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TRANSITIVE RIEMANNIAN ISOMETRY GROUPS WITH NILPOTENT RADICALS

by Carolyn GORDON

1. Introduction.

This paper addresses the problem of describing the full isometry group $\text{I}(M)$ of a homogeneous Riemannian manifold $M$ in terms of a given connected transitive subgroup $G$. This problem has been investigated by several authors in case $G$ is compact — see in particular Oniščik [6] and Ozeki [7] — and by the present author [3] for $G$ semisimple or at least reductive with compact radical. Less is known for solvable $G$, although Wilson [8] has recently established the normality of $G$ in $\text{I}(M)$ when $G$ is nilpotent. In this contribution, we utilize these results on compact, semisimple, and nilpotent groups to study the case in which $G$ is any connected Lie group with nilpotent radical. We will restrict our attention to $\text{I}_0(M)$, the identity component of $\text{I}(M)$.

We reformulate the problem in a slightly more general context. For $G$ and $M$ as above, $\text{I}_0(M)$ is the product $\text{I}_0(M) = GL$ of $G$ with the isotropy subgroup $L$ at a point of $M$. $L$ is compact and contains no normal subgroups of $\text{I}_0(M)$. We will describe all connected Lie groups of the form $A = GL$, $G$ connected with nilpotent radical and $L$ compact, omitting the latter condition on $L$.

The main results appear in Sections 2 and 3. In Section 2 we describe the Levi factors of $A$, establishing that the noncompact parts of suitable Levi factors of $G$ and $A$ coincide. A weaker relationship is obtained between the compact parts. We then examine in Section 3 the structure of the Lie algebra of $A$, paying particular attention to its radical.

Section 4 extends these results in case $G \cap L$ is trivial. In terms of our original problem, this is the case of a simply transitive isometry action of $G$.

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on a manifold $M$. Finally as a consequence of the results of Sections 2 and 3, we note in Section 5 a sufficient condition on the structure of $G$ to insure normality of $G$ in $A$.

2. Description of the Levi factors.

**Notation (2.1).** — Given connected Lie groups $A$ and $G$ with $G \subset A$, choose Levi factors $G_{ss}$ and $A_{ss}$ of $G$ and $A$ with $G_{ss} \subset A_{ss}$ (see Jacobson [5], pp. 91-93). Denote by $a$, $g$, $a_{ss}$, and $g_{ss}$ the Lie algebras of $A$, $G$, $A_{ss}$, and $G_{ss}$, respectively. Write

$$a_{ss} = a_{nc} \oplus a_c \text{ and } g_{ss} = g_{nc} \oplus g_c$$

where $a_{nc}$ and $g_{nc}$ are semisimple of the noncompact type, i.e., all simple ideals of $a_{nc}$ and $g_{nc}$ are noncompact, and $a_c$ and $g_c$ are compact. Let $A_{nc}$, $A_c$, $G_{nc}$, and $G_c$ be the connected subgroups of $A$ with Lie algebras $a_{nc}$, $a_c$, $g_{nc}$, and $g_c$. We have Levi decompositions

$$A = (A_{nc})(\text{rad}(A)) \text{ and } G = (G_{ss})(\text{rad}(G))$$

with $A_{ss} = A_{nc}A_c$ and $G_{ss} = G_{nc}G_c$.

**Theorem (2.2).** — Let the connected Lie group $A$ be a product $A = GL$ of a connected subgroup $G$ with nilpotent radical and a compact subgroup $L$. Then in the notation (2.1), $A_{nc} = G_{nc}$.

**Proof.** — We need only show that $a_{nc} = g_{nc}$. Let

$$\pi_{nc} : a \to a_{nc} \text{ and } \pi_c : a \to a_c$$

be the homomorphic projections relative to the decomposition

$$a = a_{nc} + a_c + \text{rad}(a).$$

$\pi_c(g_{nc}) = \{0\}$ since $a_c$ contains no noncompact semisimple subalgebras, so $g_{nc} \subset a_{nc}$.

Let $A' = A/(A_c \text{rad}(A))$ and let $\pi : A \to A'$ be the natural projection. For any subgroup $H$ of $A$, we will denote $\pi(H)$ by $H'$. The Lie algebra of $A'$ may be identified with $a_{nc}$ and the differential $(d\pi)_e$ with $\pi_{nc}$. $G'_{nc}$ then has Lie algebra $g_{nc}$. Letting $N = \text{rad}(G)$,

$$G' = G'_{nc}G_cN'$$

with $N'$ nilpotent, and $A' = G'L'$. 
Modding out a discrete normal subgroup if necessary, we may assume $A'$ has finite center. Let $U'$ be a maximal compact subgroup of $A'$ containing $G'$. A conjugate of $L'$ lies in $U'$, so

$$A' = G'U' = (G_{nc}'N')U'$$

by (1). Under a left-invariant Riemannian metric, $A'/U'$ is a symmetric space of non-positive sectional curvature with no Euclidean factor (see Helgason [4], pp. 241-253) on which $G_{nc}'N'$ acts transitively and effectively by isometries. We now use the characterization by Azencott and Wilson of isometry groups transitive on manifolds of non-positive sectional curvature. By [1], Proposition (2.5), given any Iwasawa subgroup $S_1$ of $G_N$, there exists a closed subgroup $S_2$ of $N'$, normal in $G_{nc}'N'$, such that $S_1S_2$ is a closed simply-connected solvable subgroup of $A'$ acting simply transitively on $A'/U'$. The Lie algebra $g_{nc} + s_2'$ of $G_{nc}'S_2'$ is a «basic isometry algebra» (see [2], pp. 27-29), so Theorem (4.6) and Proposition (5.3), part (i), of [2] together contradict the nilpotency of $s_2'$, unless $s_2' = \{0\}$. Hence $S_1$ and consequently $G_{nc}'$ act transitively on $A'/U'$, and $A' = G_{nc}'U'$. Since both $A'$ and $G_{nc}'$ are semisimple of the noncompact type, $A' = G_{nc}'$ ([3], Proposition (3.3)) and $a_{nc} = g_{nc}'$. □

We now describe $a_c$. For $L_{ss}$ the (unique) Levi factor of $L$, $hL_{ss}h^{-1} \subset A_{ss}$ for some $h \in A$. Note that $A = G(hLh^{-1})$, so there is no loss of generality in assuming that $L_{ss} \subset A_{ss}$.

**Notation (2.3).** — If $u$ is a compact Lie algebra, the unique Levi factor $[u,u]$ of $u$ will be denoted $u_{ss}$.

**Proposition (2.4).** — Let the connected Lie group $A$ be a product $A = GL$ of a connected subgroup $G$ with nilpotent radical and a compact subgroup $L$ with Lie algebra denoted by $l$. Using notation (2.1) and (2.3),

$$a_c = g_c + \pi_c(l_{ss})$$

where $\pi_c : a \rightarrow a_c$ is the projection along $a_{nc} + \text{rad}(a)$.

Replacing $L$ by a conjugate so that $l_{ss} \subset a_{ss}$,

$$a_{ss} = g_{ss} + l_{ss}.$$

**Proof.** — Since $a_c = \pi_c(g) + \pi_c(l)$ and $a_c$ is compact and semisimple, we have

$$a_c = (\pi_c(g))_{ss} + (\pi_c(l))_{ss}.$$
(see Oniščik [6], Theorem (1.1)).

\[ [g_c, a_w] = \{0\} \] by Theorem (2.2), so
\[ g_c \subset a_c \quad \text{and} \quad \pi_c(g) = g_c + \pi_c(\text{rad}(g)). \]

\( \pi_c(\text{rad}(g)) \) is a solvable ideal in the compact algebra \( \pi_c(g) \), hence is central. Thus \( (\pi_c(g))_{ss} = g_c \) and (4) now implies (2). (3) follows from (2) and Theorem (2.2). \( \square \)

We note that the work of Oniščik [6] on decompositions of compact Lie algebras may be applied to (2) to further analyze \( a_c \).

3. Description of the radical.

**Theorem (3.1).** — Let the connected Lie group \( A \) be a product \( A = GL \) of a connected subgroup \( G \) and a compact subgroup \( L \), and suppose the radical of \( G \) is nilpotent. We use notation (2.1) and denote the radicals of \( a \) and \( g \) by \( s \) and \( n \), respectively. Then :

(a) \( n \) is the sum of ideals \( n = n_1 \oplus n_2 \) where \( n_1 : = n \cap a_{ss} \) is central in \( g \) and \( [g, n] \subset n_2 \).

(b) \( s \) is a vector space direct sum \( s = u + n'_2 \) of an abelian subalgebra \( u \), compactly imbedded in \( a \), and an ideal \( n'_2 \) containing \([g, n] \).

(c) \([a, s] \subset n'_2 \) and \([g_{ss}, s] = [g_{ss}, n]\).

(d) There exists an isomorphism

\[ \psi : g_{ss} + n_1 + n'_2 \rightarrow g \]

which maps \( n'_2 \) onto \( n_2 \) and restricts to the identity map on \([g, g] + n_1 \).

**Remarks (3.2).** — (1) \( n_1 \) is in general non-trivial. For example, the unitary group \( G = U(n) \) acts transitively on the sphere \( \text{SO}(2n)/\text{SO}(2n-1) \). \( U(n) \) has non-trivial radical whereas \( A = \text{SO}(2n) \) is semisimple. Hence \( n_1 = n \neq \{0\} \).

Theorems (2.2) and (3.1) imply \( g_{nc} \oplus n'_2 \) is an \( a \)-ideal isomorphic to \( g_{nc} \oplus n_2 \). Thus one might also ask whether \( n_1 \) can be non-zero when \( g_c = \{0\} \). The answer is again yes. Let \( H \) be a connected semisimple Lie group of the noncompact type containing a connected compact semisimple subgroup \( K \). Set

\[ A = H \times K \]
\[ G = H \times N \]

\[ g_{nc} \oplus n'_2 \]
where \( N \) is a non-trivial connected abelian subgroup of \( K \), and
\[
L = \{(h,h) \in A : h \in K\}.
\]
Then \( G \) is transitive on \( A/L \) and again \( n_1 = n \neq \{0\} \).

(2) By part (b), \( n'_2 = n_2 \) in case \([g,n] = n\). However, in the proof of Proposition (5.2), we will construct a class of examples in which \( n'_2 \neq n_2 \).

**Proof of Theorem (3.1).** — The center of a Lie algebra \( h \) will be denoted \( z(h) \). We will make frequent use of the fact that if \( u \) is a compactly imbedded subalgebra of \( a \), then the operators \( \text{ad}_X \), \( X \in u \), are all skew-symmetric relative to some inner product on \( a \) and are consequently semisimple.

Let
\[
P : a \to a_{ss} \quad \text{and} \quad Q : a \to s
\]
be the projections relative to the Levi decomposition \( a = a_{ss} + s \). \( P = \pi_{nc} + \pi_c \) where as before \( \pi_{nc} : a \to a_{nc} \) and \( \pi_c : a \to a_c \) are the projections relative to \( a = a_{nc} + a_c + s \). By Theorem (2.2), \( a_{nc} = g_{nc} \), so \( \pi_{nc}(n) = \{0\} \) and \( P(n) = \pi_c(n) \). In particular, \( n_1 = n \cap a_{ss} \subseteq a_c \) and \( \text{ad}_{g_{n_1}} \) consists of semisimple operators. Hence the elements of \( \text{ad}_{g_{n_1}} \) are semisimple as well as nilpotent, i.e. \( n_1 \subset z(g) \). Moreover
\[
(1) \quad P([g,n]) = [P(g), P(n)] = [P(g), \pi_c(n)] = \{0\},
\]
the last equality following from the proof of Proposition (2.4), so \( n_1 \cap [g,n] = \{0\} \). Letting \( n_2 \) denote any complement of \( n_1 \) in \( n \) which contains \([g,n] \); (a) follows.

Let
\[
g_{nc} = f + p
\]
be a Cartan decomposition with \( f \) compactly imbedded in \( g \). Since the connected subgroup of \( \text{Int}(a) \) with Lie algebra \( \text{ad}_{g_{nc}} \) is a semisimple matrix group, it has finite center and hence \( f \) is compactly imbedded in \( a \) (see Helgason [4], pp. 252-253). \( f + a_c \) lies in a maximal compactly imbedded subalgebra \( w \) of \( a \). \( P(w) = f + a_c \), \( f + a_c \) being maximal compact in \( a_{ss} \), so \( w = (f + a_c) + (w \cap s) \) with \( (w \cap s) \subset z(w) \). After replacing \( L \) by a conjugate subgroup of \( A \), we may assume that \( f \subseteq w \). Thus \( a = w + g \) and \( s = (w \cap s) + Q(n) \). Let \( u \) be a complement of
\( \omega \cap Q(\pi) \) in \( \omega \cap s \) and set
\[
(2) \quad v = u + l + a_c.
\]
Note that \( u \subset z(v) \). We have vector space direct sums
\[
(3) \quad a = v + p + n_2 \quad \text{and} \quad s = u + Q(n_2).
\]
Denote by \( s_0 \) the 0-eigenspace in \( s \) of \( \text{ad}_a v \). Since \( v \) lies in the compactly imbedded subalgebra \( \omega \), \( s = s_0 + [v, s] \).
\[
s_0 = u + (s_0 \cap Q(n_2)). \quad \text{Set}
\]
\[
(4) \quad n'_2 = [v, s] + (s_0 \cap Q(n_2)).
\]
Then \( s = u + n'_2 \) and \( v \cap n'_2 = \{0\} \).
\[
P(n_2) \subset a_c \subset v, \quad \text{so (2) and (3) imply } s \subset n_2 + v \quad \text{with} \quad n_2 \cap v = \{0\}. \quad \text{For } X \in s, \quad \text{write}
\]
\[
X = X_p + X_n \quad X_p \in v, \quad X_n \in n_2.
\]
Claim. – For \( X \in n'_2, \quad [X_v, s] = \{0\} \).
\[
\text{For } H \in v, \quad Y \in n_2, \quad \text{write}
\]
\[
[H, Y] = \rho(H)Y - \phi(Y)(H), \quad \rho(H)Y \in n_2, \quad \phi(Y)H \in v.
\]
To prove the claim, it suffices to show that \( \rho(X_p) = 0 \), since then
\[
[X_v, s] \subset v \cap [v, s] \subset v \cap n'_2 = \{0\}.
\]
Let \( v_0 \) be the maximal \( (v + n_2) \) - ideal in \( v \) and
\[
\pi : v + n_2 \to (v + n_2)/v_0
\]
the projection. \( \pi(n_2) \) is nilpotent, \( \pi(v) \) contains no ideals of \( \pi(v + n_2) \) and \( \pi(n_2) \cap \pi(v) = \{0\} \). Hence (Wilson [8]), \( \pi(n_2) \) is an ideal in \( \pi(v + n_2) \). i.e. for \( Y \in n_2, \quad \phi(Y)(v) \subset v_0 \) and
\[
(5) \quad \rho(\phi(Y)H) = 0, \quad H \in v, \quad Y \in n_2.
\]
We suppose first that \( X \in s_0 \cap Q(n_2) \). Since \( X \in s_0, \quad [a_c, X] = \{0\} \) and for \( H \in a_c, \)
\[
0 = [H, X]_v = [H, X_n] - \phi(X_n)H.
\]
Thus by (5)
\[
\rho([H, X_n]) = \{0\}, \quad H \in a_c.
\]
But \( X_0 = -\mathcal{P}(X_0) \in \mathfrak{a}_c \) since \( X \in \mathbf{Q} \mathbf{O} (n_2) \). Noting that \( \ker \rho|_{\mathfrak{a}_c} \) is an ideal in the semisimple algebra \( \mathfrak{a}_c \), it follows that \( \rho(X_0) = 0 \).

Now let \( \nu_1 = \{ Y_0 : Y \in [\nu, s] \} \). Then

\[
[\nu, s] = [\nu_1, s] + \{ Y \in [\nu, s] : [\nu_1, Y] = \{0\} \}.
\]

Suppose \( X = [H, Y] \) for some \( H \in \nu_1 \), \( Y \in s \). Then

\[
X_0 = -\varphi(Y_0)H + [H, Y_0].
\]

\( \nu_1 \subset \mathcal{P}(n_2) + u \) by (3), \( \mathcal{P}(n_2) \) is abelian by (1), and \( u \subset z(\nu) \); hence \( \nu_1 \) is abelian and \( [H, Y_0] = \{0\} \). Thus by (5), \( \rho(X_0) = 0 \).

In view of (4) and (6) it remains only to check the case \( X \in [\nu, s] \) while \( [\nu_1, s] = \{0\} \). Since \( [\nu, s] \) is contained in the nil radical of \( \mathfrak{a} \) (see Jacobson [5], p. 51), \( \text{ad}_\mathfrak{a} X \) is nilpotent. \( X_0 \in \nu_1 \), so \( [X_0, X] = \{0\} \) and consequently \( [X_n, X] = \{0\} \). Thus if we show that \( \text{ad}_{X_0}|\mathfrak{n}_2 \) is nilpotent, it will follow that \( \text{ad}_{X_0}|\mathfrak{n}_2 \) is nilpotent. Noting that \( \text{ad}_{X_0}|\mathfrak{n}_2 \) is also semisimple since \( X_0 \in \mathfrak{w} \), the claim will be established.

For \( Y \in s \),

\[
[X_n, Y] = [X_n, Y]_{n_2} + \varphi(X_0)Y_0.
\]

Setting \( Z = [X_n, Y]_{n_2} \), (5) and (7) inductively imply

\[
(\text{ad}_{X_0})^m(Y) = (\text{ad}_{n_2} X_n)^m(Z) + (\varphi(X_0))^m(Y_0).
\]

Since \( n_2 \) is nilpotent, \( (\text{ad}_{n_2} X_n)^k = 0 \) for some \( k \), so

\[
(\text{ad}_{X_0})^k(Y) \in \nu \cap \text{nil rad}(\mathfrak{a}).
\]

But \( \nu \cap \text{nil rad}(\mathfrak{a}) \subset z(\mathfrak{a}) \) since \( \nu \) lies in a compactly imbedded subalgebra of \( \mathfrak{a} \), so \( (\text{ad}_{X_0})^{k+1} = 0 \), i.e. \( \text{ad}_{X_0}|\mathfrak{n}_2 \) is nilpotent. As noted above, the claim follows.

The claim implies

\[
[X, Y] = [X_n, Y_n], \quad X, Y \in n_2^\prime.
\]

Since \( s = u + n_2^\prime \) and \( Q|_{n_2} \) is \( 1 \times 1 \), \( \{ X_n : X \in n_2^\prime \} = n_2 \). Thus (8) and part (a) together imply

\[
[n_2^\prime, n_2^\prime] = [u, n].
\]
\([v,[n, n]] \subset n_2\) by (4), so by (9)

\[(10) \quad [v,[n, n]] \subset [n, n].\]

For \(X \in n_2\), \([g, X] \subset s\) by (1) and \([g, X] \subset s\), so \([X, g] \subset s\). But \([X, g] = \{0\}\) by the claim, and \(\text{ad}_g X\) is a semisimple operator. Hence \([X, g] = \{0\}\) and

\[(11) \quad [Y, X] = [Y, X], \quad Y \in g, \quad X \in n_2.\]

In particular,

\[(12) \quad [f + g, s] \subset n\]

since \([f + g, u] \subset [v, u] = \{0\}\). Hence

\([v, u] = [[[f, p], u]] = [f, [[[p, u]]] \subset [f, s] \subset n].\]

Thus \([g, s] \subset n \cap s\). Since \(g, s\) is semisimple and \([g, n] \subset n \cap s\) by (1),

\[(13) \quad [g, s] = [g, n \cap s] = [g, n].\]

Similarly, using (12), we obtain \([g, s] = [g, n]\) and the second statement of (c) follows.

By Theorem (2.2) and (13),

\[(14) \quad [g, n] = g + [g, n].\]

Thus,

\([v, [g, n]] = [v, [g, n \cap s]] \quad \text{by (13)}\]
\(\subset [[v, g], n \cap s] + [g, [v, n]]\]
\(\subset [g + n, n \cap s] + [g, n, s]\) by (14)
\(\subset [g, s] + [n, n].\]

Define

\[(15) \quad m = [g, s] + [n, n].\]

By (10) and the above computation, \(m\) is an \(\text{ad}_n(v)\)-invariant subspace of \(n \cap s\). Therefore

\[(16) \quad m = [v, m] \subset m \cap s_0 \subset n_2.\]
by (4), so \([g_{nc}, s] \subset n'_2\) by (15). Since
\[s = u + n'_2\]
and \([u, n'_2] \subset n'_2\),
(9), (15), and (16) show that \([n'_2, s] \subset n'_2\). Noting that
\[a = v + p + n'_2,\]
we thus have \([a, s] \subset n'_2\).

Finally define \(\psi : g_{ss} + n_1 + n'_2 \rightarrow g\) by
\[\psi(Y + X) = Y + X,\quad Y \in g_{ss} + n_1,\quad X \in n'_2.\]
\(\psi\) maps \(n'_2\) injectively onto \(n_2\) and by (8) and (11), \(\psi\) is an isomorphism.

**Corollary (3.2).** Under the hypothesis and notation of Theorem (3.1), \([n, n]\) and \([g_{nc} + n, g_{nc} + n]\) are ideals of \(a\).

**Proof.** Both subalgebras are \(g\)-ideals. \(a \subset g + p\) by (3), so the corollary follows from (10), (13) and Theorem (2.2). \(\square\)

4. The simply transitive case.

Under the notation and hypotheses of Theorem (3.1), suppose that \(G \cap L\) is trivial. Then \(G\) intersects any conjugate of \(L\) trivially, so the last statement of Proposition (2.4) implies \(n \cap a_{ss} = \{0\}\), i.e. \(n_1 = 0\) and \(n = n_2 \cong n'_2\).

**Theorem (4.1).** Let the connected Lie group \(A\) be a product of disjoint subgroups \(A = GL\) with \(L\) compact and \(G\) connected with nilpotent radical. We use the notation of (2.1) and (3.1) but write \(n'\) in place of \(n'_2\). Then \(A = G'L\) where \(G'\) is a connected normal subgroup of \(A\) with Lie algebra \(g'\) satisfying :

(i) \(g' \cap l = \{0\}\);
(ii) \(g' = g_{nc} + g'_c + n'\) for some \(a_c\)-ideal \(g'_c\) isomorphic to \(g_c\);
(iii) if \([g_c, n]\) = \(\{0\}\), then \(g' \cong g\).

**Proof.** We will continue to use the notation developed in the proof of Theorem (3.1). In particular, recall the construction of the maximal compactly imbedded subalgebra \(w\) of \(a\). The conclusions of (4.1) are not
affected when \( L \) is replaced by a conjugate subgroup of \( A \), so we may assume that \( I \subset \omega \). Then \( I_{ss} \subset [\omega, \omega] \subset a_{ss} \). Proposition (2.4) and Theorem (2.2) imply that

\[
a_{ss} = g_{ss} + I_{ss}, \quad a_{c} = g_{c} + \pi_{c}(I_{ss}),
\]

and

\[
\pi_{c}(I_{ss}) \subset I_{ss} + g_{nc}.
\]

Thus \( \pi_{c}(I_{ss}) \cap g_{c} = \{0\} \) since \( g \cap I = \{0\} \). Let \( a'_{c} \) be the minimal \( a_{c} \)-ideal containing \( g_{c} \). \( a'_{c} = g_{c} + (a'_{c} \cap \pi_{c}(I_{ss})) \), a vector space direct sum, so \( a'_{c} \) contains an \( a_{c} \)-ideal \( g'_{c} \) isomorphic to \( g_{c} \) such that

\[
a'_{c} = g'_{c} + (a'_{c} \cap \pi_{c}(I_{ss})),
\]

again a vector space direct sum (Ozeki [7]). Hence \( a_{c} = g'_{c} + \pi_{c}(I_{ss}) \) and

\[
(1) \quad a_{ss} = g_{nc} + g'_{c} + I_{ss} \quad \text{(vector space direct sum)}.
\]

Letting \( g' = g_{nc} + g'_{c} + n' \), Theorems (2.2) and (3.1) imply that \( g' \) is an \( a \)-ideal.

We now show that \( a = g' + I \). Since \( a_{ss} = g_{ss} + I_{ss} \),

\[
\mathbf{s} = Q(z(I)) + Q(\mathfrak{n}). \quad Q(z(I)) \subset Q(\omega) = \omega \cap s.
\]

The subalgebra \( \mathfrak{u} \) in (3.1) was defined to be any complement of \( \omega \cap Q(\mathfrak{n}) \) in \( \omega \cap s \). We may therefore choose \( \mathfrak{u} \) so that \( \mathfrak{u} \subset Q(z(I)) \). Then by (3.1),

\[
\mathbf{s} = \mathfrak{u} + n' = Q(z(I)) + n' \subset a_{ss} + z(I) + n'.
\]

Thus by (1), \( a = g' + I \) and \( A = G'L \), where \( G' \) is the connected normal subgroup of \( A \) with Lie algebra \( g' \). Since \( g \) and \( g' \) have the same dimension, \( g' \cap I = \{0\} \).

Finally, suppose that \([g_{c}, \mathbf{n}] = \{0\}\). Then Theorem (3.1) part (c) and the semisimplicity of \( g_{c} \) imply \([g_{c}, s] = \{0\}\). Since \( a'_{c} \) is the minimal \( a_{c} \)-ideal containing \( g_{c} \), \([a'_{c}, s] = \{0\}\) and consequently \([g'_{c}, n'] = [g'_{c}, s] = \{0\}\).

Since \( g_{nc} + n' \simeq g_{nc} + n \) by (3.1), (iii) follows.

5. A condition for normality of the transitive subgroup.

**Theorem (5.1).** — Let \( M \) be a connected homogeneous Riemannian manifold and \( I_{0}(M) \) the connected component of the identity in the group of
all isometries of $M$. Suppose that $G$ is a connected transitive subgroup of $A$ with Lie algebra $\mathfrak{g}$ satisfying $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$ and that some (hence every) Levi factor of $G$ is of the noncompact type. Then $G$ is normal in $A$.

Proof. — The condition $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$ implies that the radical $n$ of $\mathfrak{g}$ is nilpotent and that $\mathfrak{g} = [\mathfrak{g}_{nc} + n, \mathfrak{g}_{nc} + n]$, where $\mathfrak{g}_{nc}$ denotes a Levi factor of $\mathfrak{g}$. Thus Corollary (3.2) applies.

The following proposition is a partial converse to Theorem (5.1).

PROPOSITION (5.2). — Suppose that $G$ is a connected simply-connected Lie group with Lie algebra $\mathfrak{g}$ satisfying $[\mathfrak{g},\mathfrak{g}] + \mathfrak{g}$ and that $G$ is not solvable. Then there exists a Riemannian manifold $M$ such that $G$ acts simply transitively by isometries on $M$ but is not normal in $I_0(M)$.

Proof. — Let $\mathfrak{f}$ be a maximal compactly imbedded subalgebra of a Levi factor of $\mathfrak{g}$ and $\mathfrak{g}_1$ a codimension one ideal of $\mathfrak{g}$ containing $[\mathfrak{g},\mathfrak{g}]$. There exists a homomorphism $\lambda_1 : \mathfrak{g} \to \mathfrak{f}$ with kernel $\mathfrak{g}_1$. Denoting by $K$ the connected subgroup of $G$ with Lie algebra $\mathfrak{f}$, the simple-connectivity of $G$ implies the existence of a homomorphism $\lambda : G \to K$ with $(d\lambda)_e = \lambda_1$. Denote the center of $G$ by $G_z$ and set

$$D = \{(h,h) \in G \times K : h \in G_z \cap K\}.$$ Let

$$A = (G \times K)/D$$

with canonical projection $\pi : G \times K \to A$ and set

$$L = \{\pi((h,h)) : h \in K\}.$$ L $\simeq K/(G_z \cap K)$, hence is compact, and $L$ contains no normal subgroups of $A$. $M : = A/L$ may be given a left-invariant Riemannian metric, and $A$ is then identified with a subgroup of $I_0(M)$. Define an imbedding $\eta : G \to A$ by $\eta(g) = \pi((g,\lambda(g))$. $\lambda(K) = \{e\}$ since $\mathfrak{f} \subset [\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g}$, so $\eta(G) \cap L$ is trivial. Under this imbedding $G$ is a simply transitive subgroup of $I_0(M)$. However $G$ is not normal in the subgroup $A$ of $I_0(M)$.

Suppose the group $G$ in (5.2) has nilpotent radical so that $A = GL$ satisfies the hypotheses of Theorem (3.1). In the notation of (3.1), $a \simeq \mathfrak{g} \oplus \mathfrak{f}$, where $\mathfrak{f}$ is the Lie algebra of $K$. However, $\mathfrak{g}$ is imbedded in
\( \alpha \) as \( \{(X, \lambda_1(X)) : X \in g\} \). \( \lambda_1 \mid_n \) is non-trivial since \( g = g_{ss} + n \) with \( g_{ss} \subset [g, g] \subset \ker \lambda_1 \). Hence \( n \) is not an \( \alpha \)-ideal. But \( n = n_2 \) since \( G \cap L = \{e\} \), so \( n_2 \) is not equal to the \( \alpha \)-ideal \( n'_2 \). (See remark (3.2).)

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