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## ON THE WEAK $L^1$ SPACE AND SINGULAR MEASURES

by Robert KAUFMAN

### Introduction.

The class  $R$  of finite, complex measures  $\mu$  on  $(-\infty, \infty)$  such that  $\hat{\mu}(\infty) = 0$ , has been intensively investigated (since 1916). For this class  $o(1)$  is trivial and for absolutely continuous measures, we have the Riemann-Lebesgue Lemma. We investigate the corresponding  $o(1)$  condition for the partial-sum operators

$$S_T(x, \mu) \equiv \int D_T(x-t) \mu(dt),$$
$$D_T(t) \equiv (\pi t)^{-1} \sin Tt, T > 0.$$

The  $o(1)$  condition for  $S_T$  depends on the weak  $L^1$  norm, defined by

$$\|u\|_1^* \equiv \sup Y m\{|u| > Y\};$$
$$\|S_T(\mu)\|_1^* \leq C \|\mu\|, 0 < T < +\infty.$$

The weak estimate is an easy consequence of Kolmogorov's estimate for the Hilbert transform [2, Chapter II]. Elementary approximations show that when  $\mu = f(x) dx$ , then  $\lim \|S_T(\mu) - f\|_1^* = 0$ . When  $\mu$  is singular and  $\lim \|S_T(\mu) - g\|_1^* = 0$  for a certain measurable  $g$ , two conclusions can be obtained without great difficulty (see below):

- a)  $\|S_k(\mu) - S_{k+1}(\mu)\|_1^* \rightarrow 0$  whence  $\hat{\mu}(\infty) = 0$ ;
- b)  $S_T(\mu) \rightarrow 0$  in measure as  $T \rightarrow +\infty$

whence  $g = 0$  a.e. This leads us to define:

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$W_0$  is the class of measures  $\mu$  for which  $\|S_T(\mu)\|_1^* \rightarrow 0$  as  $T \rightarrow +\infty$ .

We present an elementary structural property of  $W_0$ , and then show by example that

(A) There exist  $M_0$ -sets  $F$  carrying no measure  $\mu \neq 0$  in  $W_0$ .

The sets  $F$  are defined by a purely metrical property, and they need not be especially small. Their construction is based on an idea from the theory of divergent Fourier series [3I, Chapter VIII].

(B) The set  $F_\theta$  of all sums  $\sum_0^\infty \pm \theta^m$  ( $0 < \theta < 1/2$ ) carries a measure  $\lambda \neq 0$  in  $W_0$ , provided  $F_\theta$  is an  $M_0$ -set.

To elucidate example (B) and the next one we recall that  $F_\theta$  fails to be an  $M_0$ -set (or even an  $M$ -set) unless  $\mu_\theta \in R$ , where  $\mu_\theta$  is the Bernoulli convolution carried by  $F_\theta$  and that  $\mu_\theta \in R$  except for certain algebraic numbers  $\theta$  [3II, p. 147-156]. Therefore the next example is somewhat unexpected.

(C) When  $0 < \theta < 1/2$ , then  $\mu_\theta \notin W_0$ , in fact

$$\|S_T(\mu_\theta)\|_1^* \geq c(\theta) > 0$$

for large  $T > 0$ . We observe in passing that  $\mu$  is not known to be singular for  $1/2 < \theta < 1$  except when  $\mu_\theta \notin R$ , e.g., for  $\theta^{-1} = (1 + \sqrt{5})/2$ .

## 1.

From the weak estimate for  $S_T$  it is clear that  $W_0$  is norm-closed in the space of all measures. We shall prove that when  $\mu \in W_0$  and  $\psi \in C^1 \cap L^\infty$ , then  $\psi\mu \in W_0$ ; consequently the same is true if only  $\psi \in L^1(\mu)$ . We need two lemmas; the first was already used implicitly.

LEMMA 1. — *Let  $\mu$  be a measure such that  $S_k(\mu) - S_{k+1}(\mu) \rightarrow 0$  in measure (over finite intervals). Then  $\hat{\mu}(\infty) = 0$ , i.e.,  $\mu \in R$ .*

*Proof.* —  $|D_k(t) - D_{k+1}(t)| \leq \min(1, |t|^{-1}) \equiv K(t)$ , say, and  $K \in L^2(-\infty, \infty)$ . Thus the functions  $|S_k(\mu) - S_{k+1}(\mu)|$  have a common majorant  $\int K(x-t) |\mu|(dt)$  in  $L^2$ . The hypothesis on

$S_k - S_{k+1}$  then yields  $\|S_k - S_{k+1}\|_2 \rightarrow 0$ . This means that  $\int_k^{k+1} (|\hat{\mu}(t)|^2 + |\hat{\mu}(-t)|^2) dt \rightarrow 0$  so  $\hat{\mu}(\infty) = 0$ , because  $\hat{\mu}$  is uniformly continuous.

LEMMA 2. — Let  $\mu \in \mathbb{R}$  and  $\psi \in C^1 \cap L^\infty$ . Then as  $T \rightarrow +\infty$   $\|S_T(x, \psi \cdot \mu) - \psi(x) S_T(x, \mu)\|_1^* \rightarrow 0$ .

*Proof.* — Since  $\mu$  can be approximated in norm by measures  $\mu_n \in \mathbb{R}$ , each of compact support, we can suppose that  $\mu$  itself has compact support, say  $|t| \leq a$ . Now  $S_T(\psi \cdot \mu) - \psi S_T(\mu)$  converges to 0 uniformly on  $[-a - 1, a + 1]$ , being equal to

$$\pi^{-1} \int \sin T(t - x) \cdot \varphi(x, t) \mu(dt),$$

with  $\varphi(x, t) = (t - x)^{-1} [\psi(t) - \psi(x)]$ ;  $\varphi(x, t)$  is jointly continuous. This is sufficient to obtain the uniform convergence claimed.

For  $|x| > a + 1$  we write

$$x S_T(x, \mu) = \pi^{-1} \int \sin T(t - x) \cdot \sigma(x, t) \mu(dt)$$

with  $\sigma(x, t) = x(t - x)^{-1}$ ; now  $|\sigma| \leq a + 1$  and

$$\left| \frac{\partial}{\partial t} \sigma(x, t) \right| \leq a + 1,$$

for  $|t| \leq a$ . Therefore  $x S_T(\mu, x) \rightarrow 0$  as  $T \rightarrow +\infty$ , uniformly for  $|x| \geq a + 1$ . The same applies to  $x S_T(x, \psi \cdot \mu)$ , because  $\psi \cdot \mu \in \mathbb{R}$ , and these inequalities show that  $\psi S_T(\mu) - S_T(\psi \cdot \mu) \rightarrow 0$ .

## 2. Examples.

I. Let  $F$  be a compact set in  $(-\infty, \infty)$ ,  $0 < \alpha < 1$ ,  $(\epsilon_j)$  a sequence decreasing to 0; for each  $j$ , let  $F = \cup F_k^j$ , where

$$\text{diam}(F_k^j) \leq \epsilon_j, d(F_k^j, F_\ell^j) \geq \epsilon_j^\alpha, k \neq \ell.$$

Then  $F$  carries no probability measure  $\mu$  in  $W_0$  (and hence no signed measure  $\mu \neq 0$  in  $W_0$ ).

We define the following property of a number  $\beta$  in  $[0, 1)$ , relative to  $\mu$  and the sequence of partitions  $F = \cup F_k^j$ :

(\*\*) The total  $\mu$ -measure of the sets  $F_k^j$ , such that  $\mu(F_k^j) > \epsilon_j^\beta$ , tends to 0, as  $j \rightarrow +\infty$ .

Plainly  $\beta = 0$  has property (\*\*), because  $\mu$ , being an element of  $R$ , can have no discontinuities. We shall prove that if  $\beta$  has property (\*\*), and  $0 \leq \beta < \alpha$ , then  $\gamma = \beta + (1 - \alpha)/2$  has property (\*\*). This leads to a contradiction as soon as  $\gamma > \alpha$ , since the number of sets  $F_k^j \neq \emptyset$  is  $O(\epsilon_j^{-\alpha})$ .

Assuming that  $\beta$  has property (\*\*), we form  $\lambda = \lambda_j$ , by omitting from  $F_k$  the intervals  $F_k^j$  of  $\mu$ -measure  $> \epsilon_j^\beta$ . By Kolmogorov's estimate,  $\|S_T(\lambda_j)\|_1^* \rightarrow 0$ , as  $j \rightarrow +\infty$  and  $T \rightarrow +\infty$ , independently. Let now  $\int^*$  denote an integral over the domain  $|x - t| > \epsilon_j^\alpha/2$ . Then

$$\begin{aligned} \int^* |x - t|^{-1} \lambda_j(dt) &= O(\epsilon_j^{-\alpha}), \text{ if } \beta = 0, \\ \int^* |x - t|^{-1} \lambda_j(dt) &= O(\epsilon_j^{\beta-\alpha} (\log \epsilon_j)), \text{ } 0 < \beta < \alpha. \end{aligned}$$

The first of these is obvious; the second is obtained by packing the subsets  $F_k^j$  as close to  $x$  as is consistent with the condition  $d(F_k, F_\ell) \geq \epsilon_j^\alpha$ .

For each  $k$  such that  $\lambda_j(F_k^j) > \epsilon_j^\gamma$ , we let  $\xi_k$  belong to  $F_k^j$  and consider the set defined by

$$\begin{aligned} (S_k^j): \quad \frac{1}{2} \lambda(F_k^j) \epsilon_j^\sigma < |x - \xi_k| < \lambda(F_k^j) \epsilon_j^\sigma, \\ |\sin \epsilon_j^{-\tau} (x - \xi_k)| > \frac{1}{2} \end{aligned}$$

where  $\sigma = -\beta + 3\alpha/4 + 1/4$ ,  $\tau = (1 + \gamma + \sigma)/2$ .

The number  $\lambda(F_k^j) \epsilon_j^\sigma$  lies between  $\epsilon_j^{\beta+\sigma}$  and  $\epsilon_j^{\gamma+\sigma}$ ; we note that  $\beta + \sigma > \alpha$ , and  $\gamma + \sigma = 3/4 + \alpha/4 < 1$ . Moreover  $\epsilon_j^{-\tau} \epsilon_j = o(1)$ , while  $\epsilon_j^{-\tau} \lambda(F_k^j) \epsilon_j^\sigma \rightarrow +\infty$ .

For each  $k$  in question, the Lebesgue measure of  $S_k^j$  is asymptotically  $c\lambda(F_k^j) \epsilon_j^\sigma$ , and the different sets are disjoint, because  $\lambda(F_k^j) \epsilon_j^\sigma = o(\epsilon_j^\alpha)$ . We shall prove that  $|S_T(\lambda_j)| > c' \epsilon_j^{-\sigma}$  for a certain  $c' > 0$ , with  $T = \epsilon_j^{-\tau} \rightarrow +\infty$ . This will prove that the total  $\mu$ -measure of the subsets  $F_k^j$ , such that  $\epsilon_j^\gamma < \epsilon_j \leq \epsilon_j^\beta$ , is  $o(1)$ .

When  $x \in S_k^j$ ,

$$|S_T(x) - \int_{F_k^j}^* D_T(x - t) \lambda(dt)| < \int^* |x - t|^{-1} \lambda(dt),$$

and the error term on the right is  $o(\epsilon_j^{-\sigma})$ , because  $\sigma > \alpha - \beta$ .

When  $t \in F_k^j$ ,  $t - \xi_k = o(x - \xi_k)$  because  $\gamma + \sigma < 1$ , and  $\sin T(t - x) = \sin T(\xi_k - x) + o(1)$  because  $\tau < 1$ . This easily leads to the lower bound on  $|S_T(x)|$ .

Our construction is adapted from Kolmogorov's divergent Fourier series [3I, Chapter VIII].

To complete our example, we must present a set  $F$  that is also an  $M_0$ -set. This is known for various  $M_0$ -sets, but seems to occur explicitly in [1]: there exists a closed set  $E \subseteq [0, 1]$  and a sequence of integers  $N_k \rightarrow +\infty$  such that

$$(1) |N_k x| < N_k^{-1} \pmod{1} \text{ for } x \in E, k \geq 1,$$

$$(2) \text{ The mapping } y = e^x \text{ transforms } E \text{ onto an } M_0\text{-set.}$$

Then  $y(E)$  is covered by intervals of length  $\leq 2eN_k^{-2}$ , whose distances are at least  $(N_k^{-1} - 2N_k^{-2})$ .

In the remaining examples it is occasionally convenient to write  $S_T(y)$  in place of  $S_T(y, \mu)$ , when  $\mu = \mu_\theta$ .

II. We present example (C) first, because (B) is based on an improvement in one of the inequalities used in (C). For each  $n = 0, 1, 2, 3, \dots$ ,  $F_\theta$  is a union of  $2^{n+1}$  sets  $E_k$  of diameter  $2\theta^{n+1}(1 - \theta)^{-1}$ , and mutual distances at least

$$2\theta^{n+1}(1 - 2\theta)(1 - \theta)^{-1} \equiv c_1 \theta^{n+1}; \mu(E_k) = 2^{-n-1}.$$

The lower bound on the mutual distances gives a Hölder condition on  $\mu: \mu(B) \leq c_2(\text{diam } B)^\alpha$ , where  $\alpha = -\log 2 / \log \theta < 1$ . If  $\xi_k$  is the center of  $E_k$ , we have an identity

$$\int_{E_k} f(t) \mu(dt) = 2^{-n-1} \int f(\xi_k + \theta^{n+1}t) \mu(dt).$$

For each set  $E_k$ , we define the set  $E_k^\sim$  by the inequality  $d(x, E_k) < c_1 \theta^{n+1}/3$ , so the sets  $E_k^\sim$  have distances at least  $2c_1 \theta^{n+1}/3$ . If  $x \in E_k^\sim$ , then

$$|S_T(x, \mu) - \int_{E_k} D_T(x - t) \mu(dt)| < \int_{R - E_k} |x - t|^{-1} \mu(dt),$$

and in the last integral,  $|x - t| \geq 2c_1 \theta^{n+1}/3$ . Hence, by the Hölder condition, the integral is  $\leq c_3(\theta^n)^{\alpha-1} = c_3 2^{-n} \theta^{-n}$ . The principal term can be evaluated by the identity above, and simplified to the form  $2^{-n} \theta^{-n-1} S_{T\theta^{n+1}}(\theta^{-n-1}x - \theta^{-n-1}\xi_k)$ .

We observe that

$$\lim \int S_T(x, \mu) f(x) dx = \int f(x) \mu(dx),$$

for suitable test functions  $f$ ; for example, this is true if  $f$  and  $f'$  are integrable. Since  $\mu$  is singular, we can find a test function  $f$ , such that  $\|f\|_1 < 1$  and  $|\int f(x) \mu(dx)| > 2c_3 + 2c_1^{-1}$ . Hence  $\max |D_T(\mu)| > 2c_3 + 2c_1^{-1}$  for large  $T$ , say for  $T > T_0$ .

Let  $T > \theta^{-1}T_0$ , and let  $n \geq 0$  be chosen so that  $T^* = \theta^{n+1}T$  satisfies the inequalities  $T_0 \leq T^* \leq \theta^{-1}T_0$ . Suppose that

$$|D_{T^*}(\theta^{-n-1}x - \theta^{-n-1}\xi_k)| > c_3 + c_1^{-1}.$$

Then  $d(\theta^{-n-1}x - \theta^{-n-1}\xi_k, F_\theta) < c_1/3$ , since  $\pi > 3$ , or  $d(x, \xi_k + \theta^{n+1}F_\theta) < c_1\theta^{n+1}/3$ , so  $x \in E_k^*$ . Hence

$$|D_T(x, \mu)| > c_3 \cdot 2^{-n-1}\theta^{-n-1} - c_3 2^{-n}\theta^{-n} = c_4 2^{-n}\theta^{-n}.$$

But it is easy to see that the set of  $x$ 's in question has measure at least  $c_5 2^n \theta^n$ , because  $T_0 \leq T^* \leq \theta^{-1}T_0$ , and the functions  $D_{T^*}$  have derivatives bounded by  $\theta^{-2}T_0^2$ . Hence  $\|D_T(\mu)\|_1^* \geq c_4 c_5$ .

III. The example (B) requires a complicated construction, but relies in essence on small improvements on estimates already used. To estimate  $S_T(\mu, x)$  we divide the range of integration into the subsets  $\{|x - t| < T^{-1}\}$  and  $\{|x - t| > T^{-1}\}$ . The second yields an integral  $O(T^{1-\alpha})$ , by the Hölder condition, and the first yields  $T \cdot O(T^{-\alpha}) = O(T^{1-\alpha})$  for the same reason (and the inequality  $|D_T| < T$ ).

We give another estimate on  $S_T(x, \mu)$  for large  $T$ , supposing that  $\mu \in R$ .

LEMMA 3. — To each  $\epsilon > 0$  there is a  $T_0$  such that

$$|S_T(x, \mu)| < \epsilon d(x, F_\theta)^{-1}$$

whenever  $T \geq T_0$  and  $d \equiv d(x, F_\theta) \geq \epsilon$ .

*Proof.* — Let  $\delta = d(x, F)$  and observe that

$$\delta S_T(x, \mu) = \pi^{-1} \int \sin T(x - t) \cdot \delta \cdot (x - t)^{-1} \mu(dt).$$

The function  $g(t) = \delta \cdot (x - t)^{-1}$  is bounded by 1 on  $F$ , and

$|g(t_1) - g(t_2)| \leq \delta^{-1} |t_1 - t_2|$  for numbers  $t_1, t_2$  in  $F_\theta$ . Hence the conclusion follows from our assumption that  $\mu \in R$  and the Tietze extension theorem.

The inequality of the Lemma can be written in a more useful way. When  $t \in F_\theta$ , then  $|x - t| \leq d + 2 \leq d(1 + 2\epsilon^{-1})$ . Hence  $d(x, F_\theta)^{-1} \leq (1 + 2\epsilon^{-1}) \int |x - t|^{-1} \mu(dt)$ . Suppose now that  $x \notin E_k^\sim$  so that  $d(\theta^{-n-1}x - \theta^{-n-1}\xi_k, F_0) \geq c_1 \theta^{n+1}/3$ . Using the identity for integrals over  $E_k$ , we find the following estimate:

If  $x \notin E_k^\sim$  and  $T\theta^{n+1} > T_{00}$ , then

$$\left| \int_{E_k} D_T(x - t) \mu(dt) \right| < \epsilon \int_{E_k} |x - t|^{-1} \mu(dt).$$

Consequently, when  $x \in E_\ell^\sim$  and  $T\theta^{n+1}$  is sufficiently large (depending on  $\epsilon > 0$ )

$$|S_T(x, \mu) - 2^{-n-1} \theta^{-n-1} S_{T\theta^{n+1}}(\theta^{-n-1}x - \theta^{-n-1}\xi_\ell)| < \epsilon \theta^{n(\alpha-1)}.$$

LEMMA 4. — *To each  $\epsilon > 0$  there is a  $\delta > 0$  so that, when  $\theta^{-1} < Y < \delta T^{1-\alpha}$  then  $Ym\{|S_T(x, \mu)| > Y\} < \epsilon$ .*

*Proof.* — We choose  $n \geq 0$  so that  $1 < \theta^{n+1} Y^{1/\alpha} < \theta^{-1}$ ; this leads to the inequalities  $\theta^{n(\alpha-1)} > Y$ , and  $T\theta^{n+1} > \delta^{-1}$ . For fixed  $\ell$ , we must estimate the Lebesgue measure of the set defined by

$$|S_{T\theta^{n+1}}(\mu, \theta^{-n-1}x - \theta^{-n-1}\xi_\ell)| > \frac{1}{2} \cdot 2^{n+1} \theta^{n+1} Y.$$

The right hand side exceeds  $\frac{1}{2} \theta^{-1}$ ; when  $T\theta^{n+1}$  is large, the measure of the set is at most  $\epsilon \theta^{n+1}$ ; the total for all  $\ell$  is at most  $\epsilon 2^{n+1} \theta^{n+1} < \epsilon Y^{-1}$ . Hence  $Ym\{|S_T(x, \mu)| > Y\} < \epsilon$ .

In view of the inequality  $|S_T(\mu, x)| = O(T^{1-\alpha})$ , the conclusion of the last lemma holds when  $Y > \delta^{-1} T^{1-\alpha}$ ,  $T > 1$ , for a certain  $\delta > 0$ .

In preparation for the next lemma, we recall the identity ( $n = 1, 2, 3, \dots$ )

$$\int f(t) \mu(dt) \equiv 2^{-n} \sum_{k=1}^{2^n} \int f(\xi_k + \theta^n t) \mu(dt).$$

We define  $\int f(t) \sigma_n(dt) \equiv 2^{-n} \sum_k \int f(\xi_k + \theta^{n+k} t) \mu(dt)$ . Then



$\sigma_n = g_n \cdot \mu$ , where  $g_n \geq 0$ ,  $g_n$  is continuous on  $F_\theta$  and takes the values 0 and  $2^k (1 \leq k \leq 2^n)$ . Using the formula for  $\sigma_n$  we get an identity

$$S_T(x, \sigma_n) = 2^{-n} \theta^{-n} \sum_k \theta^{-k} S_{T\theta^{n+k}}(\theta^{-n-k}x - \theta^{-n-k}\xi_k).$$

LEMMA 5. — *To each  $\epsilon > 0$ , there is an  $N > 1$  such that  $\limsup_{T \rightarrow +\infty} \|S_T(\sigma_n)\|_1^* < \epsilon$ , if  $n \geq N$ .*

*Proof.* — In calculating  $\limsup_{T \rightarrow +\infty} \|S_T(\sigma_n)\|_1^*$  we can omit  $x$ 's outside  $(-3, 3)$ , because  $\sigma_n \in R$ . In an obvious notation we write  $\sigma_n = \sum_k \sigma_{n,k}$ , and observe that, for  $T > T_{n,\epsilon}$

$$|S_T(\sigma_n)| < \max_k |S_T(\sigma_{n,k})| + \epsilon/12.$$

When  $Y > \epsilon/6$  (the others are trivial, since we suppose that  $|x| < 6$ ),

$$\begin{aligned} m\{|S_T(\sigma_n)| > 2Y\} &\leq \sum_k m\{|S_T(\sigma_{n,k})| > Y\} \\ &= \sum_k \theta^{n+k} m\{|S_{T\theta^{n+k}}(x, \mu)| > 2^n \theta^{n+k} Y\}. \end{aligned}$$

Each summand is  $O(2^{-n} Y^{-1})$  by Kolmogorov's inequality; if  $T\theta^{n+k} > 1$ , then the  $k$ -th term exceeds  $\epsilon 2^{-n} Y$  only if

$$\delta(T\theta^{n+k})^{1-\alpha} < Y < \delta^{-1}(T\theta^{n+k})^{1-\alpha},$$

by Lemma 4 and the remark after it, and this inequality occurs for at most  $2(1-\alpha)^{-1} \cdot \log \delta / \log \theta$  indices  $k = 1, \dots, 2^n$ . (We assume that  $Y > \theta^{-1}$ , since  $S_T(\sigma_n) \rightarrow 0$  almost everywhere as  $T \rightarrow +\infty$ .) This proves our lemma.

A further property of  $\sigma_n$ , obtained simply by increasing  $n$ , is the inequality  $|\sigma_n(I) - \mu(I)| < \epsilon$  for all intervals  $I$ .

The next lemma establishes a property of the functional  $\|\cdot\|_1^*$  to simplify the remaining calculations.

LEMMA 6. — *Let  $a_i = \|f_i\|_1^*$   $1 \leq i \leq N$ . Then*

$$\|\sum f_i\|_1^* \leq (\sum a_i^{1/2})^2.$$

*Proof.* — Let  $0 \leq t_i \leq 1$ , and  $\sum t_i = 1$ . Then

$$m\{|\sum f_i| \geq Y\} \leq \sum m\{|f_i| \geq t_i Y\} \leq \sum t_i^{-1} Y^{-1} a_i.$$

The minimum of the sum is  $Y^{-1}(\sum a_i^{1/2})^2$ . With a little more effort, we can obtain the bound  $c(1-p)^{-1}(\sum a_i^p)^{1/p}$ ,  $0 < p < 1$ .

We are now in a position to construct the measure  $\lambda$ . We shall find probability measures  $\lambda_k = f_k \mu$ , with  $f_k \geq 0$ ,  $\int f_k d\mu = 1$ , such that  $\|S_T(\lambda_k)\|_1^* < k^{-1}$  for  $T > T_k > T_{k-1} \dots$  and  $|\hat{\lambda}_k(u)| < k^{-2}$  for  $u > T_k$ . Lemma 5 provides  $\lambda_1$ ; let us suppose that  $\lambda_k$  and  $T_k$  are known. We find  $\sigma_k$  so that  $|\sigma_k(I) - \lambda_k(I)| < k^{-1}(1 + T_k)^{-2}$  and  $\|S_T(\sigma_k)\|_1^* < k^{-4}/25$ , and  $|\hat{\sigma}_k(u)| < k^{-1}$ , for  $u > T_{k+1}^0 > T_k$ . (The construction of  $f_{k+1}\mu$  from  $f_k\mu$  follows Lemma 5). We now set  $\lambda_{k+1} = (1 - k^{-1/2})\lambda_k + k^{-1/2}\sigma_k$ ; by Lemma 6, we have for  $T > T_{k+1}^0$

$$\|S_T(\lambda_{k+1})\|_1^{*1/2} \leq (1 - k^{-1/2})^{1/2} k^{-1/2} + k^{-2}/5.$$

When  $k = 1$ , the last bound is  $1/5$ , while  $(k + 1)^{-1} = \frac{1}{2}$ . For  $k \geq 2$ , we need the inequality

$$(1 - k^{-1/2})^{1/2} k^{-1/2} + k^{-2}/5 < (k + 1)^{-1/2},$$

which can be verified with the aid of calculus. Clearly, we have  $|\hat{\lambda}_{k+1}(u)| < (k + 1)^{-2}$  for  $T > T_{k+1}^{00}$ ; we take  $T_{k+1} = T_{k+1}^0 + T_{k+1}^{00}$ .

By the construction, and integration by parts,

$$|\hat{\lambda}_k(u) - \hat{\lambda}_{k+1}(u)| \leq k^{-3/2}(1 + T_k)^{-2}|u|;$$

consequently  $|\hat{\lambda}_k(u) - \hat{\lambda}_{k+1}(u)| \leq k^{-3/2}$  unless  $|u| > 1 + T_k$ . However, if  $|u| > T_{k+1} > T_k$ , then  $|\hat{\lambda}_k(u) - \hat{\lambda}_{k+1}(u)| < 2k^{-2}$ . Since  $|\hat{\lambda}_k - \hat{\lambda}_{k+1}| \leq 2k^{-1/2}$ , we have a limit  $\varphi(u)$ , with

$$|\varphi - \hat{\lambda}_k| = O(k^{-1/2}).$$

Hence  $\varphi = \hat{\lambda}$ , with  $\lambda$  carried by  $F_\theta$  and  $\lambda \in R$ .

In verifying that  $\lim \|S_T(\lambda)\|_1^* = 0$  we can calculate the weak norms over  $(-3,3)$ . Suppose that  $T_{k-1} \leq T \leq T_k$ ; then

$$|S_T(\lambda_k) - S_T(\lambda)| = O(k^{-1/2}).$$

Since  $T \geq T_{k-1}$ ,  $\|S_T(\lambda_{k-1})\|_1^* < (k - 1)^{-1}$ ; and finally

$$\|S_T(\lambda_k) - S_T(\lambda_{k-1})\|_1^* = O(k^{-1/2}).$$

Hence  $\|S_T(\lambda)\|_1^* = O(k^{-1/2})$  over  $(-3,3)$ .

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