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# UNIFORM BOUNDS FOR QUOTIENTS OF GREEN FUNCTIONS ON $C^{1,1}$-DOMAINS

by H. HUEBER and M. SIEVEKING

Consider a partial differential operator L on $\mathbf{R}^n$ which has the form

$$Lu = \sum_{i,j=1}^{n} a_{ij} D_{ij} u + \sum_{i=1}^{n} b_i D_i u + cu .$$

We assume that L is strictly elliptic and has Hölder continuous coefficients. We will also assume $c \leqslant 0$. Hence every bounded domain $\Omega$ has a Green function which we will denote by $G_L^\Omega$. As usual $\Delta$ denotes the Laplace operator. The aim of our paper is to prove the following theorem:

THEOREM. – *For every bounded* $C^{1,1}$*-domain* $\Omega$ *there exists a constant* C *such that we have*

$$C^{-1} G_\Delta^\Omega \leqslant G_L^\Omega \leqslant C G_\Delta^\Omega$$

*on* $\Omega \times \Omega$. *The constant* C *may be chosen depending only on the ellipticity constant of* L, *on the Hölder norms of the coefficients of* L, *on the diameter of* $\Omega$, *and on the curvature of* $\partial\Omega$.

We proceed as follows: In § 1 we introduce notations concerning the geometry of $\Omega$ and give two elementary lemmas. In § 2 we introduce the harmonic space and the adjoint harmonic space associated to L and quote some results which we will use in our proof. The proof is given in § 3. We finish our paper with some remarks which include an application of our result to the Dirichlet problem for $Lu = f$.

## 1. The geometrical situation.

For $r > 0$ and $x \in \mathbf{R}^n$ let $B_r(x)$ denote the open ball with radius $r$ and center $x$ and let $S_r(x) := B_r(x) \backslash B_{\frac{1}{3}r}(x)$. Throughout this paper $\Omega$ will denote a bounded domain in $\mathbf{R}^n$ which is of class $C^{1,1}$. The distance from $x \in \mathbf{R}^n$ to $\partial\Omega$ will be denoted by $d_x$.

For every $x_0 \in \partial\Omega$ there exists a unique inner normal $n(x_0)$ at $\partial\Omega$ with $|n(x_0)| = 1$. By definition the mapping $x_0 \longrightarrow n(x_0)$ fulfills a local Lipschitz condition, and a simple compactness argument shows that there exists a constant $C_\Omega \geqslant 1$ such that we have $|n(x_0) - n(y_0)| \leqslant C_\Omega |x_0 - y_0|$ for all $x_0, y_0 \in \partial\Omega$. Furthermore there exists $r_\Omega \in ]0,1[$ such that we have $B_{r_\Omega}(x_0 + r_\Omega n(x_0)) \subset \Omega$ and $B_{r_\Omega}(x_0 - r_\Omega n(x_0)) \subset \mathbf{R}^n \backslash \Omega$ for all $x_0 \in \partial\Omega$. We may assume $C_\Omega r_\Omega \leqslant 1$.

For every $x \in \overline{\Omega}$ with $d_x < r_\Omega$ there exists exactly one point $x_0 \in \partial\Omega$ with $x = x_0 + d_x n(x_0)$. Hence for such a point $x \in \overline{\Omega}$ the following definitions make sense:

$$x_\alpha := x_0 + \alpha n(x_0) \qquad \alpha \in \mathbf{R}$$

$$n(x) := n(x_0)$$

$$Z_\alpha(x) := \{y \in \Omega \mid |y - x_\beta| < 2^{-12}\alpha \text{ for some } \beta \in ]-\alpha, \alpha[\} \quad \alpha \in \mathbf{R}_+.$$

LEMMA 1.1. – *Let* $x, z \in \overline{\Omega}$ *with* $d_x, d_z < r_\Omega$. *Let* $\beta, \alpha, \lambda \in \mathbf{R}^+$ *such that* $0 < \beta < \alpha < \frac{1}{2} r_\Omega$ *and* $0 < \lambda \leqslant \frac{1}{10}$. *Assume further* $|z_0 - x_\beta| \leqslant \beta + 2\lambda\alpha$. *Then one has* $|z_0 - x_0| \leqslant 4\alpha\sqrt{\lambda}$.

*Proof.* – Regarding the situation in a plane which contains $x_0$, $x_{r_\Omega}$ and $z_0$ the problem becomes a two-dimensional one. The statement then follows from the theorem of Pythagoras and elementary estimates which we will leave to the reader. □

LEMMA 1.2. – a) *Let* $x \in \Omega$ *with* $d(x, \partial\Omega) < r_\Omega$, *let* $\alpha \in \left]0, \frac{1}{2} r_\Omega\right[$ *and let* $z \in \overline{Z_\alpha(x)}$. *Then we have* $|z_0 - x_0| \leqslant \frac{1}{16}\alpha$.

b) *Let* $x, y \in \Omega$ *with* $d_x < r_\Omega$ *and* $d_y < r_\Omega$. *Assume*

$$\overline{Z_\alpha(x)} \cap \overline{Z_\alpha(y)} \neq \phi$$

*for some* $\alpha \in ]0, r_\Omega[$. *Hence we have* $|x_\beta - y_\beta| < \frac{1}{4}\alpha$ *for all* $\beta \in [0, r_\Omega]$.

*Proof.* – a) Since $z \in Z_\alpha(x)$ there exists $\beta \in ]-\alpha, \alpha[$ such that $|x_\beta - z| < 2^{-12}\alpha$. Hence we have

$$|z_0 - x_\beta| \leqslant |z_0 - z| + |z - x_\beta| \leqslant |x_0 - z| + |z - x_\beta|$$
$$< |x_0 - x_\beta| + 2\,|z - x_\beta| < |\beta| + 2 \cdot 2^{-12}\alpha .$$

Let $\lambda = 2^{-12}$. Since $z_0 \notin B_{r_\Omega}(x_{r_\Omega})$ the assertion follows from Lemma 1.1 at least for $\beta \geqslant 0$. For $\beta < 0$ the assertion is obviously true.

b) Let $z \in \overline{Z_\alpha(x)} \cap \overline{Z_\alpha(y)}$. By a) we have

$$|y_0 - x_0| \leqslant |y_0 - z_0| + |z_0 - x_0| \leqslant \frac{1}{8}\,\alpha .$$

Hence for $\beta \in [0, r_\Omega]$ we get

$$|x_\beta - y_\beta| = |x_0 + \beta n(x_0) - y_0 - \beta n(y_0)|$$
$$\leqslant |x_0 - y_0| + \beta\,|n(x_0) - n(y_0)| \leqslant |x_0 - y_0|\,(1 + \beta C_\Omega)$$
$$\leqslant \frac{1}{8}\,\alpha(1 + r_\Omega C_\Omega) \leqslant \frac{1}{4}\,\alpha . \qquad \square$$

## 2. Potential theory of L.

Following [2] we say that the operator

$$L = \sum_{i,j=1}^{n} a_{ij} D_{ij} + \sum_{i=1}^{n} b_i D_i + c$$

belongs to the class $\mathcal{L}(\lambda, \alpha_0)$ with $\lambda \geqslant 1$ and $\alpha_0 \in ]0,1[$ if it fulfills the following properties:

(i) For all $x \in \mathbf{R}^n$ and all $\xi \in \mathbf{R}^n$ we have

$$\sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i \xi_j \geqslant \lambda^{-1}\,|\xi|^2$$

(ii) For all $x, y \in \mathbf{R}^n$ we have

$$\sum_{i,j=1}^{n} |a_{ij}(x) - a_{ij}(y)| + \sum_{i=1}^{n} |b_i(x) - b_i(y)| + |c(x) - c(y)| \leqslant \lambda\,|x - y|^{\alpha_0}$$

(iii) For all $x \in \mathbf{R}^n$ we have

$$\sum_{i,j=1}^{n} |a_{ij}(x)| + \sum_{i=1}^{n} |b_i(x)| + |c(x)| \leqslant \lambda .$$

From now on we will always assume $L \in \mathcal{L}(\lambda, \alpha_0)$, and furthermore we will assume $c \leqslant 0$.

The sheaf $\mathcal{H}_L$ of solutions of $Lu = 0$ is a harmonic sheaf which gives rise to a Brelot space ([3], [4], [6]). Every bounded domain V has a Green function $G_L^V$ which may be characterized by the following properties:

(i) $G_L^V(\cdot, y)$ is a potential on V for every $y \in V$. Its support is $\{y\}$.

(ii) For all $y \in V$ we have

$$\lim_{x \to y} \frac{G_L^V(x, y)}{(\sqrt{(x-y)\, B(y)\, (x-y)})^{2-n}} = (n-2)\, n\, \omega_n \sqrt{\det(a_{ij}(y))}\ (^1).$$

Here $B(y)$ is the inverse of $(a_{ij}(y))_{i,j}$ and $\omega_n$ denotes as usual the volume of the unit ball in $\mathbf{R}^n$.

R.M. Herve [6] has shown that there exists a unique Brelot space on $\mathbf{R}^n$ such that the extremal potentials on V for this adjoint harmonic space are just the functions $G_L^V(x, \cdot)$ with $x \in \Omega$. We use the symbol $L^*$ to indicate this adjoint space. If the adjoint operator of L exists as a differential operator and is denoted by $L^*$ the notation will stay consistent.

For every L-regular bounded open set $V \subset \mathbf{R}^n$ and every $f \in \mathcal{C}(\partial V)$ we denote by $H_V^L f$ the unique continuous function on $\overline{V}$ which is equal to $f$ on $\partial V$ and L-harmonic in V. For $L^*$-regular V the meaning of $H_V^{L^*} f$ is obvious. For our purposes it is enough to know that Lipschitz domains are L-regular and $L^*$-regular (cf. [1] Théorème 4.1), and we will use this without further comment.

It is clear that $\Delta \in \mathcal{L}(n, \alpha)$ for every $\alpha \in ]0, 1[$. To be able to work with the same constants for L and $\Delta$ we will henceforth assume $\lambda \geqslant n$.

(1) For $n \leqslant 2$ this formula has to be modified in the usual way. The same remark applies to the statement of Lemma 2.4.

LEMMA 2.1. – *There exists a constant* $C_1 = C_1(\lambda, \alpha_0, n)$ *such that for every* $x \in \mathbf{R}^n$, $r \in ]0,1[$ *and* $f \in \mathcal{C}_+(\partial B_r(x))$ *the following inequalities hold on* $B := B_r(x)$:

$$C_1^{-1} H_B^{\Delta} f \leqslant H_B^{L} f \leqslant C_1 H_B^{\Delta} f$$
$$C_1^{-1} H_B^{\Delta} f \leqslant H_B^{L^*} f \leqslant C_1 H_B^{\Delta} f.$$

*Proof.* – The first inequality is due to Serrin and the second one is due to Ancona. They can be found in [2] page 13 and page 20 (for $r \leqslant 1$ the constant C in Proposition 10 on page 20 does not depend on $r$). □

LEMMA 2.2. – *There exists a constant* $C_2 := C_2(\lambda, \alpha_0, n)$ *such that for every* $x \in \mathbf{R}^n$, $r \in ]0,1[$, *and every positive* L*-harmonic or* $L^*$*-harmonic function* $h$ *on* $B_r(x)$ *the following inequality holds for all* $y_1, y_2 \in B_{\frac{1}{2}r}(x)$: $h(y_1) \leqslant C_2 h(y_2)$.

*Proof.* – With the use of the ordinary Harnack inequality this statement follows immediately from Lemma 2.1. □

LEMMA 2.3. – *Let* $x \in \overline{\Omega}$ *with* $d_x < r_\Omega$ *and let* $\alpha \in \left]0, \frac{1}{2} r_\Omega\right[$. *Let* $h$ *be a positive* L*-harmonic or* $L^*$*-harmonic function in* $Z_\alpha(x)$ *which tends to zero at each* $y \in \overline{Z_\alpha(x)} \cap \partial\Omega$. *Then we have*

$$C_3^{-1} \frac{\gamma}{\alpha} h(x_\alpha) \leqslant h(x_\gamma) \leqslant C_3 \frac{\gamma}{\alpha} h(x_\alpha)$$

*for all* $\gamma \in [0, \alpha]$ *with a constant* $C_3$ *depending only on* $\lambda, \alpha_0, n$ *and* $r_\Omega$.

*Proof.* – In the following the symbol L has to be replaced by $L^*$ if $h$ is $L^*$-harmonic.

Let $\beta := 5^{-1} 2^{-12} \alpha$ and let

$$\Gamma := \{z \mid |z - x_\beta| = \beta;\ |z - x_{2\beta}| \leqslant \beta\}.$$

By Lemma 2.2 we have $h(x_\beta) \leqslant C_2 \inf h(\Gamma)$.

By [1] Théorème 2.2 and Théorème 4.1 there exists a constant $C = C(\lambda, \alpha_0, n, r_\Omega)$ such that

$$\sup h(B_{4\beta}(x_0) \cap \Omega) \leqslant C\, h(x_\beta).$$

Using the maximum principle we get

$$C_2^{-1}\, h(x_\beta)\, H^L_{B_\beta(x_\beta)}\, 1_\Gamma(x_\gamma) \leqslant h(x_\gamma) \leqslant C\, h(x_\beta)\, H^L_{S_{3\beta}(x_{-\beta})}\, 1_{\partial B_{3\beta}(x_{-\beta})}(x_\gamma)$$

for all $\gamma \in [0, \beta]$. Using Lemma 2.1 we get

$$\begin{aligned} H^L_{B_\beta(x_\beta)}\, 1_\Gamma(x_\gamma) &\geqslant C_1^{-1}\, H^\Delta_{B_\beta(x_\beta)}\, 1_\Gamma(x_\gamma) \\ &\geqslant \widetilde{C}_1^{-1}\, \beta^{-1}\, |x_\gamma - x_0| = \widetilde{C}_1^{-1}\, \frac{\gamma}{\beta} \end{aligned}$$

with $\widetilde{C}_1 = \widetilde{C}_1(\lambda, \alpha_0, n)$. A standard transportation argument together with Lemma 2.1 shows

$$H^L_{S_{3\beta}(x_{-\beta})}\, 1_{\partial B_{3\beta}(x_{-\beta})}(x_\gamma) \leqslant \widetilde{C}_2\, \frac{\gamma}{\beta}$$

with $\widetilde{C}_2 = \widetilde{C}_2(\lambda, \alpha_0, n)$ (cf. [2] Théorème 2).

Thus we have

$$C_2^{-1}\, \widetilde{C}_1^{-1}\, \frac{\gamma}{\beta}\, h(x_\beta) \leqslant h(x_\gamma) \leqslant C\, \widetilde{C}_2\, \frac{\gamma}{\beta}\, h(x_\beta)$$

for all $\gamma \in [0, \beta]$.

Lemma 2.2 implies that there exists a constant $\widetilde{C}_3$ such that we have $h(x_{\gamma_1}) \leqslant \widetilde{C}_3\, h(x_{\gamma_2})$ for all $\gamma_1, \gamma_2 \in [\beta, \alpha]$. Thus for $\gamma \in [0, \beta]$ we have

$$5{\cdot}2^{12}\, \widetilde{C}_1^{-1}\, C_2^{-1}\, \widetilde{C}_3^{-1}\, \frac{\gamma}{\alpha}\, h(x_\alpha) \leqslant h(x_\gamma) \leqslant 5{\cdot}2^{12} C\, \widetilde{C}_2\, \widetilde{C}_3\, \frac{\gamma}{\alpha}\, h(x_\alpha),$$

and for $\gamma \in [\beta, \alpha]$ we have

$$\begin{aligned} \gamma\alpha^{-1}\, \widetilde{C}_3^{-1}\, h(x_\alpha) &\leqslant \widetilde{C}_3^{-1}\, h(x_\alpha) \leqslant h(x_\gamma) \leqslant \widetilde{C}_3\, h(x_\alpha) \\ &\leqslant \gamma\alpha^{-1}(\alpha\beta^{-1})\, \widetilde{C}_3\, h(x_\alpha) = \gamma\alpha^{-1}\, 5{\cdot}2^{12}\, \widetilde{C}_3\, h(x_\alpha). \end{aligned}$$

This shows that the assertion is true with $C_3 := 5{\cdot}2^{12} C\, \widetilde{C}_1\, C_2\, \widetilde{C}_2\, \widetilde{C}_3$. □

LEMMA 2.4. – *There exists a constant* $C_4 = C_4(\lambda, \alpha_0, n)$ *such that we have*

$$C_4^{-1}\, |x - y|^{2-n} \leqslant G^\Omega_L(x, y) \leqslant C_4\, |x - y|^{2-n}$$

*for all* $x, y \in \Omega$ *with* $|x - y| \leqslant \frac{1}{2}\, d_x$ *or* $|x - y| \leqslant \frac{1}{2}\, d_y$.

*Proof.* – This Lemma is essentially due to Gilbarg/Serrin [5]. It can be found in [2] on page 14.

## 3. Proof of the theorem.

Let $K := \left\{ x \in \Omega \mid d_x \geqslant \frac{1}{16} r_\Omega \right\}$. Using Lemma 2.4 one can see that $C_4^{-2}\, G_\Delta^\Omega \leqslant G_L^\Omega \leqslant C_4^2\, G_\Delta^\Omega$ is true at least in a neighbourhood of the diagonal of $K \times K$. Since $G_L^\Omega$ and $G_\Delta^\Omega$ are strictly positive and continuous off the diagonal there exists a constant $C_5$ such that we have

$$(*)\quad C_5^{-1}\, G_\Delta^\Omega(x, y) \leqslant G_L^\Omega(x, y) \leqslant C_5\, G_\Delta^\Omega(x, y)$$

for all $x, y \in K$. Looking very carefully at Lemma 2.2 and Lemma 2.4 it is obvious that $C_5$ may be chosen depending only on $\lambda, \alpha_0, n, r_\Omega$ and the diameter of $\Omega$.

Now let $x, y \in \Omega$. Since all arguments which we will use in the following proof are also true if one interchanges the role of $\mathcal{H}_L$ and $\mathcal{H}_{L^*}$ we may assume $d_x \leqslant d_y$. The proof is divided into three cases:

$$\text{I } d_x \geqslant \frac{1}{16} r_\Omega\ ;\quad d_y \geqslant \frac{1}{4} r_\Omega$$

$$\text{II } d_x \leqslant \frac{1}{16} r_\Omega\ ;\quad d_y \geqslant \frac{1}{4} r_\Omega$$

$$\text{III } d_y \leqslant \frac{1}{4} r_\Omega\,.$$

*Case I.* – In this case we have $x, y \in K$ and our statement follows from (*).

*Case II.* – In this case we have $y \notin Z_{\frac{1}{8} r_\Omega}(x)$. With $\beta := d_x$ we get

$$\begin{aligned} G_\Delta^\Omega(x, y) &\leqslant C_3\, \frac{8\beta}{r_\Omega}\, G_\Delta^\Omega(x_{\frac{1}{8} r_\Omega}, y) \leqslant C_3 C_5\, \frac{8\beta}{r_\Omega}\, G_L^\Omega(x_{\frac{1}{8} r_\Omega}, y) \\ &\leqslant C_3^2\, C_5\, G_L^\Omega(x, y) \leqslant C_3^3 C_5\, \frac{8\beta}{r_\Omega}\, G_L^\Omega(x_{\frac{1}{8} r_\Omega}, y) \\ &\leqslant C_3^3 C_5^2\, \frac{8\beta}{r_\Omega}\, G_\Delta^\Omega(x_{\frac{1}{8} r_\Omega}, y) \leqslant C_3^4 C_5^2 G_\Delta^\Omega(x, y)\,. \end{aligned}$$

Here we have used (*), and we have applied Lemma 2.3 to the L-harmonic function $G_L^\Omega(\cdot, y)$ and the $\Delta$-harmonic function $G_\Delta^\Omega(\cdot, y)$ on $Z_{\frac{1}{8} r_\Omega}(x)$.

*Case III.* – Let $\beta := d_x$ and $\gamma := d_y$. By assumption we have $0 < \beta \leqslant \gamma \leqslant \frac{1}{4} r_\Omega$. By $\alpha$ we will denote the minimum of all positive numbers $\delta$ for which we have $\overline{Z_\delta(x)} \cap \overline{Z_\delta(y)} \neq \emptyset$. We will consider four subcases.

Subcase 1: $\alpha \geqslant \frac{1}{4} r_\Omega =: \mu$.

We have $y_\mu, x_\mu \in K$ and $Z_\mu(x) \cap Z_\mu(y) = \emptyset$. With (*) and Lemma 2.3 we get

$$\begin{aligned}
G_\Delta^\Omega(x, y) &\leqslant C_3 \frac{\beta}{\mu} G_\Delta^\Omega(x_\mu, y) \\
&\leqslant C_3^2 \frac{\beta}{\mu} \frac{\gamma}{\mu} G_\Delta^\Omega(x_\mu, y_\mu) \\
&\leqslant C_3^2 C_5 \frac{\beta}{\mu} \frac{\gamma}{\mu} G_L^\Omega(x_\mu, y_\mu) \\
&\leqslant C_3^3 C_5 \frac{\beta}{\mu} G_L^\Omega(x_\mu, y) \\
&\leqslant C_3^4 C_5 G_L^\Omega(x, y) \\
&\;\;\vdots \\
&\leqslant C_3^8 C_5^2 G_\Delta^\Omega(x, y),
\end{aligned}$$

where the dots indicate that we must repeat our arguments in the inverse order.

Subcase 2: $\frac{1}{4} r_\Omega \geqslant \alpha \geqslant \gamma$.

Obviously we have $\overline{Z_\alpha(x)} \cap \overline{Z_\alpha(y)} \neq \emptyset$, and hence Lemma 1.2 implies $|x_\alpha - y_\alpha| \leqslant \frac{1}{4} \alpha \leqslant \frac{1}{2} d_{y_\alpha}$. We apply Lemma 2.3 and Lemma 2.4 and get

$$\begin{aligned}
G_\Delta^\Omega(x, y) &\leqslant C_3^2 \frac{\beta}{\alpha} \frac{\gamma}{\alpha} G_\Delta^\Omega(x_\alpha, y_\alpha) \\
&\leqslant C_3^2 C_4^2 \frac{\beta}{\alpha} \frac{\gamma}{\alpha} G_L^\Omega(x_\alpha, y_\alpha) \\
&\leqslant C_3^4 C_4^2 G_L^\Omega(x, y) \\
&\leqslant C_3^6 C_4^2 \frac{\beta}{\alpha} \frac{\gamma}{\alpha} G_L^\Omega(x_\alpha, y_\alpha)
\end{aligned}$$

$$\leqslant C_3^6 C_4^4 \frac{\beta}{\alpha} \frac{\gamma}{\alpha} G_\Delta^\Omega(x_\alpha, y_\alpha)$$

$$\leqslant C_3^8 C_4^4 G_\Delta^\Omega(x, y).$$

Subcase 3: $\beta \geqslant \alpha$.

If $|x - y| \leqslant \frac{1}{2} \beta$ the desired inequalities follow from Lemma 2.4. Hence we may assume $|x - y| > \frac{1}{2} \beta$. Since $\beta \leqslant \gamma$ we have $y \notin Z_\beta(x)$. From Lemma 1.2 we get $|x_{\frac{1}{2}\beta} - y_{\frac{1}{2}\beta}| \leqslant \frac{1}{4} \beta$, and $|y - y_{\frac{1}{2}\beta}| \geqslant \frac{1}{2} \beta$ is clear. Hence we get

$$\begin{aligned}
G_\Delta^\Omega(x, y) &\leqslant C_3\, 2\, G_\Delta^\Omega(x_{\frac{1}{2}\beta}, y) \\
&\leqslant C_2 C_3\, 2\, G_\Delta^\Omega(y_{\frac{1}{2}\beta}, y) \\
&\leqslant C_2 C_3^2\, 2\gamma^{-1}\beta\, G_\Delta^\Omega(y_{\frac{1}{2}\gamma}, y) \\
&\leqslant C_2 C_3^2 C_4^2\, 2\gamma^{-1}\beta\, G_L^\Omega(y_{\frac{1}{2}\gamma}, y) \\
&\leqslant C_2 C_3^3 C_4^2\, 2\, G_L^\Omega(y_{\frac{1}{2}\beta}, y) \\
&\leqslant C_2^2 C_3^3 C_4^2\, 2\, G_L^\Omega(x_{\frac{1}{2}\beta}, y) \\
&\leqslant C_2^2 C_3^4 C_4^2\, G_L^\Omega(x, y) \\
&\ \ \vdots \\
&\leqslant C_2^4 C_3^8 C_4^4\, G_\Delta^\Omega(x, y).
\end{aligned}$$

To derive the first of these inequalities we applied Lemma 2.3 to the $\Delta$-harmonic function $G_\Delta^\Omega(\cdot, y)$ on $Z_\beta(x)$. In the second line we applied Lemma 2.2 to the same function in $B_{\frac{1}{2}\beta}(y_{\frac{1}{2}\beta})$. In the third inequality we used Lemma 2.3 on $Z_{\frac{1}{2}\gamma}(y)$ observing that $y \notin Z_{\frac{1}{2}\gamma}(y)$. The fourth inequality followed from Lemma 2.4. Then we applied again Lemma 2.3, then Lemma 2.2, then Lemma 2.3 once more, and the dots indicate that we must repeat all our arguments in the inverse order.

Subcase 4: $\gamma \geqslant \alpha \geqslant \beta$.

We apply subcase 3 to the points $x_\alpha$, $y$ and get

$$C_2^{-2}C_3^{-4}C_4^{-2}\,G_\Delta^\Omega(x_\alpha, y) \leqslant G_L^\Omega(x_\alpha, y) \leqslant C_2^2C_3^4C_4^2\,G_\Delta^\Omega(x_\alpha, y).$$

Since $y \notin Z_\alpha(x)$ we may apply Lemma 2.3 to $Z_\alpha(x)$, and we get

$$\begin{aligned} G_\Delta^\Omega(x, y) &\leqslant C_3\,\frac{\beta}{\alpha}\,G_\Delta^\Omega(x_\alpha, y) \\ &\leqslant C_2^2C_3^5C_4^2\,\frac{\beta}{\alpha}\,G_L^\Omega(x_\alpha, y) \\ &\leqslant C_2^2C_3^6C_4^2\,G_L^\Omega(x, y) \\ &\leqslant C_2^2C_3^7C_4^2\,\frac{\beta}{\alpha}\,G_L^\Omega(x_\alpha, y) \\ &\leqslant C_2^4C_3^{11}C_4^4\,\frac{\beta}{\alpha}\,G_\Delta^\Omega(x_\alpha, y) \\ &\leqslant C_2^4C_3^{12}C_4^4\,G_\Delta^\Omega(x, y). \end{aligned}$$

Now we see that with $C := C_2^2C_3^6C_4^2C_5$ the statement of the theorem is true. □

## 4. Remarks.

4.1. Obviously our theorem is true if the Laplace operator is replaced by any $L' \in \mathcal{L}(\lambda, \alpha_0)$ with $c' \leqslant 0$. Hence

$$C(L, L') := \inf\{c > 0 \mid c^{-1}G_{L'}^\Omega \leqslant G_L^\Omega \leqslant c\,G_{L'}^\Omega\}$$

is a well defined real number $\geqslant 1$. We can show that for sufficiently smooth domains $C(L, L')$ depends continuously on the coefficients of $L, L'$ with respect to uniform convergence. The proof is by a modification of a method presented in [7]. □

4.2. For Lipschitz domains our theorem is not true:

Let $\Omega := \{x = (x_1, x_2) \in \mathbf{R}^2 \mid 0 < x_1 < 1;\ 0 < x_2 < x_1\}$; fix $y \in \Omega$ and let $V$ denote the intersection of $\Omega$ and a small ball around zero whose closure does not contain $y$. Hence there exists a constant $C_\Delta$ such that

$$C_\Delta^{-1}(x_1^3x_2 - x_1x_2^3) \leqslant G_\Delta^\Omega(x, y) \leqslant C_\Delta(x_1^3x_2 - x_1x_2^3)$$

for all $x = (x_1, x_2) \in V$.

Let $A := \frac{\partial^2}{\partial x_1^2} + 2\,\frac{\partial}{\partial x_1 \partial x_2} + 3\,\frac{\partial^2}{\partial x_2^2}$. Then there exists a constant $C_A$ such that

$$C_A^{-1}(x_1^2 - x_1 x_2) \leqslant G_A^\Omega(x\,,y) \leqslant C_A(x_1^2 - x_1 x_2)\,.$$

This shows that $G_A^\Omega(\cdot\,,y)/G_\Delta^\Omega(\cdot\,,y)$ is unbounded near zero. □

4.3. Let X be a locally compact space with countable base, and assume that we have two harmonic sheafs $\mathcal{H}_1, \mathcal{H}_2$ on X such that $(X, \mathcal{H}_i)$ $i = 1, 2$ is a $\mathcal{P}$-Brelot space. Assume that $\Omega$ is a bounded domain in X which is regular for both structures and happens to have Green functions $G_{\mathcal{H}_i}^\Omega\,(i = 1, 2)$. Let $f := \partial\Omega \longrightarrow \mathbf{R}$ be continuous and positive, and let $u_i$ denote the solution of the Dirichlet problem for $f$ with respect to the sheaf $\mathcal{H}_i$. We know that $u_1$ is the limite of an increasing sequence $(p_n)_{n\in\mathbf{N}}$ of $\mathcal{H}_1$-potentials, and we may even assume that the potential theoretic carrier of $p_n$ is contained in $\Omega\setminus U_n$ where $(U_n)_{n\in\mathbf{N}}$ is an exhaustion of $\Omega$.

Hence

$$p_n(x) = \int G_{\mathcal{H}_1}^\Omega(x\,,y)\,\mu_n(dy)$$

with $\mu_n(U_n) = 0$.

Let

$$q_n(x) := \int G_{\mathcal{H}_2}^\Omega(x\,,y)\,\mu_n(dy)\,.$$

Assume now that there exists a constant C such that

$$C^{-1}\,G_{\mathcal{H}_1}^\Omega \leqslant G_{\mathcal{H}_2}^\Omega \leqslant C\,G_{\mathcal{H}_1}^\Omega\,.$$

Hence

$$C^{-1}\,q_n \leqslant p_n \leqslant C\,q_n\,.$$

By Azela-Ascoli a subsequence of $(q_n)_{n\in\mathbf{N}}$ is locally uniformly convergent and therefore we may assume that $(q_n)_n$ itself converges locally uniformly to an $\mathcal{H}_2$-harmonic function $q$.

Obviously we have $C^{-1}\,q \leqslant u_1 \leqslant C\,q$ and from this we derive $C^{-1}\,q \leqslant u_2 \leqslant C^{-1}\,q$. Thus we have $C^{-2}\,u_2 \leqslant u_1 \leqslant C^2\,u_2$.

This shows that our theorem implies the statement of Lemma 2.1. In a similar way we may also derive the other statements of § 2 from the theorem. This shows that our paper characterizes in a certain way all Brelot spaces on $\mathbf{R}^n$ for which the comparability statement of our theorem is true with a constant C which depends only on the diameter of $\Omega$ and the curvature of $\partial\Omega$.

4.4. For $L_1, L_2 \in \mathcal{L}(\lambda, \alpha_0)$ the following comparability statement is an easy conclusion of 4.1 and 4.3:

For any $u_1, u_2 \in \mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ with $u_1 = u_2 \geqslant 0$ on $\partial\Omega$ and $L_1 u_1 = L_2 u_2 \leqslant 0$ in $\Omega$ we have $u_1 \leqslant C u_2$ with a constant $C = C(\lambda, \alpha_0, n, \Omega)$. □

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