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A factorization theorem in Banach lattices and its application to Lorentz spaces


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A FACTORIZATION THEOREM
IN BANACH LATTICES
AND ITS APPLICATION TO LORENTZ SPACES

by Shlomo REISNER

1.

A Köthe function space is a Banach lattice of locally integrable, real-valued functions (more precisely, equivalent classes of functions, modulo equality a.e.) on a σ-finite, complete measure space \((\Omega, \Sigma, \mu)\), which satisfy the two conditions

(i) If \(|f| \leq |g|\) with \(f \in L_0(\mu), g \in L\), then \(f \in L\) and \(\|f\| \leq \|g\|\) (\(L_0(\mu)\) is the space of all \(\mu\)-measurable functions).

(ii) For every \(A \in \Sigma\) with \(\mu(A) < \infty\), the characteristic function of \(A, \chi_A\), is in \(L\).

For background on Köthe function spaces and Banach lattices in general we refer to [7], part II. We use standard notation of Banach space theory.

In particular when \(L\) is a Köthe function space, \(L^*\) is its dual space. \(L'\) is the subspace of \(L^*\) consisting of functionals \(\varphi\) for which there is \(g \in L_0(\mu)\) so that \(\varphi(f) = \int_\Omega f(t) g(t) \, dt\) for all \(f \in L\) (in the sequel we use the same letter for \(\varphi\) and \(g\)). The adjoint of a linear operator \(T\) is denoted by \(T^*\).

We say that a linear operator \(T : E \to L\) (resp. \(T : L \to E\)) where \(E\) is a Banach space and \(L\) is a Banach lattice, is \(p\)-convex (resp. \(q\)-concave) for \(1 \leq p, q \leq \infty\), if there is \(K > 0\) such that for all \(x_1, \ldots, x_n \in E\),
\[
\left\| \left( \sum \frac{|Tx_i|^p}{p} \right)^{1/p} \right\| \leq K \left( \sum \|x_i\|^p \right)^{1/p}
\]

(resp. for all \( f_1, \ldots, f_n \in L \), \( \left( \sum \|Tf_i\|^q \right)^{1/q} \leq K \left( \sum |f_i|^q \right)^{1/q} \)).

We denote \( \inf K = K^{(p)}(T) \) (resp. \( K^{(q)}(T) \)). If the identity \( I \) of \( L \) is \( p \)-convex (resp. \( q \)-concave) we say that \( L \) is a \( p \)-convex (resp. \( q \)-concave) lattice, and denote

\[
K^{(p)}(L) = K^{(p)}(I) \quad (\text{resp. } K^{(q)}(L) = K^{(q)}(I)).
\]

The following theorem was proved by Lozanovskii in [9] (for another proof in the discrete case see [5]).

**Theorem.** Let \( L \) be a Köthe function space on \((\Omega, \Sigma, \mu)\). Every \( g \in L_1(\mu) \) has, for every \( \varepsilon > 0 \), a factorization \( g = g_1 g_2 \) with \( g_1, g_2 \in L_0(\mu) \) and \( \|g_2\|_{L_1} \|g_1\|_{L_1} \leq (1 + \varepsilon) \|g\|_{L_1(\mu)} \).

We interpret this theorem as follows: The multiplication operator \( T_g : L_\infty(\mu) \to L_1(\mu) \) (\( T_g f = gf \)) has a factorization \( T_g = T_{g_2} \circ T_{g_1} \) with

\[
\|T_{g_2} : L \to L_1(\mu)\| \|T_{g_1} : L_\infty \to L\| \leq (1 + \varepsilon) \|T_g\|
\]

(if \( X, Y \) are Köthe function spaces on \((\Omega, \Sigma, \mu)\) and \( T \) is a linear operator in \( L_0(\mu) \) we denote by \( \|T : X \to Y\| \) the norm of \( T \) as an operator from \( X \) into \( Y \)).

We show in Section 2 that with this interpretation the factorization theorem has a generalization concerning \( p \)-convex \( q \)-concave Köthe function spaces. Moreover, this generalization has an inverse which makes it a characterization of \( p \)-convex, \( q \)-concave Köthe function spaces.

In Section 3 we make use of this characterization to find a necessary and sufficient condition on a non-increasing sequence \( w \) or function \( W \) in order that the Lorentz sequence space \( d(w, p) \) or function space \( L_{d(w, p)} \) be \( q \)-concave. For Köthe function spaces \( L \) and \( M \) and \( 0 < \theta < 1 \) we construct, following [2], the Köthe function space

\[
L^\theta M^{1-\theta} = \{f \in L_0(\mu) : |f| \leq \lambda g^\theta h^{1-\theta} \text{ for some } g \in L, h \in M, \|g\|_L = \|h\|_M = 1 \text{ and } \lambda \geq 0\}
\]

with the norm \( \|f\|_{L^\theta M^{1-\theta}} = \inf \{\lambda ; \lambda \text{ as above}\} \).
A result that we use in the sequel is the recent result of Pisier [10] which says that if a Köthe function space $L$ has $K^{(p)}(L) = K^{(q)}(L) = 1$ then there is a Köthe function space $X$ with $L = [L_{t}(\mu)]^{1-\theta} X^{\theta}$ with $\theta$ and $t$ such that

$$\frac{\theta}{t} + \frac{1-\theta}{1} = \frac{1}{p}; \quad \frac{\theta}{t} + \frac{1-\theta}{\infty} = \frac{1}{q}$$

(i.e. $t = \frac{1}{s'}, \theta = 1 - \frac{1}{s} = \frac{1}{s'}$).

2.

In this section $L$ is a Köthe function space on a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$. We assume that $L'$ is a norming subspace of $L^*$, i.e., that for all $f \in L$ \[ \|f\| = \sup_{\|g\|_{L^1} = 1} \int_{\Omega} fg d\mu. \]

Let $g \in L_0(\mu)$, the multiplication operator $T_g$ in $L_0(\mu)$ is defined by $T_g f = gf$.

THEOREM 1. — Let $1 < p < q < \infty$ and let $s$ be defined by $\frac{1}{s} = \frac{1}{p} - \frac{1}{q}$.

$L$ is $p$-convex and $q$-concave if and only if there is $K > 0$ so that for all $g \in L_s(\mu)$ the multiplication operator $T_g$ has a factorization as a composition of multiplication operators $T_{h_2}$ and $T_{h_1}$ in the form

\[
\begin{array}{ccc}
L_q(\mu) & \xrightarrow{T_g} & L_p(\mu) \\
& T_{h_1} \searrow & T_{h_2} \\
L & \nearrow & \\
\end{array}
\]

with $\|T_{h_2}\| \cdot \|T_{h_1}\| \leq K$. Moreover, if $K^{(p)}(L)$ and $K^{(q)}(L)$ are given, we may choose $K = (1 + \epsilon) K^{(p)}(L) K^{(q)}(L)$ with arbitrarily small $\epsilon > 0$. If, on the other hand, $K$ is given then $K^{(p)}(L) K^{(q)}(L) \leq K^2$. 
Proof. — Necessity. Suppose \( L \) is \( p \)-convex and \( q \)-concave. By the result of Pisier which is quoted in Section 1, \( L \) is \( K^{(p)}(L) \) \( K_{(q)}(L) \)-isomorphic to \([L_{\theta}(\mu)]^\theta X^{1-\theta}\) for an appropriate Köthe function space \( X \) on \((\Omega, \Sigma, \mu)\) and for \( \theta, t \) which satisfy (1).

(1) implies that \( L_p = L^{\theta}_t L_1^{1-\theta} \), \( L_q = L^{\theta}_t L_\infty^{1-\theta} \), \( L_s = L^{\theta}_t L_1^{1-\theta} \) (from here on we write \( L_p \) instead of \( L_p(\mu) \)).

Let \( g \in L_s \). 
\[
g = g_1^{\frac{1}{1-\theta}} \quad \text{(where } g_1 = g^{1-\theta})
\]
and 
\[
\|g_1\|_{L_1} = \|g\|_0^{\frac{1}{1-\theta}}.
\]

Let \( g_1 = g_{1,1} g_{1,2} \) be the factorization of \( g_1 \) through \( X \) by Lozanovskii's theorem of Section 1. If \( h_1 = g_{1,1}^{\frac{1}{s}} \), \( h_2 = g_{1,2}^{\frac{1}{s}} \) then clearly \( h_2 h_1 = g_1^{\frac{1}{s}} = g \) and also 
\[
\|T_{h_2} : L \rightarrow L_p\| \|T_{h_1} : L_q \rightarrow L\| \leq K^{(p)}(L) K_{(q)}(L) \|g\|_{L_s}(1 + \epsilon)^{1/s}
\]
(see Diagram (2)).

\[
g = g_1^{\frac{1}{1-\theta}} \quad \text{\( L\)} \quad \text{\( L_1^{1-\theta}\)}
\]

(2)

Sufficiency. — Suppose that every \( g \in L_s \) has a factorization \( g = h_2 h_1 \) with \( \|T_{h_2} : L \rightarrow L_p\| \|T_{h_1} : L_q \rightarrow L\| \leq K \|g\|_{L_s} \).

We define a positive homogeneous functional \(! \) on \( L \) by 
\[
!f! = \sup_{\|g\|_{L_s} = 1} \inf_{h_1, h_2 \in L_0} \|h_2 f\|_{L_p} \|T_{h_1} : L_q \rightarrow L\|.
\]

We denote the lattice semi-norm which is induced by this functional by \( \| \| \cdot \| \| \). 
\[
\|f\| = \inf \left\{ \sum_{k=1}^n !f_k! \mid f_k \geq 0 ; |f| = \sum_{k=1}^n f_k \right\}.
\]
We show that this is in fact a norm and that the formal inclusion map \( I : (L, \| \cdot \|) \to (L, \| \cdot \|) \) is a lattice isomorphism with \( K^{(p)}(I) K^{(q)}(I^{-1}) \leq K \). Clearly by showing this we complete the proof.

\( a) \ K^{(p)}(I) \leq K \). Let \( \{f_i\}_{i=1}^n \subset L \), then

\[
\left\| \left( \sum_{i=1}^n |f_i|^p \right)^{1/p} \right\| \leq \left( \sum_{i=1}^n \| h f_i \|_{L_p} \right)^{1/p}
\]

\[
= \sup_{\|g\|_{L_2^*} = 1} \inf_{h_1, h_2 \in L_0} \left\| h_2 \left( \sum_{i=1}^n |f_i|^p \right)^{1/p} \right\|_{L_p} \| T_{h_1} : L_q \to L \|.
\]

Now, for all \( h \in L_0 \)

\[
\left\| h \left( \sum_{i=1}^n |f_i|^p \right)^{1/p} \right\|_{L_p} \leq \left( \sum_{i=1}^n \| h f_i \|_{L_p} \right)^{1/p}
\]

\[
= \left( \sum_{i=1}^n \| f_i \|_{L_L^p} \right) \left\| h \frac{f_i}{\| f_i \|_{L_L^p}} \right\|_{L_p}^{1/p} \leq \| T_{h_1} : L \to L_p \| \left( \sum_{i=1}^n \| f_i \|_{L_L^p} \right)^{1/p}.
\]

Hence, the assumption on \( L \) yields

\[
\left\| \left( \sum_{i=1}^n |f_i|^p \right)^{1/p} \right\| \leq \left( \sum_{i=1}^n \| f_i \|_{L_L^p} \right) \sup_{\|g\|_{L_2^*} = 1} \inf_{h_1, h_2 \in L_0} \| T_{h_2} : L \to L_p \| \left\| T_{h_1} : L_q \to L \| \leq K \left( \sum_{i=1}^n \| f_i \|_{L_L^p} \right)^{1/p}.
\]

\( b) \ \ K^{(q)}(I^{-1}) \leq 1 \). To show this we show

\[
K^{(q')}((I^{-1})^*_{L'}) \leq 1, \ \left( \frac{1}{q} + \frac{1}{q'} = 1 \right).
\]

(One can verify that \( I^{-1} \) is well defined and bounded, and in particular that \( \| \cdot \| \) is a norm, by noting in the course of the following argument that for all \( g \in L' \)

\[
\sup_{\|f\|_{L'} \leq 1} \int_{\Omega} g f d\mu \leq \sup_{\|f\|_{L} \leq 1} \int_{\Omega} g f d\mu,
\]

and using the fact that \( L' \) is a norming subspace of \( L^* \). (3) implies \( K^{(q)}(I^{-1}) \leq 1 \) by [6] (th. 5) and by the fact that \( L \) is isometric to a subspace of \( (L')^* \) since \( L' \) is a norming subspace of \( L^* \). Let \( \{g_i\}_{i=1}^m \subset L' \) and \( 0 \leq f \in L \). We denote \( \varphi = \left( \sum_{i=1}^m |g_i|^{q'} \right)^{1/q'} \).
Let \( g^0 \in L_s \) with \( \|g^0\|_{L_s} = 1 \) be defined by
\[
g^0 = \frac{(f \varphi)^{1/s}}{\left( \int f \varphi \, d\mu \right)^{1/s}}.
\]
we also define \( h^0_1, h^0_2 \in L_0 \) by
\[
h^0_1 = \begin{cases} \left( \frac{g^0 \varphi}{\varphi} \right)^{1/(p+q)}; \varphi \neq 0 \\ 0; \varphi = 0 \end{cases}, \quad h^0_2 = \begin{cases} \left( \frac{g^0 \varphi'}{\varphi'} \right)^{1/(p+q)}; f \neq 0 \\ 0; f = 0. \end{cases}
\]

Then \( g_0 = h^0_2 h^0_1 \) and using Hölder's inequality we get for \( r \) with
\[
\frac{1}{r} = \frac{1}{p} + \frac{1}{q'} = 1 + \frac{1}{s}
\]
\[
\varphi(f) = \int g_0 f \varphi \, d\mu = \|g_0 f\|_{L_r} = \|h^0_2 f\|_{L_p} \|h^0_1 \varphi\|_{L_q},
\]
\[
= \inf_{h^0_2 h^0_1 = g_0} \|h_2 f\|_{L_p} \|h_1 \varphi\|_{L_q} \leq \sup_{\|g\|_{L_s} = 1} \inf_{g = h_2 h_1} \|h_2 f\|_{L_p} \|h_1 \varphi\|_{L_q}.
\]

Now, like in part a) of the proof we have
\[
\|h_1 \varphi\|_{L_q} = \left\| h_1 \left( \sum_i |g_i|^{q'} \right)^{1/q'} \right\|_{L_q} \leq \|T_{h_1} : L' \to L_{q'} \| \left( \sum_i \|g_i\|_{L_i}^{q'} \right)^{1/q'}.
\]

It is clear from the assumptions on \( L \) that
\[
\|T_{h_1} : L' \to L_{q'}\| = \|T_{h_1} : L_q \to L\|
\]
hence \( \varphi(f) \leq \left( \sum_i \|g_i\|_{L_i}^{q'} \right)^{1/q'} \|f\| \) and sub-linearity shows
\[
\varphi(f) \leq \left( \sum_i \|g_i\|_{L_i}^{q'} \right)^{1/q'} \|f\|.
\]

Therefore \( \|\varphi\|_* = \left\| \left( \sum_i |g_i|^{q'} \right)^{1/q'} \right\|_* \leq \left( \sum_i \|g_i\|_{L_i}^{q'} \right)^{1/q'} \|f\| \) (where \( \|\cdot\|_* \) is the norm dual to \( \|\cdot\| \)), q.e.d.

If \( \mu \) is a probability measure and \( g \equiv 1 \) on \( \Omega \), then \( T_g \) is the inclusion map \( i : L_q(\mu) \to L_p(\mu) \). From Theorem 1 it follows that if \( L \) is \( p \)-convex and \( q \)-concave then there is a factorization of \( i \) in the form
That is, for all \( f \in L_q(\mu) \)

\[
\|hf\|_L \leq K \|f\|_{L_q(\mu)} = K \|hf\|_{L_q \left( \frac{d\mu}{h^q} \right)}
\]

and for all \( g \in L \)

\[
\|g\|_{L_p \left( \frac{d\mu}{h^p} \right)} = \|g/h\|_{L_p(\mu)} \leq K \|g\|_L.
\]

In other words:

**Corollary 1.** - If \( \mu \) is a finite measure and \( L \) is \( p \)-convex and \( q \)-concave, then there exists \( 0 < h \in L_0(\mu) \) such that

\[
L_q \left( \frac{d\mu}{h^q} \right) \subset L \subset L_p \left( \frac{d\mu}{h^p} \right)
\]

(set-inclusions with bounded inclusion operators).

3.

In this section we demonstrate an application of the factorization Theorem 1 for the calculation of the convexity exponent of Lorentz sequence and function spaces.

Let \( w = (w_i)_{i=1}^\infty \) be a positive, non increasing sequence which tends to 0 and satisfies \( \sum_{i=1}^\infty w_i = \infty \). For \( 1 \leq p < \infty \) the Lorentz sequence space \( d(w, p) \) is defined by

\[
d(w, p) = \left\{ v = \left( v_i \right)_{i=1}^\infty \in c_0 : \|v\| = \left( \sum_{i=1}^\infty v_i^{*p} w_i \right)^{1/p} < \infty \right\}
\]

(\( v^* = (v_i^*)_{i=1}^\infty \) is the non increasing rearrangement of \( |v| \)). The space \( d(w, p) \), equipped with the norm \( \| \cdot \| \) is a Köthe sequence space which is \( p \)-convex with constant 1. It is not \( r \)-convex for
any $r > p$ since it contains subspaces isomorphic to $\ell_p$ with the unit basis elements supported on disjoint blocks. $d(w, p)$ is reflexive if and only if $p > 1$. Let $W$ be a positive, continuous, non-increasing function on $(0, \infty)$ which satisfy

$$\lim_{t \to \infty} W(t) = 0, \lim_{t \to 0} W(t) = \infty, \int_0^\infty W(t) \, dt = \infty, \int_0^1 W(t) \, dt = 1.$$  

For $1 \leq p < \infty$ the Lorentz function space $L_{w,p}(0, \infty)$ introduced in [8] is the space of all functions $f \in L_0(0, \infty)$ which satisfy

$$\|f\| = \left\{ \int_0^\infty f^*(t)^p W(t) \, dt \right\}^{1/p} < \infty$$

($f^*$ is the non-increasing rearrangement of $|f|$). If we assume only those conditions on $W$ which involve the interval $(0, 1]$ and define the norm by integration on $(0, 1]$, we get the space $L_{w,p}(0, 1]$. In the sequel I denotes $(0, \infty)$ or $(0, 1]$. We write $L_{w,p}$ instead of $L_{w,p}(I)$ if we do not specify $I$ exactly or if it is clear from the context which $I$ we deal with. $L_{w,p}$ are Köthe function spaces in which the norm is order continuous, hence $L'_{w,p} = L_{w,p}^* \cdot L_{w,p}$ is $p$-convex with constant 1; it is not $r$-convex for any $r > p$ by the same reason as that of $d(w, p)$ (cf. [3]).

An automorphism of $I$ on itself is a 1-1 (a.e.) map $\tau$ of $I$ on itself such that $\tau$ and $\tau^{-1}$ are measurable and $\tau$ preserves measure.

**Definition 1.** — Let $w = (w_i)_{i=1}^\infty$ be a positive, non-increasing sequence and let $W$ be a positive, non-increasing function defined in $I$ and integrable on finite intervals. For $p > 0$

a) We say that $w$ is $p$-regular if

$$w_n^p \sim \frac{1}{n} \sum_{i=1}^n w_i^p ; \quad n \in \mathbb{N}$$

b) We say that $W$ is $p$-regular if

$$W(x)^p \sim \frac{1}{x} \int_0^x W(t)^p \, dt ; \quad x \in I.$$ 

**Theorem 2.** — For $1 \leq p < \infty$ let $X$ be one of the spaces $d(w, p), L_{w,p}(0, 1)$ or $L_{w,p}(0, \infty)$.
a) For $p < q < \infty$ a necessary and sufficient condition for $X$ to be $q$-concave is that the sequence $w$ or the function $W$ is $\frac{s}{p}$-regular, where $s$ is defined by $\frac{1}{s} = \frac{1}{p} - \frac{1}{q}$.

b) If $q(x) = \inf\{q ; X$ is $q$-concave\} $< \infty$ then $X$ is not $q(X)$ concave.

c) A necessary and sufficient condition for the existence of $q < \infty$ so that $X$ is $q$-concave (i.e. for $X$ not to contain $\ell_\infty$ uniformly) is that $w$ or $W$ is 1-regular.

We prove Theorem 2 for function spaces; the proof for sequence spaces is analogous.

**Lemma 1.** – For a positive, non increasing function $W$ defined in $1$ and $p > 0$, the following are equivalent:

a) $W$ is $p$-regular.

b) $\sup_{x \in I} \frac{1}{x} \int_0^x \left( \frac{W(t)}{W(\chi - t)} \right)^p dt < \infty$.

If, in addition, $p \geq 1$ then a) and b) are equivalent to

c) $\sup_{x \in I} \frac{1}{x} \int_0^x \frac{W(t) dt}{\int_0^x W(t) dt} < \infty$.

The equivalence of a) and b) is very simple and we omit its proof. The equivalence of b) and c) will follow from lemmas 2) and 3) in the sequel.

**Lemma 2.** – For $0 < p < \infty$ there is $K(p) > 0$ so that if $f$ and $g$ are positive, non increasing functions on $(0, \infty)$ then for all $x > 0$

$$\frac{1}{x} \int_0^x \left( \frac{f(t)}{g(x - t)} \right)^p dt > K(p) \left( \frac{1}{x} \int_0^x f(t) dt \right)^p \left( \frac{1}{x} \int_0^x g(t) dt \right)^p .$$

Proof. – We put $\bar{g}(x) = \frac{1}{x} \int_0^x g(t) dt$. 

\[
\mu \left\{ t : \frac{1}{g(t)} \leq \frac{1}{2g(x)} \right\} = \mu \{ t : g(t) \geq 2g(x) \} \leq \frac{x}{2}
\]

(\mu\text{-Lebesgue measure}).

Therefore, in the interval \( \left( 0, \frac{x}{2} \right] \), \( \frac{1}{2g(x)} < \frac{1}{g(x-t)} \) and we get

\[
\frac{1}{x} \int_0^x \left( \frac{f(t)}{g(x-t)} \right)^p dt > \frac{1}{x} \int_0^{x/2} \left( \frac{f(t)}{g(x-t)} \right)^p dt
\]

\[
> \frac{1}{x} \int_0^{x/2} \left( \frac{f(t)}{2g(x)} \right)^p dt > \frac{1}{2x} \int_0^x \left( \frac{f(t)}{2g(x)} \right)^p dt
\]

\[
= \frac{1}{2^{p+1}} \frac{1}{x} \int_0^x (f(t))^p dt
\]

q.e.d.

**Lemma 3.** Let \( f \) be a positive, non increasing function defined in \( I \) and \( 1 < p < \infty \). Suppose for some \( K > 0 \)

\[
\sup_{x \in I} \frac{1}{x} \int_0^x (f(t))^p dt \leq K ;
\]

(4)

then there is \( N > 0 \) so that

\[
\sup_{x \in I} \frac{1}{x} \int_0^x \left( \frac{f(t)}{f(x-t)} \right)^p dt \leq N .
\]

(5)

**Proof.** We put \( \bar{f}(x) = \frac{1}{x} \int_0^x f(t) dt \). It is enough to show that for some \( c > 0 \), for all \( x \)

\[
 cf(x) \geq \bar{f}(x)
\]

(6)

since then

\[
\frac{1}{x} \int_0^x \left( \frac{f(t)}{f(x-t)} \right)^p dt \leq \frac{1}{x} \int_0^x \left( \frac{f(t)}{f(x)} \right)^p dt
\]

\[
\leq \frac{c^p}{x} \int_0^x \left( \frac{f(t)}{f(x)} \right)^p dt \leq c^p K .
\]

We prove therefore that (4) implies (6).
Let $A$ be such that \( \log \frac{\sqrt{A}}{A} \leq -\frac{1}{2} \) and suppose \( \frac{f(x_0)}{f(x_0)} \leq \frac{1}{A} \).

Let $x_1 = \sqrt{A} x_0$ (later we show that in the case $I = (0,1]$ we may assume $x_0 \leq \frac{1}{\sqrt{A}}$). Let $x_0 < t < x_1$, then

\[
\overline{f}(t) \geq \frac{1}{t} \int_0^{x_0} f(s) \, ds = \frac{x_0}{t} \quad \overline{f}(x_0) \geq \frac{1}{\sqrt{A}} \quad A \quad f(x_0) \geq \sqrt{A} \quad f(t).
\]

Hence

\[
\int_{x_0}^{x_1} f(t) \, dt \leq \frac{1}{\sqrt{A}} \int_{x_0}^{x_1} \left( \frac{1}{t} \int_0^t f(s) \, ds \right) \, dt
\]

\[
\leq \frac{1}{\sqrt{A}} \int_{x_0}^{x_1} \frac{1}{t} \int_0^t f(s) \, ds \, dt
\]

\[
= \frac{1}{\sqrt{A}} \left( \log \frac{x_1}{x_0} \right) \int_0^{x_1} f(s) \, ds
\]

\[
= \log \frac{\sqrt{A}}{A} \int_0^{x_1} f(s) \, ds \leq \frac{1}{2} \int_0^{x_1} f(s) \, ds .
\]

We get

\[
\int_0^{x_1} f(t) \, dt \leq \int_0^{x_0} f(t) \, dt + \frac{1}{2} \int_0^{x_1} f(t) \, dt \quad \text{or}
\]

\[
\int_0^{x_1} f(t) \, dt \leq 2 \int_0^{x_0} f(t) \, dt .
\]

From (4) it follows now

\[
K^{1/p} \geq \frac{1}{(x_1)} \int_0^{x_1} f(t)^p \, dt \right)^{1/p} \quad \geq \quad \frac{(x_0)^{1/p} \left( \frac{1}{x_0} \int_0^{x_0} f(t)^p \, dt \right)^{1/p}}{2 \frac{x_0}{x_1} \left( \frac{1}{x_0} \int_0^{x_0} f(t) \, dt \right)}
\]

\[
\geq \frac{1}{2} \left( \frac{x_0}{x_1} \right)^{p-1}
\]

(the last inequality by Hölder).

Whence

\[
K^{1/p} \geq \frac{1}{2} \left( \frac{1}{\sqrt{A}} \right)^{p-1} \quad \text{or} \quad A \leq (2K^{1/p})^{1-1/p}
\]

which proves the assertion. If $I = (0,1]$, the preceding argument shows that if $c \geq (2K^{1/p})^{1-1/p}$ and \( \frac{\log \sqrt{c}}{\sqrt{c}} \leq \frac{1}{2} \) then for $0 < x < \frac{1}{\sqrt{c}}$.
holds $f(x) \leq c f(x)$. On the other hand, for $\frac{1}{\sqrt{c}} \leq x \leq 1$
\[ \frac{f(x)}{f(x)} \geq \frac{f(1)}{f(\frac{1}{\sqrt{c}})} \]
which completes the proof in this case as well. \[ \text{q.e.d.} \]

**Lemma 4.** — Let \( W \) be a positive, non-increasing continuous function defined on \( I \). Then \( A = \{ p > 0 ; W \text{ is } p\text{-regular} \} \) is an open interval (if it is not empty).

**Proof.** — By Hölder's inequality \( A \) is an interval. Suppose that \( W \) is \( p\)-regular for some \( p > 0 \). Then there is \( 0 < c < 1 \) such that for all \( x \in I \)
\[ \frac{c}{x} \leq \frac{W(x)^p}{\int_0^x W(t)^p \, dt} = \frac{d}{dx} \left( \log \int_0^x W(t)^p \, dt \right) . \] (7)
For \( 0 < x_0 < x_1 \) integration yields
\[ c \log \frac{x_1}{x_0} \leq \log \frac{\int_0^{x_1} W(t)^p \, dt}{\int_0^{x_0} W(t)^p \, dt} \]
whence
\[ x_0^{-c} \int_0^{x_0} W(t)^p \, dt \leq x_1^{-c} \int_0^{x_1} W(t)^p \, dt . \] (8)
From (7) and (8) we get
\[ x_0^{1-c} W(x_0)^p \leq x_0^{1-c} \frac{1}{x_0} \int_0^{x_0} W(t)^p \, dt \leq x_1^{1-c} \frac{1}{x_1} \int_0^{x_1} W(t)^p \, dt \leq \frac{1}{c} x_1^{1-c} W(x_1)^p . \] (9)
We choose \( \epsilon > 0 \) such that \( \theta = (1 - c) (1 + \epsilon) < 1 \). Let \( p_1 = p (1 + \epsilon) \). We claim that \( W \) is \( p_1\)-regular. Raising the ends of (9) to the power \( 1 + \epsilon \) we get
\[ x_0^\theta W(x_0)^{p_1} \leq K x_1^\theta W(x_1)^{p_1} \] (10)
for some constant \( K \). Let \( x \in I \), in the interval \( \left( 0, \frac{x}{2} \right) \), \( t < x - t \) hence (10) yields...
\[
\frac{1}{x} \int_0^x \left( \frac{W(t)}{W(x - t)} \right)^{\theta_1} dt \leq \frac{2}{x} \int_0^{x/2} \left( \frac{W(t)}{W(x - t)} \right)^{\theta_1} dt \\
\leq \frac{2K}{x} \int_0^{x/2} \left( \frac{x - t}{t} \right)^{\theta} dt \leq \frac{2K}{x} \int_0^x \left( \frac{x - t}{t} \right)^{\theta} dt < M
\]

where \( M \) does not depend on \( x \), since \( \theta < 1 \). We have shown that \( \sup_{x \in 1} \frac{1}{x} \int_0^x \left( \frac{W(t)}{W(x - t)} \right)^{\theta_1} dt < \infty \) which, by Lemma 1, is equivalent to \( p_1 \)-regularity of \( W \). q.e.d.

We omit the proof of the following simple lemma.

**Lemma 5.** Let \( g \in L_0(1) \). The multiplication operator \( T_g \) is a bounded operator from \( L_{W,p} \) into \( L_p \) if and only if

\[
\|T_g: L_{W,p} \rightarrow L_p\| = \sup_{x \in 1} \int_0^x W(t) dt < \infty.
\]

**Proof of Theorem 2.** Sufficiency. Suppose \( W \) is \( s/p \)-regular. By Theorem 1 and \( p \)-convexity of \( L_{W,p} \), it is enough to show that for some \( K > 0 \), for every \( g \in L_s \), there is \( h \in L_0 \) such that

\[
\text{support } g \subset \text{support } h
\]

and

\[
\|T_h: L_{W,p} \rightarrow L_p\| \|T_{g/h}: L_q \rightarrow L_{W,p}\| < K \|g\|_{L_s}.
\]

\[
\text{we put } \frac{g}{h}(t) = 0 \text{ if } h(t) = 0.
\]

Since \( L_{W,p} \) is rearrangement invariant we may assume \( g \) is positive and non-increasing. We take \( h = W^{1/p} \). Of course \( \|T_h: L_{W,p} \rightarrow L_p\| = 1 \).

We show that for positive non-increasing \( g \in L_s \)

\[
\|T_{g/h}: L_q \rightarrow L_{W,p}\| < K \|g\|_{L_s},
\]

i.e. that for all \( \varphi \in L_q \) and every automorphism \( \sigma \) of \( I \) on itself

\[
\left\{ \int_I \left( \frac{\varphi(\sigma(t)) g(\sigma(t))}{W(\sigma(t))^{1/p}} \right)^p W(t) dt \right\}^{1/p} < K \|g\|_{L_s}.
\]
Since \[ \left\{ \int_1 \left( \frac{\varphi(\sigma(t))}{W(\sigma(t))^{1/p}} \right)^{1/p} W(t) \, dt \right\}^{1/p} \leq \|\varphi\|_{L_q} \left\{ \int_1 \left( \frac{g(\sigma(t))}{W(\sigma(t))^{1/p}} \right)^{s/p} W(t)^{q/p} \, dt \right\}^{1/s} \]
it is enough to show that for all \( \sigma \)
\[ \left\{ \int_1 g(t)^s \left( \frac{W(\sigma(t))}{W(t)} \right)^{s/p} \, dt \right\}^{1/s} \leq K \|g\|_{L_s} \]
which is equivalent to the fact that for all non-increasing \( 0 \leq g \in L_1 \) and all \( \sigma \)
\[ \int_1 g(t) \left( \frac{W(\sigma(t))}{W(t)} \right)^{s/p} \, dt \leq K \|g\|_{L_1} . \tag{12} \]
In fact, if \( g = \frac{1}{x} \chi_{(0,x]} \) for some \( x \in I \) then, by Lemma 1 and \( s/p \)-regularity of \( W \)
\[ \int_1 g(t) \left( \frac{W(\sigma(t))}{W(t)} \right)^{s/p} \, dt = \frac{1}{x} \int_0^x \left( \frac{W(\sigma(t))}{W(t)} \right)^{s/p} \, dt \leq \frac{1}{x} \int_0^x \left( \frac{W(t)}{W(x-t)} \right)^{s/p} \, dt \leq K . \]
Now, for other positive, non-increasing functions \( g \in L_1 \), (12) follows from the fact that the convex hull of the functions \( \frac{1}{x} \chi_{(0,x]} \) is dense in \( L_1 \)-norm in \( \{f; \|f\|_{L_1} \leq 1, 0 \leq f - \text{non increasing}\} \).

\textbf{Necessity.} – Assume \( L_{W,p} \) is \( q \)-concave \((q < \infty)\). Since it is also \( p \)-convex, it is necessary that for every \( g \in L_s \) there is \( h \in L_0 \) such that (11) holds. In particular, from Lemma 5 we conclude (applying (11) to \( g = \frac{1}{x^{1/s}} \chi_{(0,x]} \)) that for every \( x \in I \) there is \( h \in L_0 \) so that
\[ \frac{\int_0^x h^*(t)^p \, dt}{\int_0^x W(t) \, dt} \leq K^p \quad \text{and} \quad \|T_{g/h}: L_q \to L_{W,p} \| \leq 1 . \tag{13} \]
We may, of course, assume that \( h \) is non increasing and that support \( h \subset (0,x] \). For all \( \varphi \in L_q \) we have
Let $0 < \epsilon < x$. We define $W_\epsilon$ to be equal to the constant $W(\epsilon)$ in $(0, \epsilon]$ and to $W(t)$ for $t \geq \epsilon$. Let the bounded function $\varphi$ be defined by

$$\varphi(x - t) = \frac{1}{x^{1/q}} \left( \frac{W_\epsilon(t)}{h(x - t)^p} \right)^{\frac{s}{p}} ; \quad 0 < t \leq x$$

$$\varphi(t) = 0 \quad ; \quad x < t .$$

Then $$\left( \frac{\varphi}{h} \right)^*(t) = \frac{\varphi(x - t)}{h(x - t)}$$ and by (14)

$$\| \varphi \|_{L_q} \geq \left\{ \frac{1}{x^{p/s}} \int_0^x \frac{1}{x^{p/q}} \frac{(W_\epsilon(t)^{s/p} W(t)}{(h(x - t)^p)^{s/p} h(x - t)^p} \, dt \right\}^{1/p}$$

$$\geq \left\{ \frac{1}{x} \int_0^x \frac{(W_\epsilon(t)^{s/p} W(t)}{(h(x - t)^p)^{s/p} h(x - t)^p} \, dt \right\}^{1/p}$$

$$= \| \varphi \|_{L_q} \left\{ \frac{1}{x} \int_0^x \left( \frac{W_\epsilon(t)}{h(x - t)^p} \right)^{s/p} \, dt \right\}^{1/s} .$$

It follows now from Lemma 2 and from (13) that

$$1 \geq \frac{1}{x} \int_0^x \left( \frac{W_\epsilon(t)}{h(x - t)^p} \right)^{s/p} \, dt \geq K \left( \frac{s}{p} \right) \frac{1}{x} \int_0^x W_\epsilon(t)^{s/p} \, dt$$

$$\geq \frac{K \left( \frac{s}{p} \right)}{K^s} \frac{1}{x} \int_0^x W(t)^{s/p} \, dt$$

$$\geq \frac{1}{x} \int_0^x W(t)^{s/p} \, dt \leq \frac{K^p}{K^s} \left( \frac{s}{p} \right)^{p/s} .$$

By Lemma 1 this is equivalent to $\frac{s}{p}$-regularity of $W$. This proves part a) of the theorem. Part b) follows from part a) and Lemma 4.
Part c). If $L_{w,p}$ is $q$-concave with $q < \infty$ then $W$ is $\frac{s}{p}$-regular with $\frac{s}{p} > 1$. Therefore it is also 1-regular (it is also easy to construct directly subspaces of $L_{w,p}$ which are uniformly isomorphic to $l^q_w$, if $W$ is not 1-regular).

On the other hand, if $W$ is 1-regular, then from Lemma 4 it follows that it is $r$-regular for some $r > 1$. Part a) implies now that $L_{w,p}$ is $q$-concave for some $q < \infty$. (We remark that this last argument provides an alternative proof for the isomorphic parts of Theorem 3.1 in [4] and Theorem 1 in [1]; i.e. for 1-regularity being a necessary and sufficient condition for $L_{w,p}$ or $d(w,p)$ to be isomorphic to a uniformly convex space when $p > 1$). q.e.d.

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