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MEASURABLE FUNCTIONALS ON FUNCTION SPACES

by J.P.R. CHRISTENSEN (*) and J.K. PACHL

1. Introduction.

All vector spaces are over the field **R** of reals. If E and F are two vector spaces in duality, we denote by $w(E,F)$ the weak topology on E. In a topological space, the Baire sigma-algebra is the one generated by continuous real-valued functions.

We are mainly interested in the following property of a dual pair of vector spaces (typically, E will be a space of signed measures and F will be a space of functions).

DEFINITION. – *Let* E *and* F *be two vector spaces in duality. Say that* E *is Baire saturated with respect to* F, *in short* F*-saturated, if every* $w(F,E)$ *Baire-measurable linear form on* F *is* $w(F,E)$ *continuous (and hence identifies with an element in* E).

Obviously, if E is F-saturated then it is $w(E,F)$ sequentially complete. The example at the end of this section shows that the converse is not true.

In section 2 we show that various spaces of measures, in duality with function spaces, are saturated. (It is known that these spaces of measures are weakly sequentially complete). We conjecture that the space of bounded (signed) Radon measures on an arbitrary complete separable metric space is Baire saturated with respect to the space of bounded uniformly continuous functions. We prove the conjecture for the spaces with the (ℓ^1) property. (Every locally compact metric group has the ((ℓ^1) property; no infinite dimensional Banach space has the (ℓ^1) property).

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In section 3 we study invariant functionals on Banach spaces, and give a sufficient condition for their measurability (Theorem 3). We show that Theorem 3 applies to the space of bounded uniformly continuous functions on a metric space.

Section 4 illustrates how the results in sections 2 and 3 can be used in the study of invariant means. Our results are parallel to those of Granirer [9].

The following example shows that a vector space E can be $w(\mathrm{E},\mathrm{F})$ sequentially complete without being F-saturated. Note, however, that E is not a space of measures in this example.

Example. – Let E be the space of bounded Baire-measurable functions on $\mathbf{R}$ and let F be the space of bounded (signed) countably additive measures on the Baire sigma-algebra in $\mathbf{R}$. The duality is $\langle g,\mu\rangle = \int g\, d\mu$ for $g \in \mathrm{E}$ and $\mu \in \mathrm{F}$.

Clearly E is $w(\mathrm{E},\mathrm{F})$ sequentially complete. To show that E is not F-saturated, define a linear form h on F as follows :

For each $\mu \in \mathrm{F}$ denote by μ_d the discrete (=atomic) part of μ, and put $h(\mu) = \mu_d(\mathbf{R})$. The form $h : \mathrm{F} \to \mathbf{R}$ is not $w(\mathrm{F},\mathrm{E})$ continuous. We show that h is $w(\mathrm{F},\mathrm{E})$ Baire-measurable.

Let Q be a countable dense subset of $\mathbf{R}$. If $s \in \mathbf{R}$ then

$$\{\mu \in \mathrm{F} \mid h(\mu^+) > s\} = \bigcup_{j=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{k_0=1}^{\infty} \bigcap_{k=k_0}^{\infty} \bigcup_{(r_1,\ldots,r_n)\in \mathrm{Q}^n} \left\{\mu \mid \mu\left(\bigcup_{i=1}^{n}\left[r_i, r_i+\frac{1}{k}\right]\right) > s+\frac{1}{j}\right\}.$$

Hence $\{\mu \mid h(\mu^+) > s\}$ is a $w(\mathrm{F},\mathrm{E})$ Baire set, and so is the set

$$\{\mu \mid h(\mu^-) < s\} = \bigcup_{j=1}^{\infty}\left[\mathrm{F}\setminus\left\{\mu \mid h((-\mu)^+) > s - \frac{1}{j}\right\}\right].$$

Finally,

$$\{\mu \in \mathrm{F} \mid h(\mu) > s\} = \bigcup_{r\in\mathrm{Q}} [\{\mu \mid h(\mu^+) > r\} \cap \{\mu \mid h(\mu^-) < r-s\}]$$

is a $w(\mathrm{F},\mathrm{E})$ Baire set for any $s \in \mathbf{R}$. This proves that h is $w(\mathrm{F},\mathrm{E})$ Baire measurable.

2. Saturated spaces of measures.

We use the following property (similar to the universal BP-measurability [1]) : Say that a real-valued function f on a topological space X is CBP-measurable if for every continuous map g from a *compact* space K into X and for every open set G of reals, the set $(f \circ g)^{-1}(G) \subseteq K$ has the Baire property (i.e. belongs to the BP-field, in the terminology of [1]). Obviously, every Baire measurable function is CBP-measurable.

Denote by $\mathbf{P}(D)$ the set of all subsets of a set D. Identify $\mathbf{P}(D)$ with the product $\{0,1\}^D$; as such, $\mathbf{P}(D)$ is a compact space. Write $\mathbf{N} = \{1,2,3,\ldots\}$.

Our starting point is the following lemma. It is essentially proved in ([1], p. 96); see also ([16], p. 159).

LEMMA 2.1. – *Let* D *be an infinite set and let* $m : \mathbf{P}(D) \to \mathbf{R}$ *be a finitely additive map. If* m *is* CBP-*measurable then*

(i) m *is countably additive;*

(ii) $m(A) = \sum_{a \in A} m(\{a\})$ *for every* $A \subseteq D$.

Proof. – Assertion (i) is proved in ([1], Th. 5.7) for the case $D = \mathbf{N}$. It follows that (i) holds for any D : If $\{A_n\}$ is a sequence of pairwise disjoint subsets of D then the map from $\mathbf{P}(\mathbf{N})$ into $\mathbf{P}(D)$ defined by $B \longmapsto \bigcup_{n \in B} A_n$, $B \subseteq \mathbf{N}$, is continuous; hence

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n)$$

by the result for $D = \mathbf{N}$.

To prove (ii), observe that the set $S = \{a \in D \mid m(\{a\}) \neq 0\}$ is countable by (i). Thus it suffices to prove that $m(B) = 0$ for every $B \subseteq D \backslash S$. To this end, we proceed as in ([1], p. 96-98), but use the topological zero-one law for arbitrary (i.e. not necessarily countable) products, due to Oxtoby ([14], Th. 4) : m is constant on a dense G_δ subset of $\mathbf{P}(D \backslash S)$, and we conclude as in [1] that m is identically 0 on $\mathbf{P}(D \backslash S)$. □

Write $g_n \nearrow f$ or $(-g_n) \searrow (-f)$ if g_n, $n = 1,2,\ldots$, and f are real-valued functions such that $g_1 \leqslant g_2 \leqslant \cdots \leqslant g_n \leqslant \cdots$ and $\lim g_n = f$ pointwise.

DEFINITIONS. — *Let* X *be a nonempty set and* F *a linear space of real-valued functions on* X. *A linear form* $\mu : \mathrm{F} \to \mathbf{R}$ *is countably additive if* $\lim \mu(f_n) = 0$ *whenever* $f_n \in \mathrm{F}$, $f_n \searrow 0$. *Say that* F *has the interpolation property if* F *contains every function* f *for which there are* $g_n \in \mathrm{F}$ *and* $h_n \in \mathrm{F}$ *such that* $g_n \nearrow f$ *and* $h_n \searrow f$.

THEOREM 1. — *Let* X *be a nonempty set and* F *a linear space of real-valued functions on* X. *Denote by* E *the space of countably additive linear forms on* F. *If* F *has the interpolation property then every* $w(\mathrm{F},\mathrm{E})$ *CBP-measurable linear form on* F *is countably additive; in particular,* E *is then* F-*saturated.*

Proof. — Take any $w(\mathrm{F},\mathrm{E})$ CBP-measurable linear form $\mu : \mathrm{F} \to \mathbf{R}$, and a sequence $f_n \searrow 0$, $f_n \in \mathrm{F}$. For every $\mathrm{A} \subseteq \mathbf{N}$ put

$$g_{\mathrm{A}} = \sum_{i \in \mathrm{A}} (f_i - f_{i+1}).$$

Since F has the interpolation property, $g_{\mathrm{A}} \in \mathrm{F}$ for each $\mathrm{A} \subseteq \mathbf{N}$. The map from $\mathbf{P}(\mathbf{N})$ into F defined by $\mathrm{A} \longmapsto g_{\mathrm{A}}$, $\mathrm{A} \subseteq \mathbf{N}$, is $w(\mathrm{F},\mathrm{E})$ continuous; hence the finitely additive map $m : \mathbf{P}(\mathbf{N}) \to \mathbf{R}$ defined by $m(\mathrm{A}) = \mu(g_{\mathrm{A}})$, $\mathrm{A} \subseteq \mathbf{N}$, is CBP-measurable. By Lemma 2.1, m is countably additive and

$$\lim \mu(f_n) = \lim m(\{n, n+1, n+2, \ldots\}) = 0. \qquad \square$$

Theorem 1 applies to several dual pairs of vector spaces arising in measure theory :

1. X is a Hausdorff completely regular topological space and $\mathrm{F} = \mathbf{C}(\mathrm{X})$ is the space of continuous functions on X; by Hewitt's theorem ([18], I-Th. 23), E is the order dual of $\mathbf{C}(\mathrm{X})$. (F has the interpolation property because if a function is both upper semicontinuous and lower semicontinuous then it is continuous.)

2. X is a Hausdorff completely regular topological space and $\mathrm{F} = \mathbf{C}_b(\mathrm{X})$ is the space of bounded continuous functions on X; E is the

space of bounded (signed) countably additive measures on X ([18], Part I-Th. 20).

3. X is a measurable space and F is the space of bounded measurable functions on X; again, E is the space of bounded (signed) countably additive measures on X.

4. $F = \ell^\infty$ and $E = \ell^1$; this, of course, is a special case of both 2. and 3. More generally, we can take a sigma-finite measure μ and $F = L^\infty(\mu)$, $E = L^1(\mu)$; strictly speaking, Theorem 1 does not apply in this case because elements of $L^\infty(\mu)$ are classes of functions rather than functions. Nevertheless, if we interpret all relations as valid almost everywhere then the proof of Theorem 1 demonstrates that $L^1(\mu)$ is $L^\infty(\mu)$-saturated. The latter result is also proved in ([1], Th. 5.8).

Unfortunately, spaces of *uniformly* continuous functions have not the interpolation property in many interesting cases. In fact, the space $U_b(X)$ of bounded uniformly continuous functions on a uniform space X has the interpolation property if and only if X is an Aleksandrov space ([4], Th. 1 on p. 38). Hence the proof of Theorem 1 does not establish the following conjecture :

CONJECTURE. — *Let* $U_b(X)$ *be the space of bounded uniformly continuous real-valued functions on a complete separable metric space* X. *Then the space of bounded (signed) Radon measures on* X *is* $U_b(X)$*-saturated.*

If the conjecture is true, it will yield another proof of the fact that $\mathbf{M}_R(X)$, the space of bounded Radon measures on X, is $w(\mathbf{M}_R(X), U_b(X))$ sequentially complete (see [15] and a short proof in [2]).

In Theorem 2 we prove that the conjecture is true if X has the (ℓ^1) property. We also prove (in Lemma 4.1) that every locally compact metric group has the (ℓ^1) property. Since Zahradník [19] proved that no infinite dimensional Banach space has the (ℓ^1) property, it would be interesting to have a proof (or a disproof) for $X = \ell^2$.

DEFINITION (adapted from [5], [19]). — *Let* X *be a metric space. A family* $\{p_\alpha\}_{\alpha \in D}$ *of real-valued functions on* X *is a partition of unity if* $p_\alpha \geqslant 0$ *for each* $\alpha \in D$ *and* $\sum_{\alpha \in D} p_\alpha(x) = 1$ *for each* $x \in X$. *A partition of unity* $\{p_\alpha\}_{\alpha \in D}$ *is* ℓ^1*-continuous if the map from* X *into the Banach space* $\ell^1(D)$ *defined by* $x \to \{p_\alpha(x)\}_{\alpha \in D}$ *is uniformly continuous. Say that* X *has the* (ℓ^1) *property if for every* $\varepsilon > 0$ *there exist a set* D *and an* ℓ^1*-continuous* *partition of unity* $\{p_\alpha\}_{\alpha\in \mathrm{D}}$ *on* X *such that*

$$\operatorname{diam}(\{x\in \mathrm{X} : p_\alpha(x)\neq 0\}) < \varepsilon$$

for each $\alpha\in \mathrm{D}$.

Let (X,d) be a metric space. The space $\mathbf{U}_b(\mathrm{X})$ of bounded real-valued uniformly continuous functions on X is complete in the norm

$$\|f\| = \sup\{|f(x)|\,|\,x\in \mathrm{X}\}.$$

THEOREM 2. — *Let* (X,d) *be a complete metric space with the* (ℓ^1) *property. Every* $w(\mathbf{U}_b(\mathrm{X}),\mathbf{M}_\mathrm{R}(\mathrm{X}))$ *CBP-measurable linear form on* $\mathbf{U}_b(\mathrm{X})$ *is a (signed) Radon measure; in particular, the space* $\mathbf{M}_\mathrm{R}(\mathrm{X})$ *is* $\mathbf{U}_b(\mathrm{X})$*-saturated.*

The proof employs several lemmas :

LEMMA 2.2. — *Let* $q : \mathrm{X}\to \ell^1(\mathrm{D})$ *be a uniformly continuous map such that the set* $q(\mathrm{X})\subseteq \ell^1(\mathrm{D})$ *is bounded. For each* $\alpha\in \mathrm{D}$ *define* $p_\alpha\in \mathbf{U}_b(\mathrm{X})$ *by* $p_\alpha(x) = q(x)(\alpha)$, $x\in \mathrm{X}$. *If* $\mu : \mathbf{U}_b(\mathrm{X})\to \mathbf{R}$ *is a* $w(\mathbf{U}_b,\mathbf{M}_\mathrm{R})$ *CBP-measurable linear form then*

$$\mu\Big(\sum_{\alpha\in \mathrm{D}} p_\alpha\Big) = \sum_{\alpha\in \mathrm{D}} \mu(p_\alpha).$$

Proof. — For every $\mathrm{A}\subseteq \mathrm{D}$ define $g_\mathrm{A} = \sum_{\alpha\in \mathrm{A}} p_\alpha$. Since q is uniformly continuous, the map from $\mathbf{P}(\mathrm{D})$ into $\mathbf{U}_b(\mathrm{X})$ defined by $\mathrm{A}\longmapsto g_\mathrm{A}$, $\mathrm{A}\subseteq \mathrm{D}$, is $w(\mathbf{U}_b,\mathbf{M}_\mathrm{R})$ continuous. Hence the finitely additive map $m : \mathbf{P}(\mathrm{D})\to \mathbf{R}$ defined by $m(\mathrm{A}) = \mu(g_\mathrm{A})$, $\mathrm{A}\subseteq \mathrm{D}$, is CBP-measurable.

By Lemma 2.1,

$$\mu\Big(\sum_{\alpha\in \mathrm{D}} p_\alpha\Big) = m(\mathrm{D}) = \sum_{\alpha\in \mathrm{D}} m(\{\alpha\}) = \sum_{\alpha\in \mathrm{D}} \mu(p_\alpha). \qquad \square$$

LEMMA 2.3. — *The lattice operations in* $\mathbf{U}_b(\mathrm{X})$ *are* $w(\mathbf{U}_b(\mathrm{X}),\mathbf{M}_\mathrm{R}(\mathrm{X}))$ *continuous on every* $w(\mathbf{U}_b(\mathrm{X}),\mathbf{M}_\mathrm{R}(\mathrm{X}))$ *compact subset of* $\mathbf{U}_b(\mathrm{X})$.

Proof. — It suffices to show that the map $f\longmapsto f^+$ is continuous on every $w(\mathbf{U}_b(\mathrm{X}),\mathbf{M}_\mathrm{R}(\mathrm{X}))$ compact set, because

$$f\vee g = f + (g-f)^+$$

and

$$f\wedge g = f - (f-g)^+.$$

First observe that the map $f \longmapsto f^+$ is $w(\mathbf{U}_b(X), \mathbf{M}_R(X))$ *sequentially* continuous on $\mathbf{U}_b(X)$; this follows from the Lebesgue dominated convergence theorem.

Now consider the particular case where X is compact. Then every $w(\mathbf{U}_b(X), \mathbf{M}_R(X))$ compact subset of $\mathbf{U}_b(X)$ is $w(\mathbf{U}_b, \mathbf{M}_R)$ metrizable. From this and from the sequential continuity of $f \longmapsto f^+$ it follows that the map is continuous on every $w(\mathbf{U}_b, \mathbf{M}_R)$ compact set.

Finally, the general situation (when X is an arbitrary complete metric space) can be reduced to the previous special case : Take any net $\{f_\alpha\}_\alpha$ contained in a $w(\mathbf{U}_b(X), \mathbf{M}_R(X))$ compact subset of $\mathbf{U}_b(X)$, such that $\lim f_\alpha = f$ in $w(\mathbf{U}_b, \mathbf{M}_R)$, $f \in \mathbf{U}_b(X)$. We want to prove that $\lim \mu(f_\alpha^+) = \mu(f^+)$ for every $\mu \in \mathbf{M}_R(X)$. For a compact set $K \subseteq X$, denote by g_α, g and μ_K the restrictions of f_α, f and μ to K. The net $\{g_\alpha\}_\alpha$ is contained in a $w(\mathbf{U}_b(K), \mathbf{M}_R(K))$ compact subset of $\mathbf{U}_b(K)$, and $\lim g_\alpha = g$ in $w(\mathbf{U}_b(K), \mathbf{M}_R(K))$; hence $\lim \mu_K(g_\alpha^+) = \mu_K(g^+)$ as proved above. Since μ is approximated by μ_K's we have $\lim \mu(f_\alpha^+) = \mu(f^+)$, q.e.d.

LEMMA 2.4. – *If* μ *is a* $w(\mathbf{U}_b(X), \mathbf{M}_R(X))$ *CBP-measurable linear form on* $\mathbf{U}_b(X)$, *then* μ *is* $\|.\|$ *continuous and both* μ^+ *and* μ^- *are* $w(\mathbf{U}_b(X), \mathbf{M}_R(X))$ *CBP-measurable.*

Proof. – Suppose that μ is not $\|.\|$ continuous : there are $f_n \in \mathbf{U}_b(X)$, $n = 1, 2, \ldots$, such that $\|f_n\| \leqslant 1/2^n$ and $|\mu(f_n)| > 1$ for each n. Since the norm $\|.\|$ on $\mathbf{U}_b(X)$ is complete, for each $A \subseteq \mathbf{N}$ we can define $g_A \in \mathbf{U}_b(X)$ by

$$g_A = \sum_{n \in A} f_n .$$

The finitely additive map $m : \mathbf{P}(\mathbf{N}) \to \mathbf{R}$ defined by $m(A) = \mu(g_A)$, $A \subseteq \mathbf{N}$, is CBP-measurable. Hence Lemma 2.1 yields

$$\mu(g_N) = \sum_{n=1}^{\infty} \mu(f_n)$$

which is impossible because $|\mu(f_n)| > 1$ for each n. The contradiction proves that μ is $\|.\|$ continuous.

Now we show that $\mu^+ : \mathbf{U}_b(X) \to \mathbf{R}$ (defined by

$$\mu^+(f) = \sup \{\mu(g) \mid 0 \leqslant g \leqslant f\}$$

for $f \in \mathbf{U}_b^+(X))$ is CBP-measurable. It follows that $\mu^- = (-\mu)^+$ is CBP-measurable as well.

For $n = 1,2,\ldots$ choose f_n such that $0 \leqslant f_n \leqslant n$ and

$$\mu(f_n) + \frac{1}{n} > \sup\{\mu(f) \mid 0 \leqslant f \leqslant n\};$$

put $\mu_n(g) = \mu(f_n \wedge g)$ for $g \in \mathbf{U}_b^+(X)$. The functions $\mu_n : \mathbf{U}_b^+(X) \to \mathbf{R}$ are not affine, but each μ_n is $w(\mathbf{U}_b(X), \mathbf{M}_{\mathbf{R}}(X))$ CBP-measurable (by Lemma 2.3) and $\mu^+(g) = \lim \mu_n(g)$ for each $g \in \mathbf{U}_b^+(X)$. Since $\mu^+(g) = \mu^+(g^+) - \mu^+(g^-)$ for $g \in \mathbf{U}_b(X)$, μ^+ is $w(\mathbf{U}_b(X), \mathbf{M}_{\mathbf{R}}(X))$ CBP-measurable on $\mathbf{U}_b(X)$. □

Proof of Theorem 2. – Let μ be a $w(\mathbf{U}_b(X), \mathbf{M}_{\mathbf{R}}(X))$ CBP-measurable linear form on $\mathbf{U}_b(X)$. We may (and shall) assume, in view of Lemma 2.4, that μ is positive.

To prove that μ is a Radon measure, we have to show that for every $\varepsilon > 0$ there is a compact set $K \subseteq X$ such that if $f \in \mathbf{U}_b(X)$, $0 \leqslant f \leqslant 1$ and $f(x) = 0$ for $x \in K$ then $\mu(f) < \varepsilon$.

Take any $\varepsilon > 0$. Since X has the (ℓ^1) property, for each $n = 1, 2, \ldots$ there is an ℓ^1-continuous partition of unity $\{p_\alpha^n\}_{\alpha \in D_n}$ on X such that $\operatorname{diam}(\operatorname{supp} p_\alpha^n) < 1/n$ for every $\alpha \in D_n$. (We write $\operatorname{supp} p = \operatorname{cl}\{x \in X \mid p(x) \neq 0\}$.) By Lemma 2.2, for each n we can find a finite set $F_n \subseteq D_n$ such that

$$\mu\left(\sum_{\alpha \in D_n \setminus F_n} p_\alpha^n\right) < \frac{\varepsilon}{2^n}.$$

Put $K_n = \bigcap_{k=1}^{n} \bigcup_{\alpha \in F_k} \operatorname{supp} p_\alpha^k$ and $K = \bigcap_{n=1}^{\infty} K_n$. The set K is closed in X and precompact; hence K is compact.

We show that if $f \in \mathbf{U}_b(X)$ and $f(x) = 0$ for all $x \in K$ then there is n such that $|f(x)| < \varepsilon$ for all $x \in K_n$. Assume that it is not so : Then

$$L_n = K_n \cap \{x \in X \mid |f(x)| \geqslant \varepsilon\} \neq \varnothing$$

for every n; hence $\{L_n \mid n = 1,2,\ldots\}$ is the base of a filter, and there exists an ultrafilter $\mathscr{U}$ containing it. For every n there is $\alpha \in F_n$ such that

$\operatorname{supp} p_\alpha^n \in \mathcal{U}$. Hence $\mathcal{U}$ is a Cauchy filter and

$$\bigcap \{\operatorname{cl} U \mid U \in \mathcal{U}\} = \{x_0\}$$

for some $x_0 \in X$; clearly $|f(x_0)| \geqslant \varepsilon$. On the other hand, $f(x_0) = 0$ because $x_0 \in K$. This contradiction proves that for some n we have $|f(x)| < \varepsilon$ for all $x \in K_n$.

Let $f \in U_b(X)$ be a function such that $0 \leqslant f \leqslant 1$ and $f(x) = 0$ for $x \in K$. For some n we have $f(x) < \varepsilon$ for all $x \in K_n$; moreover,

$$f(x) \leqslant 1 \leqslant \sum_{k=1}^{n} \sum_{\alpha \in D_k \setminus F_k} p_\alpha^k(x)$$

for all $x \in X \setminus K_n$. Hence

$$\mu(f) \leqslant \mu(\varepsilon) + \sum_{k=1}^{n} \mu\left(\sum_{\alpha \in D_k \setminus F_k} p_\alpha^k\right) < \varepsilon . \mu(1) + \varepsilon,$$

q.e.d.

3. Invariant functionals on K-analytic spaces.

When T is a set of mappings from a set F into F and h is a function defined on F, we say that h is T-invariant if $h(x) = h(t(x))$ for all $x \in F$ and all $t \in T$.

The rest of this paper is devoted to applications of the following abstract result. (K-analytic spaces are called analytic by Frolík [6].)

THEOREM 3. — *Let* F *be a Banach space and let* τ *be a linear Hausdorff topology on* F *such that the closed unit ball* B *in* F *is* K-*analytic in* τ. *Let* T *be a countable set of (not necessarily linear !)* τ-*continuous mappings from* F *into* F. *If the space of* T-*invariant functionals* $h \in (F, \|.\|)'$ *is norm separable then every* T-*invariant* $h \in (F, \|.\|)'$ *is* τ-*Baire measurable.*

The proof of Theorem 3 is based on two lemmas :

LEMMA 3.1. — *Let* F *be a normed linear space and let* τ *be a Hausdorff linear topology on* F *such that the closed unit ball* B *in* F *is* K-*analytic in* τ. *If a subset* Z *of* F *is K-analytic in* τ, *then so is* cl Z, *the norm closure of* Z.

Proof. – $\mathrm{cl}\ Z = \bigcap_{n=1}^{\infty} \left(Z + \frac{1}{n} B\right)$. □

LEMMA 3.2. – *If h is a linear form on a Hausdorff topological vector space F such that $h^{-1}(0)$ is K-analytic, then h is Baire measurable (hence $h^{-1}(0)$ is a Baire set).*

Proof. – The zero linear form is Baire measurable. If $h \neq 0$, take $x \in F$ such that $h(x) = 1$; for any $Q \subseteq \mathbf{R}$ we have

$$h^{-1}(Q) = h^{-1}(0) + \{r.x : r \in Q\}.$$

If $Q \subseteq \mathbf{R}$ is open then $\{r.x : r \in Q\}$ and $\{r.x : r \in \mathbf{R} \backslash Q\}$ are K-analytic subsets of F, and so are $h^{-1}(Q)$ and $F \backslash h^{-1}(Q) = h^{-1}(\mathbf{R} \backslash Q)$. It follows that $h^{-1}(Q)$ is a Baire set ([6], 5.8). □

Proof of Theorem 3. – The set

$$Z = \{f - t(f) : f \in F \text{ and } t \in T\}$$

is K-analytic in τ, because

$$Z = \bigcup_{n=1}^{\infty} \bigcup_{t \in T} \{f - t(f) : f \in nB\}$$

and each $\{f - t(f) : f \in nB\}$ is a continuous image of the K-analytic space nB.

The linear span $L(Z)$ of Z is K-analytic in τ, and so is $\mathrm{cl}(L(Z))$, the norm closure of $L(Z)$, by Lemma 3.1. A linear form $h \in (F, \|.\|)'$ is T-invariant if and only if $h(x) = 0$ for each $x \in \mathrm{cl}(L(Z))$. Hence the space of T-invariant functionals $h \in (F, \|.\|)'$ is isometrically isomorphic to $(F/\mathrm{cl}(L(Z)))'$. Since it is separable, its predual $F/\mathrm{cl}(L(Z))$ is separable as well.

Let $h \in (F, \|.\|)'$ be T-invariant. It can be factored as $h = g \circ p$ where $p : F \to F/\mathrm{cl}(L(Z))$ is the canonical projection and $g \in (F/\mathrm{cl}(L(Z)))'$. The set $g^{-1}(0) \subseteq F/\mathrm{cl}(L(Z))$ is norm separable; we will prove that $h^{-1}(0)$ is K-analytic in τ (cf. Th. 2 in [17]). Take a countable set $C_0 \subseteq g^{-1}(0)$ such that $\mathrm{cl}\ C_0 = g^{-1}(0)$ and find a countable set $C \subseteq F$ such that $p(C) = C_0$. The linear span $L(C)$ of C is K-analytic and $h^{-1}(0) = \mathrm{cl}(L(C) + L(Z))$. Hence $h^{-1}(0)$ is K-analytic in τ. By Lemma 3.2, h is τ-Baire measurable. The proof of Theorem 3 is complete.

Remark. – The theorem can be strengthened in the following way : Replace the countable set T by a family $\{t_y\}_{y \in Y}$ of maps from F into F such that

(i) Y is a K-analytic space;

(ii) for $n = 1, 2, \ldots$, the map $(f, y) \to t_y(f)$ from $nB \times Y$ into F is jointly continuous (relative to the topology τ on nB and F).

The proof remains essentially the same.

We want to use Theorem 3 for $F = \mathbf{U}_b(X)$, the space of bounded uniformly continuous functions on a complete metric space X. Note that if X is not compact then $\check{X}$, the Samuel compactification of X (cf. [11]), is not quasi-Eberlein : X is sequentially closed in $\check{X}$, and by Th. 14 in [17] the space $\check{X}$ is not quasi-Eberlein. It follows that the Banach space $\mathbf{U}_b(X)$ is not weakly K-analytic ([17], Th. 4).

Nevertheless, we are going to prove that $\mathbf{U}_b(X)$ is « weakly compactly generated » in a weaker topology :

For a metric space (X,d), put

$$\mathbf{Lip} = \{f : X \to \mathbf{R} \mid \|f\| \leqslant 1 \text{ and } |f(x)-f(y)| \leqslant d(x,y) \text{ for } x, y \in X\}.$$

The set **Lip** is $w(\mathbf{U}_b(X), \mathbf{M}_{\mathbf{R}}(X))$ compact.

The following lemma is well known, but we have not been able to trace its origin.

LEMMA 3.3. – *Let* (X,d) *be a metric space. The set* $\bigcup_{n=1}^{\infty} n\,\mathbf{Lip}$ *is norm dense in* $\mathbf{U}_b(X)$.

Proof. – Take any $f \in \mathbf{U}_b(X)$ and any $\varepsilon > 0$. There is $\delta > 0$ such that $|f(x)-f(y)| \leqslant \varepsilon$ whenever $d(x,y) \leqslant \delta$. We prove that if

$$n \geqslant \max(\|f\|, 2\|f\|/\delta)$$

then there is $g \in n\mathbf{Lip}$ such that $\|g-f\| \leqslant \varepsilon$: Put

$$g = \sup\{h \in n\mathbf{Lip} \mid h \leqslant f\}.$$

Clearly $g \in n\mathbf{Lip}$. For any $x \in X$ define $h_x \in n\,\mathbf{Lip}$ by

$$h_x(y) = \max(-n, f(x) - \varepsilon - nd(y,x));$$

then $h_x \leqslant f$ and $h_x(x) \geqslant f(x) - \varepsilon$. Hence $f \geqslant g \geqslant f - \varepsilon$, q.e.d.

THEOREM 4. — *Let* X *be a metric space. The closed unit ball* B *in* $\mathbf{U}_b(X)$ *is* K-*analytic in the topology* $w(\mathbf{U}_b(X),\mathbf{M}_R(X))$.

Proof (cf. ([17], Th. 1)). — Let $G = (\mathbf{M}_R(X), \|.\|)'$ be the norm dual of $\mathbf{M}_R(X)$. Identify $\mathbf{U}_b(X)$ with its canonical image in G, and denote by B″ the unit ball in G. We have $\mathbf{Lip} \subseteq B \subseteq B''$ and both **Lip** and B″ are $w(G,\mathbf{M}_R(X))$ compact.

By Lemma 3.1, $\mathrm{cl}\left(\bigcup_{n=1}^{\infty} n\mathbf{Lip}\right)$, the norm closure of $\bigcup_n n\mathbf{Lip}$ in G, is K-analytic in the topology $w(G,\mathbf{M}_R(x))$. From Lemma 3.3 it follows that

$$B = B'' \cap \mathrm{cl}\left(\bigcup_{n=1}^{\infty} n\,\mathbf{Lip}\right);$$

hence B is K-analytic in the topology $w(\mathbf{U}_b(X),\mathbf{M}_R(X))$.

4. Invariant finitely additive measures.

In this section we list several applications of Theorem 3. By no means is our list exhaustive; we only show several random examples, to illustrate how the theorem can be used.

Our results are to an extent parallel to those of Granirer [9]. The difference is this : Granirer proves (among other things) that if the space of invariant functionals (finitely additive measures) is norm separable and nonempty then many invariant functionals are « smooth ». We prove that, under stronger assumptions, *every* invariant functional is « smooth ».

We begin with a result about Markov operators, and then concentrate on invariant means on semigroups.

Let μ be a sigma-finite measure and T a countable set of Markov operators from $L^1(\mu)$ into $L^1(\mu)$; cf. ([9]. pp. 73-75). For each $t \in T$ denote by t' the transposed map from $L^\infty(\mu)$ into $L^\infty(\mu)$ and put $T' = \{t' : t \in T\}$.

THEOREM 5. — *Let* μ *be a sigma-finite measure and* T *a countable set of Markov operators from* $L^1(\mu)$ *into* $L^1(\mu)$. *If the space of* T′-*invariant functionals* $h \in (L^\infty(\mu))'$ *is norm-separable then every* T′-*invariant* $h \in (L^\infty(\mu))'$ *belongs to* $L^1(\mu)$.

Proof. — Use Theorem 3 with $F = L^{\infty}(\mu)$ and $\tau = w(L^{\infty}(\mu), L^{1}(\mu))$. The closed unit ball in F is τ-compact. The space $L^{1}(\mu)$ is $L^{\infty}(\mu)$-saturated (section 2 above), hence every τ-Baire measurable linear form on $L^{\infty}(\mu)$ belongs to $L^{1}(\mu)$. □

Similar results can be proved for other saturated function spaces (cf. section 2 above). Here we restrict our attention to functions defined on semigroups.

First, let X be a countable semigroup. For each $x \in X$ define $L_x : \ell^{\infty}(X) \to \ell^{\infty}(X)$ by $L_x(f)(y) = f(xy)$ for $f \in \ell^{\infty}(X)$, $y \in X$. Denote by $\mathbf{Inv}(X) = \mathbf{Inv}(\ell^{\infty}(X), X)$ the space of $\{L_x : x \in X\}$-invariant linear norm-continuous functionals on $\ell^{\infty}(X)$. By ([13], 3.2), $\mathbf{Inv}(X)$ is an abstract L-space; its positive elements of norm 1 are called the left-invariant means on X (cf. [10]). Clearly, the set of left-invariant means is norm separable if and only if $\mathbf{Inv}(X)$ is norm separable.

THEOREM 6. — *Let* X *be a countable semigroup. If* $\mathbf{Inv}(X)$ *is norm separable then* $\mathbf{Inv}(X) \subseteq \ell^{1}(X)$.

Proof. — The closed unit ball in $\ell^{\infty}(X)$ is $w(\ell^{\infty}, \ell^{1})$ compact, and the space $\ell^{1}(X)$ is $\ell^{\infty}(X)$-saturated. Apply Theorem 3. □

Theorem 6 is due to Granier (it follows directly from the results in [7] and [8]), who also gives an intrinsic characterization of the semigroups X satisfying $\mathbf{Inv}(X) \subseteq \ell^{1}(X)$. Klawe [12] proved similar results for uncountable semigroups. Since our method relies heavily on separable descriptive theory of sets, the nonseparable results are presently beyond its reach.

Note that our proof of Theorem 6 can be used to prove analogous results for other kinds of invariance, such as two-sided invariance ([10], p. 3), inversion-invariance or inner invariance [3].

Next we apply Theorem 3 to the space of invariant finitely additive measures on uniformly continuous functions on a topological group. For the sake of brevity, say that a metric space X is *saturated* if the space $\mathbf{M}_{R}(X)$ is $\mathbf{U}_{b}(X)$-saturated. Every saturated metric space is complete; conversely, our conjecture in section 2 states that every separable complete metric space is saturated (it is even possible that *every* complete metric space is saturated).

The next lemma together with Theorem 2 implies that every closed metric subspace of a locally compact metric group is saturated.

LEMMA 4.1. — *If* X *is a locally compact group whose topology is given by a right-invariant metric* d, *then* (X,d) *has the* (ℓ^1) *property.*

Proof. — Denote the unit in X by e, and find $r > 0$ such that $V_r = \{x \in X : d(x,e) \leqslant r\}$ is compact. Take any $\varepsilon > 0$ such that $3\varepsilon < \min(r,1)$.

Let k be the maximum cardinality of an ε-discrete subset of V_r. (A set $Y \subseteq X$ is ε-discrete if $d(x,y) \geqslant \varepsilon$ for all $x, y \in Y, x \neq y$). Since V_r is compact, k is finite.

Choose a maximal ε-discrete subset D of X and for each $\alpha \in D$ put

$$q_\alpha(x) = (2\varepsilon - d(x,\alpha))^+ \,, \; x \in X.$$

The function $q(x) = \sum_{\alpha \in D} q_\alpha(x)$ satisfies $q(x) \geqslant \varepsilon$ for every $x \in X$. Moreover, for each $x \in X$ at most k terms $q_\alpha(x)$ are non-zero, because $Dx^{-1} \cap V_r$ is an ε-discrete subset of V_r, and $q_\alpha(x) = 0$ whenever $\alpha x^{-1} \notin V_r$. It follows that $\varepsilon \leqslant q(x) \leqslant k$ for $x \in X$, and

$$|q(x) - q(y)| \leqslant 2k\, d(x,y)$$

and

$$\left|\frac{1}{q(x)} - \frac{1}{q(y)}\right| \leqslant \frac{2k}{\varepsilon^2} d(x,y)$$

for $x, y \in X$.

Put $p_\alpha(x) = q_\alpha(x)/q(x)$ for $x \in X$. Then $\{p_\alpha\}_{\alpha \in D}$ is a partition of unity. We have

$$\begin{aligned} |p_\alpha(x) - p_\alpha(y)| &\leqslant \left|\frac{q_\alpha(x)}{q(x)} - \frac{q_\alpha(y)}{q(x)}\right| + \left|\frac{q_\alpha(y)}{q(x)} - \frac{q_\alpha(y)}{q(y)}\right| \\ &\leqslant \frac{1}{\varepsilon} d(x,y) + k\frac{2k}{\varepsilon^2} d(x,y) \\ &\leqslant \frac{2k^2+1}{\varepsilon^2} d(x,y), \end{aligned}$$

which gives

$$\sum_{\alpha \in D} |p_\alpha(x) - p_\alpha(y)| \leqslant 2k\frac{2k^2+1}{\varepsilon^2} d(x,y);$$

hence the partition of unity is ℓ^1-continuous. Finally,

$$\operatorname{diam}(\operatorname{supp} p_\alpha) = \operatorname{diam}(\operatorname{supp} q_\alpha) \leqslant 4\varepsilon$$

for every $\alpha \in D$.

This proves that (X,d) has the (ℓ^1) property. □

Let X be a topological group, with the topology defined by a right-invariant metric d. Denote by $\hat{X}_d$ the completion of (X,d). Every function in $\mathbf{U}_b(X,d)$ extends uniquely to a function in $\mathbf{U}_b(\hat{X}_d)$; thus the left translations L_x, $x \in X$, act on $\mathbf{U}_b(\hat{X}_d)$. Denote the space of $\{L_x : x \in X\}$-invariant linear norm continuous functionals on $\mathbf{U}_b(\hat{X}_d)$ by $\mathbf{Inv}(\mathbf{U}_b(\hat{X}_d), X)$. Again, $\mathbf{Inv}(\mathbf{U}_b(\hat{X}_d), X)$ is an abstract L-space ([13], 3.2).

THEOREM 7. — *Let* X *be a topological group whose topology is defined by a separable right-invariant metric* d. *If* $\hat{X}_d$ *is saturated and* $\mathbf{Inv}(\mathbf{U}_b(\hat{X}_d), X)$ *is norm separable then* $\mathbf{Inv}(\mathbf{U}_b(\hat{X}_d), X) \subseteq \mathbf{M}_R(\hat{X}_d)$.

Proof. — For every $f \in \mathbf{U}_b(\hat{X}_d)$, the map $x \longmapsto L_x f$ from X into $\mathbf{U}_b(\hat{X}_d)$ is norm continuous. It follows that a linear norm continuous functional on $\mathbf{U}_b(\hat{X}_d)$ is $\{L_x : x \in X\}$-invariant if it is $\{L_t : t \in T\}$-invariant for some dense subset T of X.

Since X is separable, there is a countable dense subset $T \subseteq X$. Apply Theorem 3 to $F = \mathbf{U}_b(\hat{X}_d)$ and $\tau = w(\mathbf{U}_b(\hat{X}_d), \mathbf{M}_R(X))$, together with Theorems 2 and 4. □

Theorem 7 is analogous to, but independent of, Th. 6 in ([9], p. 65).

Observe again that the same proof yields analogous results for the spaces of inversion-invariant, two-side invariant or inner invariant functionals (or any combination of these).

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