1}(\zeta))) \\
&= -\langle I^{-1}(\xi), T_A L_{A^{-1}}(w)\rangle + \langle I^{-1}(\zeta), T_A L_{A^{-1}}(v)\rangle \\
&\quad + \langle \Omega, [T_A L_{A^{-1}}(v), T_A L_{A^{-1}}(w)]\rangle \\
&= -\kappa(\xi, \beta) + \kappa(\zeta, \alpha) + \kappa(M, [\alpha, \beta]).
\end{aligned}$$

Since $E \circ \psi \mid \mathcal{J}^{-1}(\kappa(a\mathfrak{e}_3, .)) = F$, the formula for F follows.

To check the formula for X_F it is enough to observe that X_F is tangent to $\mathcal{M}_a$ and that

$$\rho(\Gamma, M)(X_F(\Gamma, M), (-[\beta, \Gamma], \zeta)) = dF(\Gamma, M)(-[\beta, \Gamma], \zeta)$$

for any $(-[\beta, \Gamma], \zeta) \in T_{(\Gamma, M)}\mathcal{M}_a$, which is easily verified.

The last part of the theorem is an immediate consequence of theorems 4.3.3. and 4.5.5. in [1]. The formulas for the effective potential and its projection appear already in Iacob [6], though not in the context of reduction. □

1.3. We shall prove below that the reduced manifolds $\mathcal{M}_a$ are among the generic adjoint orbits of the semi-direct product $so(3)_{ad} \times \overline{so(3)}$ where $\overline{so(3)}$ denotes the vector space underlying the Lie algebra $so(3)$.

If G is a Lie group with Lie algebra $\mathfrak{g}$, let $\overline{\mathfrak{g}}$ denote the vector space underlying $\mathfrak{g}$, regarded as abelian Lie group. The semi-direct product $G_{Ad} \times \overline{\mathfrak{g}}$ is the Lie group with underlying manifold $G \times \mathfrak{g}$, composition law $(g_1, \xi_1)(g_2, \xi_2) = (g_1 g_2, \xi_1 + \mathrm{Ad}_{g_1}\xi_2)$, identity element $(e, 0)$, and inverse $(g, \xi)^{-1} = (g^{-1}, -\mathrm{Ad}_{g^{-1}}\xi)$. The Lie algebra of $G_{Ad} \times \overline{\mathfrak{g}}$ is the Lie algebra semi-direct product $G_{ad} \times \overline{\mathfrak{g}}$ with bracket

$$[(\xi_1, \eta_1), (\xi_2, \eta_2)] = ([\xi_1, \xi_2], [\xi_1, \eta_2] + [\eta_1, \xi_2]). \tag{1.1}$$

The adjoint action is given by

$$\mathrm{Ad}_{(g,\zeta)}(\xi, \eta) = (\mathrm{Ad}_g \xi, \mathrm{Ad}_g \eta + [\zeta, \mathrm{Ad}_g \xi]). \tag{1.2}$$

If $\mathfrak{g}$ is semi-simple and κ denotes the Killing form up to a constant factor, define the bi-invariant, symmetric, non-degenerate two-form κ_s on $G_{ad} \times \overline{\mathfrak{g}}$ by

$$\kappa_s((\xi_1, \eta_1), (\xi_2, \eta_2)) = \kappa(\xi_1, \eta_2) + \kappa(\xi_2, \eta_1). \tag{1.3}$$

Remark that $\kappa \times \kappa$ is *not* bi-invariant. If $s$grad denotes the gradient with respect to κ_s, then $s\mathrm{grad} = (\mathrm{grad}_2, \mathrm{grad}_1)$, where $(\mathrm{grad}_1, \mathrm{grad}_2)$ is the usual gradient with respect to $\kappa \times \kappa$.

By the Kirillov-Kostant-Souriau theorem, the adjoint orbits of a semi-simple Lie algebra are symplectic manifolds; for $G_{ad} \times \overline{\mathfrak{g}}$ the orbit symplectic form is

$$\sigma(\xi,\eta)([(\xi,\eta),(\zeta_1,\zeta_1')],[(\xi,\eta),(\zeta_2,\zeta_2')]) = -\kappa_s((\xi,\eta),[(\zeta_1,\zeta_1'),(\zeta_2,\zeta_2')]) . \quad (1.4)$$

If f, g: $G \times G \longrightarrow \mathbf{R}$, the Hamiltonian vector field on the orbit equals

$$X_f(\xi,\eta) = -([\mathrm{grad}_2 f(\xi,\eta),\xi],[\mathrm{grad}_2 f(\xi,\eta),\eta] + [\mathrm{grad}_1 f(\xi,\eta),\xi]) \quad (1.5)$$

and the Poisson bracket is

$$\{f, g\}(\xi,\eta) = -\kappa(\xi,[\mathrm{grad}_2 f(\xi,\eta),\mathrm{grad}_1 g(\xi,\eta)]) - \kappa(\xi,[\mathrm{grad}_1 f(\xi,\eta),\mathrm{grad}_2 g(\xi,\eta)]) - \kappa(\eta,[\mathrm{grad}_2 f(\xi,\eta),\mathrm{grad}_2 g(\xi,\eta)]) . \quad (1.6)$$

All these formulas follow easily from the general ones displayed in e.g. Ratiu [11], [12].

THEOREM 1.2. – *All adjoint orbits of* $SO(3)_{\mathrm{Ad}} \times \overline{so(3)}$ *in* $so(3)_{\mathrm{ad}} \times \overline{so(3)}$ *are given by*

a) $\{(\Gamma, M) \in so(3) \times so(3) \mid \kappa(\Gamma, M) = \text{constant}, \kappa(\Gamma,\Gamma) = \text{constant}\}$,

if $\Gamma \neq 0$; *this orbit is four-dimensional;*

b) $\{(0, M) \in so(3) \times so(3) \mid \kappa(M, M) = \text{constant}\}$; *this orbit is two-dimensional, unless* $M = 0$ *in which case it reduces to a point.*

The symplectic structure of these orbits is

a) $\sigma(\Gamma, M)((-[\alpha_1,\Gamma],\zeta_1),(-[\alpha_2,\Gamma],\zeta_2)) = -\kappa(\zeta_1,\alpha_2) + \kappa(\zeta_2,\alpha_1) + \kappa(M,[\alpha_1,\alpha_2])$, for $\alpha_1,\alpha_2,\zeta_1,\zeta_2 \in so(3)$ *satisfying* $\kappa(\zeta_i,\Gamma) = \kappa(M,[\alpha_i,\Gamma])$, $i = 1,2$;

b) $\sigma(0, M)((0,[\alpha_1, M]), (0,[\alpha_2, M])) = \kappa(M,[\alpha_1,\alpha_2])$.

In particular, the manifolds $\mathfrak{M}_a$ *are orbits of type a).*

The proof is a straightforward verification using (1.1)-(1.4).

Formula (1.5) shows that the Euler-Poisson equations can be considered on all orbits of $so(3)_{\mathrm{ad}} \times \overline{so(3)}$, not only on $\mathfrak{M}_a$. The Hamiltonian is $F(\Gamma, M) = \frac{1}{2}\kappa(M,\Omega) + \kappa(\Gamma,\chi)$, $M = \Omega J + J\Omega$, with $\mathrm{grad}_1 F(\Gamma, M) = \chi$, $\mathrm{grad}_2 F(\Gamma, M) = M$, and Hamilton's equations on the orbit are the *Euler-Poisson equations*

$$\dot{\Gamma} = [\Gamma,\Omega], \quad \dot{M} = [M,\Omega] + [\Gamma,\chi]$$

with the invariants $\kappa(M, \Gamma) = \text{constant}$ and $\kappa(\Gamma, \Gamma) = \text{constant}$ corresponding to the momentum along $\hat{e}_3$ and the intensity of the gravitational field respectively. The fact that the Euler-Poisson equations are Hamiltonian on adjoint orbits of $so(3)_{ad} \times \overline{so(3)}$ has been independently observed by Iacob and Sternberg [7].

2. The Lagrange top and its complete integrability.

A *Lagrange top* is an axially symmetric rigid body with center of mass on the axis of symmetry, moving about a fixed point (the origin of $\mathbf{R}^3$) under the influence of gravity. Hence $J_1 = J_2 = \lambda$, $J_3 = \mu$, $\chi = (0, 0, \chi_3)$ in the body frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$.

2.1. A strictly three dimensional method of finding one additional conserved quantity for this problem (besides E and $\mathcal{J}$) is the following (Iacob [6]). Let $K \cong S^1$ be the isotropy subgroup at $\hat{\mathbf{e}}_3$ of the adjoint action of SO(3) on $so(3)$. Define a new left-action of K on $SO(3) \times so(3)$ by $\Psi(C, (A, \Omega)) = (AC^{-1}, \mathrm{Ad}_C \Omega)$. The Lie algebra $\mathcal{K}$ of K is the centralizer of $\hat{\mathbf{e}}_3$ in $so(3)$ and is isomorphic to $\mathbf{R}\mathbf{e}_3$. The infinitesimal generator of Ψ given by $\xi \in \mathcal{K}$ is $(A, \Omega) \longmapsto (-TL_A(\xi), [\xi, \Omega])$. The relations

$$I \circ \mathrm{Ad}_C = \mathrm{Ad}_C \circ I$$

$$\langle \Omega_1, \Omega_2 \rangle = (\lambda + \mu)\kappa(\Omega_1, \Omega_2) - (\mu - \lambda)\kappa(\Omega_1, \hat{\mathbf{e}}_3)\kappa(\Omega_2, \hat{\mathbf{e}}_3)$$

for any $\Omega_1, \Omega_2 \in so(3)$, $C \in K$, prove that $\langle . , . \rangle$ is K-invariant, which in turn shows that E and θ are K-invariant. Thus by Noether's theorem applied to the action Ψ of K, there exists a momentum mapping $\mathcal{L} : SO(3) \times so(3) \longrightarrow \mathcal{K}^*$,

$$\mathcal{L}(A, \Omega)(\xi) = -\langle \xi, \Omega \rangle = -2\lambda\Omega_3\xi$$

which is a conserved quantity ([1] theorems 4.2.2, 4.2.10). The *Lagrange momentum* $\mathcal{L}$ represents the momentum of the body along its symmetry axis $\mathbf{e}_3$.

Remark that the actions Φ and Ψ commute, i.e.

$$\Phi_B \circ \Psi_C = \Psi_C \circ \Phi_B$$

for any $B \in H$, $C \in \mathcal{K}$, that $\mathcal{L}$ is H-invariant, and that $\mathcal{J}$ is K-invariant. Thus regarding $\mathcal{L}$, $\mathcal{J}$ real valued we conclude $\{\mathcal{L}, \mathcal{J}\} = 0$

([1] theorem 3.3.13). Since $\mathcal{L}, \mathcal{J}$ are integrals of the motion $\{\mathcal{J}, E\} = 0$, $\{\mathcal{L}, E\} = 0$. Therefore $\mathcal{L}$ induces a function $\mathcal{L}_a$ on the manifold $\mathfrak{M}_a$ which Poisson commutes with F, the Hamiltonian on $\mathfrak{M}_a$ induced by E ([1], § 4.3, [11], § 2). The K-action Ψ induces a K-action on $\mathfrak{M}_a$ given by

$$(C, (\Gamma, M)) \longrightarrow (\mathrm{Ad}_C \Gamma, I(\mathrm{Ad}_C I^{-1}(M)))$$

whose momentum mapping is $\mathcal{L}_a$ ([1], § 4.3, [9]).

Since $\Gamma = \mathrm{Ad}_{A^{-1}} \hat{e}_3$, the trivial relation $\| A^{-1} \hat{e}_3 \| = 1$ becomes $\kappa(\Gamma, \Gamma) = 1$ and we recovered in this way the four classical integrals of the Euler-Poisson equations for the Lagrange top on $so(3) \times so(3)$:

$$\kappa(\Gamma, \Gamma) = \Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2 = 1, \quad \kappa(\Gamma, M) = \Gamma_1 M_1 + \Gamma_2 M_2 + \Gamma_3 M_3 = a$$

$$\mathcal{L}_a(\Gamma, M) = -M_3 = \text{constant}$$

$$F(\Gamma, M) = (M_1^2 + M_2^2)/2(\lambda + \mu) + M_3^2/4\lambda + \chi_3 \Gamma_3 = \text{constant.}$$

From theorem 1.1 we see that their geometrical significance is distinct: the first two define the reduced manifold $\mathfrak{M}_a$ as a submanifold of $so(3) \times so(3)$, whereas $\mathcal{L}_a$ and F are genuine constants of the motion for the Euler-Poisson equations on the 4-dimensional symplectic manifold $\mathfrak{M}_a$, F being the Hamiltonian.

In this three dimensional case the solution of the Lagrange top problem can be obtained "by hand" ([14], chapter 4, N° 71). We shall explain below the group theory underlying these classical computations. If $b \in \mathbf{R}$, $\mathcal{L}_a^{-1}(\kappa(-b\hat{e}_3, .)) = \{(\Gamma, M) \in \mathfrak{M}_a \mid M_3 = b\}$, where $\kappa(-b\hat{e}_3, .) \in \mathcal{K}^*$. On the open, dense K-invariant set $S = \{(\Gamma, M) \in \mathcal{L}_a^{-1}(\kappa(-b\hat{e}_3, .)) \mid \Gamma_1 M_2 - \Gamma_2 M_1 \neq 0\}$, the action of K is free and thus one can form a new reduced symplectic 2-manifold S/K diffeomorphic to $(-1, 1) \times (0, \infty)$ by the diffeomorphism induced from the smooth K-invariant map $\varphi(\Gamma, M) = (\Gamma_3, \sqrt{M_1^2 + M_2^2})$ of S onto $(-1, 1) \times (0, \infty)$. Taking (x, r) as coordinates on $(-1, 1) \times (0, \infty)$, φ is given by $x = \Gamma_3$, $r = \sqrt{M_1^2 + M_2^2}$ and since the Hamiltonian R on $(-1, 1) \times (0, \infty)$ induced by F is characterized by $R \circ \varphi = F \mid S$, we have

$$R(x, r) = r^2/2(\lambda + \mu) + x\chi_3 + b^2/4\lambda .$$

But the Hamiltonian vector field X_R on $(-1, 1) \times (0, \infty)$ is uniquely determined by the relation $T\varphi \circ X_R = X_F \circ \varphi$; so putting

$(x, r) = \varphi(\Gamma, M)$ and observing that

$$T_{(\Gamma,M)}\varphi(-[\alpha,\Gamma],\xi) = (-[\alpha,\Gamma]_3, (M_1\xi_1 + M_2\xi_2)/r)$$

for $\qquad \widetilde{M} = (M_1, M_2, M_3), [\alpha,\Gamma]^{\sim} = ([\alpha,\Gamma]_1, [\alpha,\Gamma]_2, [\alpha,\Gamma]_3)$, $\widetilde{\xi} = (\xi_1, \xi_2, \xi_3)$, $r^2 = M_1^2 + M_2^2$, a short computation gives

$$X_R(x, r) = (\Gamma_1 M_2 - \Gamma_2 M_1)(1/(\lambda + \mu), -\chi_3/r).$$

We shall express now $\Gamma_1 M_2 - \Gamma_2 M_1$ only in terms of (x, r). Since $\Gamma_1 M_1 + \Gamma_2 M_2 = a - xb$ (because $\kappa(\Gamma, M) = a$, $M_3 = b$ on S), $\Gamma_1^2 + \Gamma_2^2 = 1 - x^2$, $M_1^2 + M_2^2 = r^2$, it follows

$$\begin{aligned}\Gamma_1 &= \{M_1(a - xb) \pm \sqrt{(r^2 - M_1^2)[r^2(1 - x^2) - (a - xb)^2]}\}/r^2\\ \Gamma_2 &= \{(a - xb)\sqrt{r^2 - M_1^2} \mp M_1\sqrt{r^2(1 - x^2) - (a - xb)^2}\}/r^2\\ M_2 &= \sqrt{r^2 - M_1^2}\end{aligned}$$

whence

$$\Gamma_1 M_2 - \Gamma_2 M_1 = \pm\sqrt{r^2(1 - x^2) - (a - xb)^2}$$

and so

$$X_R(x, r) = \pm\sqrt{[r^2(1 - x^2) - (a - xb)^2}(1/(\lambda + \mu), -\chi_3/r).$$

Thus Hamilton's equations on $(-1,1) \times (0, \infty)$ are

$$\begin{aligned}\dot{x} &= \pm\sqrt{r^2(1 - x^2) - (a - xb)^2}/(\lambda + \mu)\\ \dot{r} &= \mp\chi_3\sqrt{r^2(1 - x^2) - (a - xb)^2}/r\end{aligned} \tag{2.1}$$

with the energy integral $r^2/2(\lambda + \mu) + x\chi_3 + b^2/4\lambda =$ constant, i.e.

$$r^2/2(\lambda + \mu) + x\chi_3 = c = \text{constant}. \tag{2.2}$$

Since $(\Gamma_1, \Gamma_2, \Gamma_3)$ is on the unit sphere, one has $|x| \leqslant 1$ and thus we must have $c - x\chi_3 > 0$, i.e. $c > |\chi_3|$. From the first equation (2.1) we get $r^2 = [(\lambda + \mu)^2\dot{x}^2 + (a - xb)^2]/(1 - x^2)$, which together with (2.2) gives

$$(\lambda + \mu)^2\dot{x}^2 = -(a - xb)^2 - 2(\lambda + \mu)\chi_3(x - x^3) + 2c(\lambda + \mu)(1 - x^2), \tag{2.3}$$

i.e. the time t is an elliptic integral in the variable x and the Hamiltonian system (2.1) is thus linearized on the circle S^1, the real part of the complex one-dimensional torus defined by the elliptic curve

$$y^2 = -(a - xb)^2 - 2(\lambda + \mu)\chi_3(x - x^3) + 2c(\lambda + \mu)(1 - x^2).$$

Formula (2.3) is identical to the one in Whittaker ([13], chapter 4, Nr. 71, page 157), obtained there by different means.

Let us return now to the Euler-Poisson equations for the Lagrange top on $\mathcal{M}_a$. By the Liouville-Arnold theorem its solution is a linear flow on that part of the level surface $\mathcal{L}_a(\Gamma, M) = -b$, $F(\Gamma, M) = c + b^2/4\lambda$ in $\mathcal{M}_a$, i.e. $M_3 = b$,

$$\text{i.e. } M_3 = b, \quad M_1^2 + M_2^2 = 2(\lambda + \mu)(c - \chi_3 \Gamma_3),$$

where $\mathcal{L}_a$ and F are independent. We will show in *2.3.* that this is the case whenever $\Gamma_1 M_2 - \Gamma_2 M_1 \neq 0$. Since $\Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2 = 1$, let $\Gamma_3 = \cos\theta$ and then $\Gamma_1^2 + \Gamma_2^2 = \sin^2\theta$. If

$$s^2 = M_1^2 + M_2^2 = 2(\lambda + \mu)(c - \chi_3 \Gamma_3),$$

then $0 < s^2 < 2(\lambda + \mu)(c + |\chi_3|)$ and we see that the parameters (θ, s) on the cylinder $S^1 \times (0, \sqrt{2(\lambda + \mu)(c + |\chi_3|)})$ determine uniquely (Γ, M) on the level surface for $\Gamma_1 M_2 - \Gamma_2 M_1 \neq 0$. Thus the flow of the *Euler-Poisson equations for the Lagrange top linearizes on a 2-dimensional cylinder;* the S^1-part of the linearization is given by (2.3). We shall recover this last result in § 3 by an application of the van Moerbeke-Mumford linearization procedure [10].

2.2. Proposition 2.1. below, which has been generalized by Ratiu [12] to n dimensions, is the first step towards proving complete integrability of the Euler-Poisson equations

$$\dot{\Gamma} = [\Gamma, \Omega], \quad \dot{M} = [M, \Omega] + [\Gamma, \chi], \quad M = \Omega J + J\Omega \qquad (2.4)$$

for the Lagrange top on the generic adjoint orbit of $so(3)_{\mathrm{ad}} \times \overline{so(3)}$.

PROPOSITION 2.1. – *If* $\chi \neq 0$ *the Euler-Poisson equations* (2.4) *can be written in the form*

$$(\Gamma + Mh + Ch^2)^{\cdot} = [\Gamma + Mh + Ch^2, \Omega + \chi h] \qquad (2.5)$$

for some constant $C \in so(3)$, h *a formal parameter, if and only if* (2.4) *describes the motion of a Lagrange top. In this case* $C = (\lambda + \mu)\chi$.

Proof. – It is straightforward to check that if in (2.4) $J_1 = J_2 = \lambda$, $J_3 = \mu$, $\tilde{\chi} = (0, 0, \chi_3)$, (2.5) holds with $C = (\lambda + \mu)\chi$. Conversely, if (2.5) holds, then necessarily $[C, \Omega] + [M, \chi] = 0$, $[C, \chi] = 0$, for any $\Omega \in so(3)$. The first relation implies

$$C_1 = (J_1 + J_2)\chi_1 = (J_1 + J_3)\chi_1\,,$$
$$C_2 = (J_2 + J_3)\chi_2 = (J_1 + J_2)\chi_2\,,$$
$$C_3 = (J_1 + J_3)\chi_3 = (J_2 + J_3)\chi_3\,,$$

whence $J_2\chi_1 = J_3\chi_1$, $J_3\chi_2 = J_1\chi_2$, $J_1\chi_3 = J_2\chi_3$. Since $\chi \neq 0$, assume e.g. $\chi_3 \neq 0$. Then $J_1 = J_2 = \lambda$, $\lambda\chi_1 = \mu\chi_1$, $\lambda\chi_2 = \mu\chi_2$, where we denote $J_3 = \mu$. If $\lambda \neq \mu$ then $\chi_1 = \chi_2 = 0$, and $C = (\lambda + \mu)\chi$; we recover the Lagrange top. If $\lambda = \mu$, $C = 2\lambda\chi$ and we get a symmetric top. But due to the transitivity of the SO(3)-action on the two-sphere in $\mathbf{R}^3$, there exists an orthogonal change of the body frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ to one in which $\tilde{\chi} = (0, 0, \chi_3)$, $\chi_3 \neq 0$; since $J = \lambda \mathrm{Id}$, this change of frame does not affect the form of the equations (2.4) and we showed that a symmetric top is a special Lagrange top. In n dimensions this is no longer true (Ratiu [12]). □

The function $\frac{1}{2}\,\mathrm{Tr}(\Gamma + Mh + Ch^2)^2$ is conserved by the flow (2.5) and hence also the coefficients of h in the expansion of $\frac{1}{2}\,\mathrm{Tr}(\Gamma + Mh + Ch^2)^2$. The expressions $\frac{1}{2}\,\mathrm{Tr}(\Gamma^2)$ and $\mathrm{Tr}(\Gamma M)$ are by theorem 1.2 orbit invariants, and $\frac{1}{2}\,\mathrm{Tr}(C^2)$ is a constant; these coefficients define thus identically zero Hamiltonian vector fields. The two remaining coefficients are

$$f_1(\Gamma, M) = \frac{1}{2}\,\mathrm{Tr}(M^2 + 2\Gamma C), \qquad f_2(\Gamma, M) = \mathrm{Tr}(MC)$$

with $\mathrm{grad}_1 f_1(\Gamma, M) = C$, $\mathrm{grad}_2 f_1(\Gamma, M) = M$, $\mathrm{grad}_1 f_2(\Gamma, M) = 0$, $\mathrm{grad}_2 f_2(\Gamma, M) = C$. By (1.6), $\{f_1, f_2\}(\Gamma, M) = 0$, defining on the four-dimensional generic orbit two constants in involution.

On the reduced manifolds $\mathcal{M}_a$

$$f_1 = -\,2(\lambda + \mu)\,F + \frac{\mu - \lambda}{2\lambda}\,\mathcal{L}_a^2\,,$$

$$f_2 = 2(\lambda + \mu)\,\chi_3\,\mathcal{L}_a\,,$$

where $\mathcal{L}_a$ is the real-valued function on $\mathcal{M}_a$ induced by $\mathcal{L}$. The above expressions show $\{F, \mathcal{L}_a\} = 0$ on $\mathcal{M}_a$, which has been obtained already in 2.1 by a different method.

2.3. We shall prove that on every generic orbit the vector fields $X_{f_1}(\Gamma, M) = ([M, \Gamma], [C, \Gamma])$, $X_{f_2}(\Gamma, M) = ([C, \Gamma], [C, M])$ (see (1.5)) are independent for almost all (Γ, M). Consider the dense set on the orbit for which $\Gamma_1 M_2 - \Gamma_2 M_1 \neq 0$. If

$$X_{f_1}(\Gamma, M) = \alpha X_{f_2}(\Gamma, M) \quad \text{for some } \alpha \in \mathbf{R},$$

then $M_2\Gamma_3 - M_3\Gamma_2 = -\alpha C_3\Gamma_2$, $M_3\Gamma_1 - M_1\Gamma_3 = \alpha C_3\Gamma_1$, $\Gamma_2 = \alpha M_2$, $\Gamma_1 = \alpha M_1$, whence $\alpha = 0$. *The Euler-Poisson equations for the Lagrange top are a completely integrable system on the Lie algebra* $so(3)_{\mathrm{ad}} \times \overline{so(3)}$.

Since $\quad X_{f_1} = -2(\lambda + \mu) X_F + \dfrac{\mu - \lambda}{\lambda} \mathcal{L}_a X_{\mathcal{L}_a}$,

$X_{f_2} = 2(\lambda + \mu)\chi_3 X_{\mathcal{L}_a}$ if follows that X_F and $X_{\mathcal{L}_a}$ are generically independent on each reduced manifold $\mathfrak{M}_a$ and hence X_E, $X_{\mathfrak{J}}$, $X_{\mathcal{L}}$ are generically independent on $SO(3) \times so(3)$. Hence *the Lagrange top is a completely integrable system.*

2.4. We shall prove that equation (2.5) is Hamiltonian on certain invariant submanifolds of the Kac-Moody extension $\widetilde{so(3)}$ of $so(3)$ and exhibit its Hamiltonian function.

If $\mathfrak{g}$ is a Lie algebra let $\widetilde{\mathfrak{g}} = \left\{ \sum_{n=-\infty}^{+\infty} \xi_n h^n \mid \text{the sum is finite} \right\}$ denote the Kac-Moody extension of $\mathfrak{g}$ with bracket

$$\left[\sum_{k=-\infty}^{+\infty} \xi_k h^k , \sum_{n=-\infty}^{+\infty} \eta_n h^n \right] = \sum_{p=-\infty}^{+\infty} \Big(\sum_{k+n=p} [\xi_k , \eta_n] \Big) h^p .$$

If κ denotes a bilinear, symmetric, non-degenerate, bi-invariant two form on $\mathfrak{g}$, then $\widetilde{\kappa}$ given by

$$\widetilde{\kappa}\Big(\sum_k \xi_k h^k , \sum_n \eta_n h^n \Big) = \sum_{k+n=-1} \kappa(\xi_k , \eta_n)$$

is a bilinear, symmetric, bi-invariant, and weakly non-degenerate two-form on $\widetilde{\mathfrak{g}}$. Remark that

$$\mathcal{K} = \left\{ \sum_{n=0}^{\infty} \xi_n h^n \right\}, \quad \mathcal{N} = \left\{ \sum_{n=-\infty}^{-1} \xi_n h^n \right\},$$

are both Lie subalgebras, $\widetilde{\mathfrak{g}} = \mathcal{K} \oplus \mathcal{N}$, $\mathcal{K}^{\perp} = \mathcal{K}$, $\mathcal{N}^{\perp} = \mathcal{N}$, where orthogonality is taken with respect to $\tilde{\kappa}$. Regarding $\mathcal{K}^{\perp}$ as the dual of $\mathcal{N}$ via $\tilde{\kappa}$, the formal Lie group $\mathrm{Id} + \mathcal{N}$ acts on $\mathcal{K}^{\perp}$ and hence defines a Hamiltonian structure (e.g. Ratiu [11]). If $f, g : \tilde{\mathcal{G}} \longrightarrow \mathbf{R}$ are functions having gradients with respect to $\tilde{\kappa}$, the Hamiltonian vector field and the Poisson bracket are

$$X_{f|\mathcal{K}^{\perp}}(\tilde{\xi}) = -\Pi_{\mathcal{K}^{\perp}}[\Pi_{\mathcal{N}}(\operatorname{grad} f)(\tilde{\xi}), \tilde{\xi}], \quad \tilde{\xi} \in \mathcal{K}^{\perp}, \tag{2.6}$$

$$\{f|\mathcal{K}^{\perp}, g|\mathcal{K}^{\perp}\} = -\tilde{\kappa}([\Pi_{\mathcal{N}}(\operatorname{grad} f)(\tilde{\xi}), \Pi_{\mathcal{N}}(\operatorname{grad} g)(\tilde{\xi})], \tilde{\xi}), \quad \tilde{\xi} \in \mathcal{K}^{\perp}. \tag{2.7}$$

Moreover, if f, g are ad-invariant on $\tilde{\mathcal{G}}$, i.e. $[(\operatorname{grad} f)(\tilde{\xi}), \tilde{\xi}] = 0$ for all $\tilde{\xi} \in \tilde{\mathcal{G}}$, then $\{f|\mathcal{K}^{\perp}, g|\mathcal{K}^{\perp}\} = 0$ and (2.6) becomes

$$X_{f|\mathcal{K}^{\perp}}(\tilde{\xi}) = [\Pi_{\mathcal{K}}(\operatorname{grad} f)(\tilde{\xi}), \tilde{\xi}], \quad \tilde{\xi} \in \mathcal{K}. \tag{2.8}$$

For the proof see Ratiu [11]; the last statement is known as the Adler-Kostant-Symes involution theorem.

In the general considerations above take $\mathcal{G} = so(3)$ and consider the submanifold $Q_C = \{\xi + \eta h + Ch^2 \mid \xi, \eta \in so(3)\} \subset \mathcal{K}$. A straightforward computation shows that if $f : \widetilde{so(3)} \longrightarrow \mathbf{R}$, $(\operatorname{grad} f)(\xi + \eta h + Ch^2) = \sum_n f_n h^n$, formula (2.3) yields

$$X_{f|\mathcal{K}}(\xi + \eta h + Ch^2) = [\eta, f_{-1}] + [C, f_{-2}] + [C, f_{-1}]h \in T_{\xi+\eta h+Ch^2}(Q_C),$$

i.e. Q_C is an invariant submanifold of $\mathcal{K}^{\perp} = \mathcal{K}$. Note however that the vector representative of $[C, f_{-1}]$ has zero third component and that $\kappa(C, [\eta, f_{-1}] + [C, f_{-2}]) + \kappa([C, f_{-1}], \eta) = 0$ which proves that Q_C is foliated by lower dimensional invariant submanifolds of the form $\{\xi + \eta h + Ch^2 \mid \frac{1}{2}\kappa(\eta, \eta) + \kappa(C, \xi) = \text{constant}, \eta_3 = \text{constant}\}$. These manifolds are generically 4-dimensional; if $\eta_1 = \eta_2 = 0$ they are 2-dimensional, unless $\xi = 0$ in which case they reduce to a point. They are the orbits of $\mathrm{Id} + \mathcal{N}$ acting on Q_C; compare this to theorem 1.2.

Fix now such an arbitrary manifold

$$\left\{ \xi + \eta h + Ch^2 \mid \frac{1}{2}\kappa(\eta, \eta) + \kappa(C, \xi) = \text{constant}, \eta_3 = \text{constant} \right\}$$

By (2.8) the Hamiltonian of (2.5) must be an ad-invariant function H such that $\Pi_{\mathcal{K}}(\operatorname{grad} H)(\Gamma + Mh + Ch^2) = -(\Omega + \chi h)$. We have

$$\Omega + \chi h = \frac{1}{\lambda + \mu} (\mathrm{M} + \mathrm{C}h) + \left(1 - \frac{2\lambda}{\lambda + \mu}\right) \frac{\mathrm{M}_3}{2\lambda(\lambda + \mu)\chi_3} \mathrm{C}$$

$$= \frac{1}{\lambda + \mu} \Pi_{\mathcal{K}}((\Gamma + \mathrm{M}h + \mathrm{C}h^2) h^{-1}) + \left(1 - \frac{2\lambda}{\lambda + \mu}\right)$$

$$\frac{\mathrm{M}_3}{2\lambda(\lambda + \mu)\chi_3} \Pi_{\mathcal{K}}((\Gamma + \mathrm{M}h + \mathrm{C}h^2) h^{-2})$$

which suggests choosing H as

$$\mathrm{H}(\xi) = -\frac{1}{2} \tilde{\kappa} \left(\xi^2, \frac{1}{\lambda + \mu} \mathrm{Id}\, h^{-1} + \left(1 - \frac{2\lambda}{\lambda + \mu}\right) \frac{\mathrm{M}_3}{4\lambda(\lambda + \mu)\chi_3} h^{-2}\right),$$
$$\xi \in \widetilde{so(3)}$$

which is clearly ad-invariant and satisfies the required condition, thus proving that *(2.5) is a Hamiltonian system on the invariant submanifolds* $\{\xi + \eta h + \mathrm{C}h^2 \mid \frac{1}{2} \kappa(\eta, \eta) + \kappa(\mathrm{C}, \xi) = \text{constant}, \eta_3 = \text{constant}\} \subset \mathcal{K}$.

Since the integrals $f_1(\mathrm{Z}) = \tilde{\kappa}(\mathrm{Z}, \mathrm{Z}h^{-3})$, $f_2(\mathrm{Z}) = \tilde{\kappa}(\mathrm{Z}, \mathrm{Z}h^{-4})$, $\mathrm{Z} \in \widetilde{so(3)}$, f_1, f_2 are ad-invariant and thus by the Adler-Kostant-Symes involution theorem, their Poisson brackets are zero on the above invariant manifolds. Note however that the symplectic structure is *different* from the usual one (1.6), namely identifying $\mathrm{Q_C}$ with $so(3) \times so(3)$ by $\xi + \eta h + \mathrm{C}h^2 \longmapsto (\xi, \eta)$, the new Poisson bracket is

$$\{f, g\}_{\mathrm{C}} (\xi, \eta) = -\kappa(\mathrm{C}, [\mathrm{grad}_2 f(\xi, \eta), \mathrm{grad}_1 g(\xi, \eta)])$$
$$- \kappa(\mathrm{C}, [\mathrm{grad}_1 f(\xi, \eta), \mathrm{grad}_2 g(\xi, \eta)])$$
$$- \kappa(\eta, [\mathrm{grad}_1 f(\xi, \eta), \mathrm{grad}_1 g(\xi, \eta)]).$$

In the generalization to n dimensions, the interplay between these two symplectic structures plays a crucial role in the proof of complete integrability and also gives rise to Lenard relations; see Ratiu [12].

3. The linearization of the flow for the Lagrange top.

From equation (2.5) it follows that the algebraic curve

$$p(z, h) = \det(\Gamma + \mathrm{M}h + \mathrm{C}h^2 - z\mathrm{Id}) = -zq(z, h) = 0$$

is isospectral, i.e. it is conserved under the flow of the Lagrange top. In this section we show in two ways how to linearize this flow on the one-dimensional invariant complex torus (elliptic curve) defined by $q(z, h) = 0$.

3.1. The infinite band matrix

$$\mathcal{M} = \begin{bmatrix} \ddots & \ddots & \ddots & & & & \\ 0 & \Gamma & \mathrm{M} & \mathrm{C} & 0 & 0 & 0 \\ 0 & 0 & \Gamma & \mathrm{M} & \mathrm{C} & 0 & 0 \\ 0 & 0 & 0 & \Gamma & \mathrm{M} & \mathrm{C} & 0 \\ & & & \ddots & \ddots & \ddots & \end{bmatrix}, \quad \mathcal{M} : (\mathbf{R}^3)^{\mathbf{Z}} \longrightarrow (\mathbf{R}^3)^{\mathbf{Z}}$$

and the shift operator $\mathcal{S} : (\mathbf{R}^3)^{\mathbf{Z}} \longrightarrow (\mathbf{R}^3)^{\mathbf{Z}}$, $\mathcal{S}(\mathrm{F}_i) = \mathrm{F}_{i+1}$ $\mathrm{F}_i \in \mathbf{R}^3$, $i \in \mathbf{Z}$ commute and their common spectrum $\{(z, h) \mid \mathcal{M}(\mathrm{F}) = z\mathrm{F}, \mathcal{S}(\mathrm{F}) = h\mathrm{F}\}$ is thus given by the solutions of

$$p(z, h) = \det(\Gamma + \mathrm{M}h + \mathrm{C}h^2 - z\,\mathrm{Id}) = 0 .$$

In fact we have $\mathrm{F}_i = h^i \mathrm{F}_0$, $(\Gamma + \mathrm{M}h + \mathrm{C}h^2)\,\mathrm{F}_0 = z\mathrm{F}_0$, so that the components f_1, f_2, f_3 of F_0, normalized by $f_1 = 1$, as a solution of a homogeneous linear system, are given by

$$f_k = \frac{\Delta_{1k}}{\Delta_{11}} = \frac{\Delta_{kk}}{\Delta_{k1}}, \quad k = 1,2, \tag{3.1}$$

where Δ_{ij} denotes the (i, j) th minor of the matrix

$$\Gamma + \mathrm{M}h + \mathrm{C}h^2 - z\,\mathrm{Id} .$$

The matrix

$$\mathrm{U} = \begin{bmatrix} 0 & i/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & i/\sqrt{2} \\ 1 & 0 & 0 \end{bmatrix}$$

diagonalizes C and we have

$$\mathrm{A} = \mathrm{U}^{-1}(\Gamma + \mathrm{M}h + \mathrm{C}h^2)\,\mathrm{U} = \begin{bmatrix} 0 & \beta & i\beta^* \\ -\beta^* & -\omega & 0 \\ i\beta & 0 & \omega \end{bmatrix} \tag{3.2}$$

with

$$\begin{cases} \beta = y + hx\,,\ \beta^* = \bar{y} + h\bar{x}\,,\ x = \dfrac{1}{\sqrt{2}}(M_1 - iM_2) \\ \quad = \dfrac{\lambda+\mu}{\sqrt{2}}(\Omega_1 - i\Omega_2)\,,\ y = \dfrac{1}{\sqrt{2}}(\Gamma_1 - i\Gamma_2) \\ \omega = -i(\Gamma_3 + M_3 h + C_3 h^2) = -i(\Gamma_3 + 2\lambda\Omega_3 h + (\lambda+\mu)\chi_3 h^2). \end{cases} \tag{3.3}$$

The equation $p(z,h) = -z(z^2 - \omega^2 + 2\beta\beta^*) = -zq(z,h) = 0$ essentially defines the elliptic curve X :

$$z^2 = \omega^2 - 2\beta\beta^* \equiv P_4(h)\,, \tag{3.4}$$

$P_4(h)$ being a quartic polynomial in h. Let P, Q cover $h = \infty$. The divisor structure of z, h on X is

$$\begin{aligned} (h) &= -P - Q + R_1 + R_2 \\ (z) &= -2P - 2Q + 4 \text{ zeros}. \end{aligned}$$

To find the divisor structure of the eigenvector (f_1, f_2, f_3), $f_1 = 1$, remark that by (3.1)

$$\begin{aligned} f_2 &= \frac{\Delta_{12}}{\Delta_{11}} = \frac{\beta^*(\omega - z)}{\omega^2 - z^2} = \frac{\beta^*}{\omega + z} = \frac{\omega - z}{2\beta} \\ f_3 &= \frac{\Delta_{13}}{\Delta_{11}} = \frac{i\beta(\omega + z)}{-(\omega^2 - z^2)} = -\frac{i\beta}{\omega - z} = \frac{\omega + z}{2i\beta^*} \end{aligned} \tag{3.5}$$

so that $f_2 f_3 = -i/2$, i.e. $(f_2) + (f_3) = 0$. Expand $z = \pm\sqrt{\omega^2 - 2\beta\beta^*}$ about P and Q

$$z = \pm iC_3 h^2\left(1 + \frac{M_3}{C_3}h^{-1} + 0(h^{-2})\right) \quad \begin{pmatrix} + \text{ at } P \\ - \text{ at } Q \end{pmatrix}$$

so that by (3.3)

$$\omega + z = \begin{cases} 0(1) & , \text{ at } P \\ -2iC_3 h^2 + 0(h), & \text{at } Q. \end{cases}$$

Hence by (3.5) f_2 has a simple pole at P and a simple zero at Q and f_3 has a simple zero at P and a simple pole at Q.

On the affine part of X, (3.5) implies that f_2 has a pole at the point ν given by $\omega + z = 0$, $\beta = 0$, i.e. $h = -y/x$,

$z = -\omega|_{h=-y/x}$. Similarly f_2 has a zero at $\overline{\nu}$ defined by $\omega - z = 0$, $\beta^* = 0$, i.e. $h = -\overline{y}/\overline{x}$, $z = \omega|_{h=-\overline{y}/\overline{x}}$. This combined with $(f_2) + (f_3) = 0$ yields

$$(f_2) = -\mathrm{P} + \mathrm{Q} - \nu + \overline{\nu}\,, \ (f_3) = \mathrm{P} - \mathrm{Q} + \nu - \overline{\nu}\,. \tag{3.6}$$

THEOREM 3.1. – *There exists a one-to-one correspondence between points on the curve* X *and matrices of the form* $\Gamma + \mathrm{M}h + \mathrm{C}h^2$, *for* Γ, M *and* C *as above, modulo rotations* $(x, y) \longmapsto (e^{i\theta} x, e^{i\theta} y)$.

Proof. – The map associates to each $\Gamma + \mathrm{M}h + \mathrm{C}h^2$ the divisor $\nu \in \mathrm{Jac(X)} = \mathrm{X}$, $\nu = (\omega|_{h=-y/x}, -y/x)$. By (3.3) changing (x, y) to $(e^{i\theta} x, e^{i\theta} y)$ does not affect ν.

The inverse of this map is constructed in the following way. $z^2 = \omega^2 - 2\beta\beta^*$ determines uniquely the coefficients of h; in particular C_3, M_3 are known (see (3.3)). The divisor ν determines $-y/x$ and $\omega|_{h=-y/x}$. Therefore by (3.3)

$$\Gamma_3 = (i\omega - \mathrm{M}_3 h - \mathrm{C}_3 h^2)|_{h=-y/x}$$

is determined. This uniquely defines ω as a polynomial in h. But then $z^2 - \omega^2 = 2\beta\beta^* = -2(|y|^2 + (y\overline{x} + \overline{y}x) + |x|^2 h^2)$ is defined, i.e. $|y|^2$, $|x|^2$, $y\overline{x} + \overline{y}x$, and $\overline{x}y = y|x|^2/x$ are known. This implies that x, y are determined up to a rotation

$$(x, y) \longmapsto (e^{i\theta} x, e^{i\theta} y). \qquad \square$$

3.2. The linearization of the Lagrange top flow on $\mathrm{Jac(X)} = \mathrm{X}$ can be established by a direct computation which is strictly three dimensional, or by a modification of a general method of Adler and van Moerbeke [4] which can be extended to higher dimensions. First we show the direct method.

Equation (2.4) is equivalent to the following system in (x, y)

$$\begin{cases} \dot{x} = -i\,\dfrac{\lambda - \mu}{2\lambda(\lambda + \mu)}\,\mathrm{M}_3 x + i\,\dfrac{\mathrm{C}_3}{\lambda + \mu}\,y\,, \ \mathrm{C}_3\,, \mathrm{M}_3\,, \mathrm{Tr}(\Gamma^2) = \text{constant} \\ \dot{y} = i\,\dfrac{\mathrm{M}_3}{2\lambda}\,y - i\,\dfrac{\Gamma_3}{\lambda + \mu}\,x\,, \ \Gamma_3^2 = -\dfrac{1}{2}\,\mathrm{Tr}(\Gamma^2) - \dfrac{1}{2}\,|y|^2. \end{cases} \tag{3.7}$$

In particular

$$\dot{x}y - \dot{y}x = \frac{i}{\lambda + \mu}(C_3 y^2 - M_3 xy + \Gamma_3 x^2). \tag{3.8}$$

Any linear flow on Jac(X) = X is of the form $\dot{h} = cz$ for some constant c, where the divisor $\nu = (h, z)$. But since $h = -y/x$, $z = \omega|_{h=-y/x}$, this linear flow is (use (3.3))

$$\begin{aligned}\dot{h} &= c\omega|_{h=-y/x} = cix^{-2}(-\Gamma_3 x^2 + M_3 xy - C_3 y^2) \\ &= x^{-2}(\dot{x}y - \dot{y}x).\end{aligned}$$

Choosing $c = \dfrac{1}{\lambda + \mu}$, *the Lagrange-top is seen to be a linear flow on the elliptic curve* Jac(X) = X; *it can be expressed as*

$$\frac{1}{\lambda + \mu} t = \int_{h_0}^{h} \frac{dh}{(\omega^2 - 2\beta\beta^*)^{1/2}}. \tag{3.9}$$

This formula is modulo a change of variables the elliptic integral defined by (2.3).

3.3. We linearize now the Lagrange top flow with the aid of a general procedure which can be used for the n-dimensional case (Ratiu [12]). The method of van Moerbeke and Mumford [10] is to be adapted to this singular problem. Therefore consider the curve X_ϵ given by

$$p_\epsilon(z, h) = \det(A^\epsilon - z\,\mathrm{Id}) = -z^3 + \epsilon h^2 z^2 + (\omega^2 - 2\beta\beta^*)z - \epsilon h^2 \omega^2 = 0,$$

where

$$A^\epsilon = \begin{bmatrix} \epsilon h^2 & \beta & i\beta^* \\ -\beta^* & -\omega & 0 \\ i\beta & 0 & \omega \end{bmatrix}.$$

In the limit $\epsilon \longrightarrow 0$, X_ϵ tends to the reducible curve $p(z, h) = 0$. Apply the theorem of Adler and van Moerbeke [4] to X_ϵ and obtain a Lax equation with linear flow on $\mathrm{Jac}(X_\epsilon)$; finally it will be shown that in the limit $\epsilon \longrightarrow 0$ this flow converges to the Lagrange top flow (3.9).

The curve X_ϵ has genus 4, is a six-sheeted covering of the complex h-line and is non-singular. At $z = \infty$, $h = \infty$, there are three branch points P_1, P_2, P_3 where the following asymptotic estimates hold

$$\frac{z}{h^2} \sim \begin{cases} \epsilon & , \text{ at } P_1 \\ -i\chi_3(\lambda + \mu), & \text{at } P_2 \\ i\chi_3(\lambda + \mu) & , \text{ at } P_3 \end{cases} \tag{3.10}$$

(see theorem 1 in Adler, van Moerbeke [4]). The holomorphic differential $\omega^{(\epsilon)} = z\left(\frac{\partial}{\partial z} p_\epsilon(z, h)\right)^{-1} dh$ on X_ϵ tends to a holomorphic differential on X given by

$$z(-3z^2 + \omega^2 - 2\beta\beta^*)^{-1}\, dh = -z(2z^2)^{-1}\, dh = -dh/2z\,.$$

By Adler and van Moerbeke (see [4]), the meromorphic function $(\lambda + \mu)^{-1} zh^{-1}$ defines the isospectral deformation

$$\dot{A}^\epsilon = [A^\epsilon, (A^\epsilon h^{-1}(\lambda + \mu)^{-1})_+] \tag{3.11}$$

where "+" means taking the polynomial part; it is linearized on $\mathrm{Jac}(X_\epsilon)$ as follows

$$\sum_{i=1}^{4} \int_{\nu_i^\epsilon(0)}^{\nu_i^\epsilon(t)} \omega_k^{(\epsilon)} = \sum_{j=1}^{3} \mathrm{Res}_{P_j}((\lambda + \mu)^{-1} zh^{-1}\omega_k)\, t, \quad k = 1,2,3,4\,, \tag{3.12}$$

where $\{\omega_k^{(\epsilon)} \mid 1 \leqslant k \leqslant 4\}$ is a basis of holomorphic differentials on X_ϵ. As $\epsilon \longrightarrow 0$, X_ϵ tends to the reducible curve $p(z, h) = 0$, P_2 and P_3 go over to P, Q respectively, $A^\epsilon \longrightarrow A$ and

$$(A^\epsilon h^{-1}(\lambda + \mu)^{-1})_+ \longrightarrow (Ah^{-1}(\lambda + \mu)^{-1})_+$$

$$= \begin{bmatrix} 0 & \dfrac{x}{\lambda + \mu} & i\dfrac{\bar{x}}{\lambda + \mu} \\ -\dfrac{\bar{x}}{\lambda + \mu} & i\dfrac{2\lambda}{\lambda + \mu}\Omega_3 & 0 \\ i\dfrac{x}{\lambda + \mu} & 0 & -i\dfrac{2\lambda}{\lambda + \mu}\Omega_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & i\chi_3 & 0 \\ 0 & 0 & -i\chi_z \end{bmatrix} h$$

Thus conjugating (3.11) by U and letting $\epsilon \longrightarrow 0$ we get the equation $(\Gamma + Mh + Ch^2)^{\cdot} = [\Gamma + Mh + Ch^2, \Omega' + \chi h]$, where $\widetilde{\Omega}' = \left(\Omega_1, \Omega_2, \dfrac{2\lambda}{\lambda + \mu}\Omega_3\right)$. Since $M_3 = M_1\Omega_2 - M_2\Omega_1 = 0$, defining $M' = \Omega' J + J\Omega'$, we get $\widetilde{M}' = \left(M_1, M_2, \dfrac{2\lambda}{\lambda + \mu} M_3\right)$ and also $(\Gamma + M'h + Ch^2) = [\Gamma + M'h + Ch^2, \Omega' + \chi h]$ which is the equation of the Lagrange top. Hence we proved that the limit of

(3.11) is, modulo a change of variables, the Lagrange top equation.

To this limit there corresponds (3.12) with $\epsilon \longrightarrow 0$. Take $\omega^{(\epsilon)}$ in (3.12) and remark that by (3.1) the expressions of f_2, f_3 for X_ϵ are

$$f_2 = \frac{\beta^*}{\omega + z}, \quad f_3 = -\frac{i\beta}{\omega - z}.$$

These do *not* depend on ϵ and thus $\lim_{\epsilon \to 0} \nu_i^\epsilon(t)$ exists and we proved that the limit of the left-hand side of (3.12) is a sum of Abelian integrals in the holomorphic differential $-dh/2z$ on X.

Using the expansion (3.10) of z and the formula for ω in (3.3) we have

$$\begin{aligned}
&\lim_{\epsilon \to 0} \operatorname{Res}_{P_1} (\omega_2 z h^{-1} (\lambda + \mu)^{-1}) \\
&\qquad = \frac{1}{\lambda + \mu} \lim_{\epsilon \to 0} \operatorname{Res}_{P_1} \frac{z^2 h^{-1} dh}{-3z^2 + 2\epsilon h^2 z + \omega^2 - 2\beta\beta^*} \\
&\qquad = \frac{1}{\lambda + \mu} \lim_{\epsilon \to 0} \operatorname{Res} \frac{\epsilon^2 h^3 dh}{-3\epsilon^2 h^4 + 2\epsilon^2 h^4 - \chi_3^2 (\lambda + \mu)^2 h^2} \\
&\qquad = 0
\end{aligned}$$

$$\begin{aligned}
&\lim_{\epsilon \to 0} \operatorname{Res}_{P_{2,3}} (\omega_2 z h^{-1} (\lambda + \mu)^{-1}) \\
&\quad = \frac{1}{\lambda + \mu} \lim_{\epsilon \to 0} \operatorname{Res} \frac{(\mp i\chi_3(\lambda + \mu) h)(\mp i\chi_3(\lambda + \mu) h^2)\, dh}{-3\chi_3^2 (\lambda + \mu)^2 h^4 \mp 2\epsilon(\lambda + \mu) h^4 - \chi_3^2 (\lambda + \mu)^2} \\
&\quad = \frac{-1}{\lambda + \mu} \operatorname{Res} \frac{dh}{h} \\
&\quad = \frac{1}{2(\lambda + \mu)},
\end{aligned}$$

the last equality being obtained taking $h = 1/t$ as local parameter at ∞. Thus the limit of the right hand side of (3.12) equals $\dfrac{1}{\lambda + \mu} t$ and we proved once again that *the Lagrange top flow linearizes on* $\operatorname{Jac}(X) = X$.

Recall that the flow of the Lagrange top lives on $p(z, h) = -zq(z, h) = 0$ and the above procedure linearizes it on the X-part only. However $p(z, h) = 0$ is a complex 2-dimensional cylinder with generator the line $z = 0$. Thus the flow of the

Lagrange top problem linearizes on the real part, i.e. on a cylinder in $\mathbf{R}^3$; this result was obtained directly in 2.1. by computing the regular part of the energy surface $\mathcal{L}_a$ = constant, F = constant.

Note added in proof. – For an alternate proof of Theorem 1.1. using momentum maps, see Holmes, Marsden: "Horseshoes and Arnold diffusion in Hamiltonian systems with symmetry", Indiana Journal of Math. (1982). This paper also contains the following result: a heavy rigid body which is almost a Lagrange top, i.e. $I_1 - I_2$ is small, has horseshoes in its phase portrait.

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Manuscrit reçu le 15 janvier 1981.

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