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Esterlè’s proof of the tauberian theorem for Beurling algebras

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1. Introduction.

In [5], J. Esterle gave a new proof of the Wiener Tauberian theorem for the algebra $L^1(\mathbb{R})$ by using some results from complex analysis and from the theory of radical Banach algebras. In this note, we show that a proof with the same idea also establishes the analogous result for Beurling algebras.

We first give the basic properties of the algebras of Beurling that we are considering.

Let $\varphi$ be a non-negative, measurable function on $\mathbb{R}$, and set

$$L_\varphi^1 = \{ f : \| f \| = \int_{-\infty}^{\infty} |f(t)| e^{\varphi(t)} \, dt < \infty \}.$$

Then $L_\varphi^1$ is a Banach space: as usual, we equate functions equal almost everywhere. If

$$\varphi(s+t) \leq \varphi(s) + \varphi(t) \quad (s, t \in \mathbb{R}),$$

then $L_\varphi^1$ is a commutative Banach algebra with respect to convolution multiplication defined by the equation

$$(f \ast g)(t) = \int_{-\infty}^{\infty} f(t-s)g(s) \, ds \quad (f, g \in L_\varphi^1).$$

These algebras were introduced by Beurling in 1938 [1].
Condition (1) ensures the existence of the finite limits \( \alpha = \lim_{t \to -\infty} \varphi(t)/t \) and \( \beta = \lim_{t \to -\infty} \varphi(t)/t \). Let \( \Pi \) be the open strip \( \{-\infty < \text{Re} \, z < -\beta\} \), and let \( \overline{\Pi} \) be the closed strip \( \{-\alpha \leq \text{Re} \, z \leq -\beta\} \) of \( \mathbb{C} \): if \( \alpha = \beta \), then \( \overline{\Pi} \) is a line. For \( f \in L_\varphi^1 \), we define the Laplace transform, \( \hat{f} \), of \( f \) on \( \overline{\Pi} \) by
\[
\hat{f}(z) = \int_{-\infty}^{\infty} f(t)e^{-zt} \, dt \quad (z \in \overline{\Pi}).
\]
The integral converges absolutely for \( z \in \overline{\Pi} \). Let \( A_0(\overline{\Pi}) \) denote the uniform algebra of functions which are continuous on \( \overline{\Pi} \), analytic on \( \Pi \), and which converge uniformly to zero as \( z \to \infty \) with \( z \in \overline{\Pi} \). Then \( \hat{f} \in A_0(\overline{\Pi}) \).

It is well known (for example, see [6], §18) that the character space, or space of maximal modular ideals, of \( L_\varphi^1 \) can be identified with \( \overline{\Pi} \), and that the map \( f \mapsto \hat{f} \) is a monomorphism of \( L_\varphi^1 \) into \( A_0(\overline{\Pi}) \).

Let \( I \) be a closed ideal of \( L_\varphi^1 \). We are interested in conditions on \( I \) which ensure that \( I = L_\varphi^1 \). Let
\[
Z(I) = \{z \in \overline{\Pi} : \hat{f}(z) = 0 \quad (f \in I)\}.
\]

Clearly, a necessary condition for the equality \( I = L_\varphi^1 \) is that \( Z(I) = \emptyset \). Wiener posed the problem for the algebra \( L^1(\mathbb{R}) \) (for which \( \varphi = 0 \)), and he proved that, if \( Z(I) = \emptyset \), then \( I = L^1(\mathbb{R}) \). This is Wiener's Tauberian theorem; of course, the formulation of Wiener was different.

**Definition.** — Let \( L_\varphi^1 \) be a Beurling algebra. Then spectral analysis holds for \( L_\varphi^1 \) if each proper closed ideal of \( L_\varphi^1 \) is contained in a maximal modular ideal of \( L_\varphi^1 \).

Clearly, spectral analysis holds for \( L_\varphi^1 \) if and only if \( I = L_\varphi^1 \) for each \( I \) with \( Z(I) = \emptyset \), and Wiener's theorem is that spectral analysis holds for \( L^1(\mathbb{R}) \).

It was shown by Beurling in [1] that spectral analysis holds for the algebra \( L_\varphi^1 \) provided that the weight \( \varphi \) satisfies (1) and the additional condition that
\[
(2) \quad \int_{-\infty}^{\infty} \frac{\varphi(t)}{1 + t^2} \, dt < \infty.
\]
(Note that this condition implies that \( \alpha = \beta = 0 \), and so in this case we are identifying the character space of \( L_\varphi^1 \) with the imaginary axis.)
Modern proofs of the theorem of Beurling use only the fact, ensured by (2), that the Banach algebra \( L^1_\varphi \) is regular, in the sense that, given \( y_0 \in \mathbb{R} \) and a neighbourhood \( U \) of \( y_0 \), there exists \( f \in L^1_\varphi \) with \( f(iy_0) = 1 \) and \( f(iy) = 0 \) \( (y \notin U) \): see [6], § 40, for example, for a proof of the theorem given that \( L^1_\varphi \) is regular. Indeed, Gurarii ([7], page 24) states, « all proofs of Wiener's theorem known to us make essential use of this fact of regularity, and... it is hardly possible to manage without it. » Following the ideas of Esterle in [5], we shall prove Beurling's result without using the regularity of \( L^1_\varphi \). It is not claimed that the present proof is any shorter than the usual one.

It is perhaps worth recalling how the regularity of \( L^1_\varphi \) follows from condition (2). The starting point is a result which is essentially Theorem XII of [10]: if \( \varphi \) is a non-negative, measurable function on \( \mathbb{R} \), then a necessary and sufficient condition that there exists a function \( f \) which is bounded and analytic in the open upper half-plane \( \Pi^+ \) and which is such that \( \lim_{y \to 0^+} |f(x + iy)| = \exp(-\varphi(x)) \) for almost all \( x \) is that \( \varphi \) satisfies (2).

To show the sufficiency of (2), suppose that \( \varphi \) satisfies this condition, and define \( u \) on \( \Pi^+ \) by

\[
u(x + iy) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\varphi(t) \, dt}{(x-t)^2 + y^2}.
\]

Then \( u \) is harmonic on \( \Pi^+ \) and has non-tangential limits agreeing with \( \varphi \) at almost every point of \( \mathbb{R} \). Let \( v \) be the harmonic conjugate of \( u \), and set \( f = \exp(-u - iv) \). This function \( f \) has the required properties.

To conclude the proof that \( L^1_\varphi \) is regular if \( \varphi \) satisfies condition (2), take \( y_0 \in (a,b) \subset \mathbb{R} \). Construct a function \( f_0 \) which is analytic and bounded in \( \Pi^+ \) and which is such that

\[
|f_0(x)| < \frac{e^{-\varphi(x)}}{1 + x^2} \quad (x \in \mathbb{R}).
\]

Let \( f_1(z) = f_0(z)/(z+i) \), so that \( f_1|\mathbb{R} \in L^1_\varphi \). Also, \( |f_1(z)| \to 0 \) as \( z \to \infty \) in \( \Pi^+ \), and so \( f_1(iy) = 0 \) for \( y \leq 0 \). We can clearly choose \( \alpha \in \mathbb{R} \) so that, if \( g_1(x) = f_1(x)e^{i\alpha x} \), then \( \hat{g}_1(iy_0) \neq 0 \) and \( \hat{g}_1(iy) = 0 \) \( (y < a) \). Similarly, there exists \( g_2 \in L^1_\varphi \) with \( \hat{g}_2(iy_0) \neq 0 \) and \( \hat{g}_2(iy) = 0 \) \( (y > b) \). If \( h = g_1 \ast g_2 \), then \( h \in L^1_\varphi \), \( h(iy_0) \neq 0 \), and \( h(iy) = 0 \) \( (y \notin (a,b)) \). This shows that \( L^1_\varphi \) is regular.

In fact, the Banach algebra \( L^1_\varphi \) is regular if and only if condition (2)
holds. The strongest result of this type is the famous theorem of Beurling
and Malliavin [2] which shows that, if $\varphi$ is a non-negative, measurable
function on $\mathbb{R}$, then the following two conditions on $\varphi$ are equivalent:

(i) for each $a > 0$, the Banach space $L^1_\varphi$ contains a non-zero element
whose Fourier transform has support in $[-ia,ia]$;

(ii) $\varphi$ satisfies (2) and the condition that

$$\text{ess sup} \{ |\varphi(s+t) - \varphi(s)| : s \in \mathbb{R} \} < \infty \quad (t \in \mathbb{R}).$$

Let $\varphi$ be a function satisfying (1), and let $\alpha$ and $\beta$ be the limits
defined above. The algebra $L^1_\varphi$ is termed analytic if $\beta > \alpha$. If $\alpha = \beta = 0$,
then $L^1_\varphi$ is quasi-analytic if the integral in (2) diverges, and $L^1_\varphi$ is non-quasi-
analytic if condition (2) holds. Thus, our theorem is that spectral analysis
holds in the non-quasi-analytic case.

In fact, spectral analysis fails in both the analytic and in the quasi-
analytic cases. This was first proved by Vretblad in [11] provided that $\varphi$
satisfies some slight extra conditions. We are grateful to Professor Yngve
Domar for pointing out that the proof of Theorem 4 in [4] implicitly shows
this result without any extra conditions on $\varphi$. Thus, spectral analysis
holds for the Beurling algebra $L^1_\varphi$ if and only if $\varphi$ satisfies condition (2).

In the special case that $\varphi(t) = a|t|$ for a positive constant $a$, the
family of all proper closed ideals of $L^1_\varphi$ which are not contained in any
maximal modular ideal was described by Korenblum ([9]). The family does
not seem to have been fully described in more general cases: see [7] and

2. The proof.

**Theorem.** — Let $\varphi$ be a non-negative, measurable function on $\mathbb{R}$ which
satisfies (1) and (2). Then spectral analysis holds for the Banach algebra $L^1_\varphi$.

The proof of this theorem depends heavily on a recent result given in
[8] which we first describe. We write $\Delta$ for the open unit disc, and, for each
$\sigma \in \mathbb{R}$, we write $\Pi_\sigma$ for the open right half-plane $\{(x,y) : x > \sigma\}$.

**Lemma 1.** — Let $k$ be a positive, continuous, increasing function on $[0,1)$.
Let $f$ be analytic on $\Delta$ and satisfy the condition that

$$\log |f(re^{i\theta})| \leq k(r) \quad (re^{i\theta} \in \Delta).$$
If
\[(4) \quad \int_0^1 \left( \frac{k(r)}{1 - r} \right)^2 \, dr < \infty, \]
then either \( f = 0 \), or \( \limsup_{r \to 1} (1 - r) \log |f(r)| > -\infty \).

Proof. — Theorem 5 of [8] shows that, under the hypotheses (3) and (4), there exists an analytic function \( g \) on \( \Delta \) such that :

(i) \( g \) is real and increasing on \([0,1)\), with \( g(r) \to 1 \) as \( r \to 1 \) ;
(ii) \( g(\Delta) \subset \Delta \);
(iii) \( \sup \{|1 - g(r)|/|1 - r| : r \in [0,1]\} < \infty \);
(iv) \( f \circ g \) has bounded (Nevanlinna) characteristic in \( \Delta \).

It follows from (ii) and (iii) by the theory of the angular derivative that
\[(5) \quad \lim_{r \to 1^-} \frac{1 - g(r)}{1 - r} \exists \text{ in } (0,\infty). \]
(The existence of this limit can also be seen from the explicit construction of \( g \) in [8], pp. 192-193.)

Suppose that \( f \neq 0 \). By (iv), there exist bounded, non-zero, analytic functions, say \( h_1 \) and \( h_2 \), on \( \Delta \) such that \( f \circ g = h_1/h_2 \) on \( \Delta \). If \( \limsup_{r \to 1^-} (1 - r) \log |(f \circ g)(r)| = -\infty \), then \( \limsup_{r \to 1^-} (1 - r) \log |h_1(r)| = -\infty \), and so, by a result of Phragmén-Lindelöf type ([3], 1.4.3, transferred from \( \Pi_0 \) to \( \Delta \)), \( h_1 = 0 \), a contradiction. It follows that \( \limsup_{r \to 1^-} (1 - r) \log |(f \circ g)(r)| > -\infty \).

The lemma follows from the existence of the finite non-zero limit given by (5).

Condition (4) in the above lemma is necessary in the sense that, if the integral in (4) diverges, then there exists a non-zero analytic function \( f \) on \( \Delta \) satisfying (3) and such that \( (1 - r) \log |f(r)| \to -\infty \) as \( r \to 1 \) : see [8], Theorem 4.

We transform this result to the half-plane \( \Pi_1 \). Throughout, if \( K \) is a positive, continuous function on \([1,\infty)\), we set
\[ J(K) = \int_1^\infty \left( \frac{K(R)}{R^3} \right)^2 \, dR. \]
LEMMA 2. — Let $K$ be a positive, continuous, increasing function on $[1, \infty)$ such that $\int K < \infty$.

Let $F$ be analytic on $\Pi_1$, and let $F$ satisfy the condition that

$$\log |F(\rho e^{i\psi})| \leq K\left(\frac{\rho}{\cos \psi}\right) \quad (\rho e^{i\psi} \in \Pi_1).$$

Then either $F = 0$, or $\limsup_{\rho \to \infty} \rho^{-1} \log |F(\rho)| > -\infty$.

Proof. — Let $\zeta = \xi + i\eta = \rho e^{i\psi}$ belong to $\Pi_1$, and let $z = (\zeta - 3)/((\zeta + 1)$ define a conformal map of $\Pi_1$ onto $\Delta$. Then $\zeta = (3 + z)/(1 - z)$. Let $f(z) = F(\xi)$, so that $f$ is an analytic function on $\Delta$. If $|z| = r < 1$, then

$$r^2 = \frac{|\zeta - 3|^2}{\zeta + 1} = 1 - \frac{8(\xi - 1)}{(\xi + 1)^2 + \eta^2} > 1 - \frac{8\xi}{\xi^2 + \eta^2},$$

so that

$$\frac{\rho}{\cos \psi} = \frac{\xi^2 + \eta^2}{\xi} < \frac{8}{1 - r^2} < \frac{8}{1 - r}.$$

Hence, $\log |f(re^{i\theta})| \leq k(r)$ for $re^{i\theta} \in \Delta$, where

$$k(r) = K\left(\frac{8}{1 - r}\right).$$

Then $k$ is a positive, continuous, increasing function on $[0,1)$, and

$$\int_0^1 \left(\frac{k(r)}{1 - r}\right)^{\frac{1}{2}} dr = 8^{\frac{1}{2}} \int_0^\infty \left(\frac{K(R)}{R^3}\right)^{\frac{1}{2}} dR,$$

and so $k$ satisfies condition (4). By Lemma 1, either $f = 0$ or $\limsup_{r \to 1-} (1 - r) \log |f(r)| > -\infty$. In the former case, $F = 0$, and in the latter case, $\limsup_{\rho \to \infty} \rho^{-1} \log |F(\rho)| > -\infty$, as required.

If $F$ is an analytic function on $\Pi_0$ such that $\sup \{\exp (-|z|^\alpha)|F(z)|\} < \infty$ for some $\alpha < 1$, then, by applying Lemma 2 with $K(R) = R^2$, we can deduce that either $F = 0$, or
lim sup $\rho^{-1} \log |F(\rho)| > -\infty$. This is Corollary 2.2 of [5], and the theorem of Esterle followed from that Corollary. The present more general result will require the stronger Lemma 2.

Now, following [5], we introduce the functions $a^\zeta$:

$$a^\zeta(t) = \frac{1}{\sqrt{\pi \zeta}} \exp \left( -\frac{t^2}{\zeta} \right) \ (\zeta \in \Pi_0, \ t \in \mathbb{R}).$$

Since $\varphi(t) = O(|t|)$ as $|t| \to \infty$, we have $a^\zeta \in L^1_\varphi$ for each $\zeta \in \Pi_0$. It is well known and straightforward to check that the map $\zeta \mapsto a^\zeta$, $\Pi_0 \to L^1_\varphi$, is a semigroup monomorphism and an analytic map. We must calculate $\|a^\zeta\|$ in $L^1_\varphi$. We first give a technical lemma.

**Lemma 3.** Let $\varphi$ be a non-negative, measurable function on $\mathbb{R}$ satisfying (1) and such that $\int_0^\infty (1+t^2)^{-1} \varphi(t) \, dt < \infty$.

(i) If $\varphi_1(t) = \max \{ \varphi(s) : 0 \leq s \leq t \}$ $(t \in \mathbb{R}^+)$, then $\varphi_1$ is monotone increasing on $\mathbb{R}^+$, $\varphi_1(t) \geq \varphi(t)$ $(t \in \mathbb{R}^+)$, and $\int_1^\infty t^{-2} \varphi_1(t) \, dt < \infty$.

(ii) If $\varphi_2(t) = t \max \{ s^{-1} \varphi_1(s) : s \geq t \}$ $(t \in \mathbb{R}^+)$, then $t^{-2} \varphi_2(t)$ is a monotone decreasing function of $t$ on $\mathbb{R}^+$, $\varphi_2(t) \geq \varphi_1(t)$ $(t \in \mathbb{R}^+)$, and $\int_1^\infty t^{-2} \varphi_2(t) \, dt < \infty$.

**Proof.** These results are obvious or are proved clearly in Lemmas 3.3 and 3.4 of [7]; they are originally due to Beurling.

**Lemma 4.** Let $\varphi$ be a non-negative, measurable function on $\mathbb{R}$ satisfying (1) and (2). Then there exists a positive, continuous, increasing function $K$ on $[1, \infty)$ with $\int K < \infty$ such that

(7) $$\log \|a^\zeta\| \leq K \left( \frac{\rho}{\cos \psi} \right) \ (\zeta = \rho e^{i\psi} \in \Pi_1).$$

Here, $\|a^\zeta\|$ is calculated in $L^1_\varphi$.

**Proof.** Let $\zeta = \rho e^{i\psi} \in \Pi_1$. We have

$$\|a^\zeta\| = \frac{1}{\sqrt{\pi \rho}} \int_{-\infty}^{\infty} \exp \left( -\frac{t^2}{\rho} \cos \psi + \varphi(t) \right) dt.$$
Since $\rho \geq 1$,
\[
||a^2|| \leq \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left( -\frac{t^2}{R} + \varphi(t) \right) dt
\]
\[
= \exp K(R), \text{ say,}
\]
where $R = \rho/\cos \psi \geq 1$. Clearly, replacing $K$ by $\sup \{K, 0\}$, we can suppose that $K$ is positive, continuous, and increasing on $[1, \infty)$. To show that $J(K) < \infty$, it suffices to show that $J(\log^+ \kappa) < \infty$, where
\[
\kappa(R) = \int_{0}^{\infty} \exp \left( -\frac{t^2}{R} + \varphi(t) \right) dt = R^{\frac{1}{2}} \int_{0}^{\infty} \exp \left( -s^2 + \varphi(R^2 s) \right) ds.
\]
Let $\varphi_1$ and $\varphi_2$ be as specified in Lemma 3. We can suppose that $\varphi_2(1) = 1$. For each $R \geq 1$, let
\[
\mu(R) = \sup \{t : 2\varphi_2(t)R \geq t^2\}, \quad \nu(R) = R^{-\frac{1}{2}} \mu(R).
\]
Then $\nu(R)$ is the supremum of the solutions of the inequality $\varphi_2(R^2 s) \geq \frac{1}{2} s^2$. Since $\varphi(t) = O(t)$ as $t \to \infty$, $\mu(R) = O(R)$ as $R \to \infty$.

If $s \geq \nu(R)$, then $\varphi(R^2 s) \leq \varphi_2(R^2 s) \leq \frac{1}{2} s^2$, and so
\[
\int_{\nu(R)}^{\infty} \exp \left( -s^2 + \varphi(R^2 s) \right) ds \leq \int_{0}^{\infty} \exp \left( -\frac{1}{2} s^2 \right) ds < \infty.
\]

If $s \leq \nu(R)$, then $\varphi(R^2 s) \leq \varphi_1(R^2 s) \leq \varphi_1(\mu(R)) \leq \varphi_2(\mu(R)) \leq \frac{1}{2} R^{-1}(\mu(R))^2$, and so
\[
\int_{0}^{\nu(R)} \exp \left( -s^2 + \varphi(R^2 s) \right) ds \leq R^{-\frac{1}{2}} \mu(R) \exp \left[ \frac{(\mu(R))^2}{2R} \right].
\]

Thus, $\log \kappa(R) \leq \frac{1}{2} R^{-1}(\mu(R))^2 + O(\log R)$ as $R \to \infty$, and so
\[
J(\log^+ \kappa) \leq \int_{1}^{\infty} \frac{\mu(R)}{R^2} dR + O(1) \text{ as } R \to \infty.
\]
Using the definition of $\mu(R)$ and Lemma 3, we see that
\[ \int_1^\infty \frac{\mu(R)}{R^2} dR - 1 = \int_1^\infty \frac{d\mu(R)}{R} = 2 \int_1^\infty \frac{\varphi_2(t)}{t^2} dt < \infty. \]
Thus, $J(\log^+ \kappa) < \infty$, as required.

**Lemma 5.** — If $A$ is a radical Banach algebra, and if $(a^t)$ is a continuous semigroup in $A$ over $R^+$, then $\lim_{t \to \infty} t^{-1} \log ||a^t|| = -\infty$.

**Proof.** — This is [5], Lemma 2.3.

We now conclude the proof of the theorem.

Let $I$ be a closed ideal of $L_\phi^1$. We must show that, if $I$ is not contained in a maximal modular ideal of $L_\phi^1$, then $I = L_\phi^1$. Let $A = L_\phi^1/I$. Then the hypothesis is that $A$ is a radical Banach algebra.

Let $(a^t)$ be the analytic semigroup in $L_\phi^1$ given above, and let $[a^t]$ be the coset of $a^t$ in $A$. Let $\lambda \in A'$, the dual space of $A$, and set
\[ \Phi(\zeta) = \langle [a^t], \lambda \rangle \quad (\zeta \in \Pi_0). \]
Then $\Phi$ is an analytic function over $\Pi_0$, and
\[ |\Phi(\zeta)| \leq ||\lambda|| ||[a^t]|| \leq ||\lambda|| ||a^t|| \quad (\zeta \in \Pi_0). \]

By Lemma 4, there is a function $K$ such that $J(K) < \infty$ and such that $\log |\Phi(\zeta)| \leq K(R)$ for $\zeta \in \Pi_1$, where $\zeta = \rho e^{i\psi}$ and $R = \rho/\cos \psi$. By Lemma 5, $\lim_{\rho \to \infty} \rho^{-1} \log |\Phi(\rho)| = -\infty$, and so, by Lemma 2, $\Phi = 0$. This shows that $[a^t] = 0$ in $A$, and hence that $a^t \in I$ for $\zeta \in \Pi_0$. However, for each $f \in L_\phi^1$, $f = \lim_{\rho \to 0^+} f \ast a^\rho$, and so $f \in I$. Thus $I = L_\phi^1$, as required.

The use of Lemma 2 in the above theorem seems to be necessary. For example, consider the case that $\varphi(t) = |t|^\beta$, where $0 < \beta < 1$, and take $(a^t)$ as above. Then the best estimate of $||a^t||$ in terms of $\rho = |\zeta|$ which we can obtain is that $\log ||a^t|| = O(\rho^{2\beta/(2-\beta)})$ as $\rho \to \infty$ with $\zeta \in \Pi_1$: here we are using the fact that $1/\cos \theta \leq \rho$ for $\zeta \in \Pi_1$. We can thus apply [5], Corollary 2.2, only if $2\beta/(2-\beta) < 1$, that is, if $\beta < 2/3$, whereas the result holds if $\beta < 1$. 


BIBLIOGRAPHY


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